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On the Topology of Surface Singularities $\{z^n = f(x, y)\}$ for f Irreducible

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1. Introduction

Let $(X, 0) \subset (\mathbb{C}^k, 0)$ be the germ of a complex analytic normal surface singularity. The intersection of X with a sufficiently small sphere centered at the origin in $\mathbb{C}^k$ is a compact connected oriented 3-manifold Σ, called the *link* of $(X, 0)$, that does not depend upon the embedding in $\mathbb{C}^k$. Let Γ be the dual resolution graph of a good resolution of the singularity. The homeomorphism type of the link can be recovered from Γ; conversely, Neumann [8] proved that (aside from a few exceptions) the homeomorphism type of the link determines the minimal good resolution graph. One interesting class of normal surface singularities is the set of those for which the link is a *rational homology sphere* ($\mathbb{Q}$HS) (i.e., $H_1(\Sigma, \mathbb{Q}) = 0$). The link is a $\mathbb{Q}$HS if and only if any good resolution graph Γ of $(X, 0)$ is a tree of rational curves.

The work of Neumann and Wahl (described in Section 2; see also [10; 18]) provides a method for generating analytic data for singularities from topological data. Starting with a resolution graph Γ that satisfies certain conditions, known as the "semigroup and congruence conditions", one can produce defining equations for a normal surface singularity with resolution graph Γ. The singularities that result from this algorithm are called *splice quotients.* If the link Σ is a $\mathbb{Z}$HS ($H_1(\Sigma, \mathbb{Z}) = 0$), then only the semigroup conditions are relevant, and the singularities produced by the algorithm are said to be *of splice type*. This work has led to a recent interest in the properties of splice quotients and related topics (see [3; 6; 13; 14; 17]), and there are still many unanswered questions.

One of the first questions that arises is: How many singularities with $\mathbb{Q}$HS link are splice quotients? There are two layers to the problem, topological and analytic. If one has a singularity that satisfies the necessary topological conditions (which depend only on the resolution graph), then there exist splice quotients with that topological type—but it is a separate issue to determine whether the singularity is analytically isomorphic to a splice quotient. Originally, one wondered whether all $\mathbb{Q}$-Gorenstein singularities with $\mathbb{Q}$HS link would turn out to be splice quotients. However, the first counterexamples were found in the paper of Luengo-Velasco, Melle-Hernández, and Némethi [3]. The authors give an example of a hypersurface singularity for which the resolution graph does not satisfy the semigroup

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conditions; they also give an example of a singularity for which the semigroup and congruence conditions are satisfied yet the analytic type is not a splice quotient. On the other hand, there are nice classes of singularities for which *all* analytic types are splice quotients: weighted homogeneous singularities, as shown by Neumann in [7], and rational and $\mathbb{Q}$HS-link minimally elliptic singularities, as shown by Okuma in [13].

A natural class of surface singularities to study (after weighted homogeneous, rational, and minimally elliptic) is the class of hypersurface singularities defined by an equation of the form $z^n = f(x, y)$. If $\{f(x, y) = 0\}$ defines a reduced curve with a singularity at the origin in $\mathbb{C}^2$ then, for $n > 1$, the surface $X_{f,n} := \{z^n = f(x, y)\}$ has an isolated (hence normal) singularity at the origin in $0 \in \mathbb{C}^3$. For f irreducible, the resolution graph of $(X_{f,n}, 0)$ can be constructed from n together with a finite set of pairs of positive integers associated to f, known as the *topological pairs* $\{(p_i, a_i) \mid 1 \le i \le s\}$ and defined in [2] (a variant of the more commonly known *Puiseux pairs*). The topological pairs completely determine the topology of the plane curve singularity. If there is only one topological pair ($s = 1$), then any such $(X_{f,n}, 0)$ with $\mathbb{Q}$HS link has the topological type of a weighted homogeneous singularity and thus has the topological type of a splice quotient. In [9], Neumann and Wahl prove that the link of $(X_{f,n}, 0)$ is a $\mathbb{Z}$HS if and only if f is irreducible and all p_i and a_i are relatively prime to n. (The result is incorrectly stated in [9], where the pairs in question are mistakenly identified as the Newton pairs instead of the topological pairs.) For this case, Neumann and Wahl prove in [12] that any such $(X_{f,n}, 0)$ is of splice type. That is, not only are the semigroup conditions satisfied but, moreover, every $(X_{f,n}, 0)$ with $\mathbb{Z}$HS link is isomorphic to one that results from Neumann and Wahl's construction.

The main result of this paper is a complete characterization of the $(X_{f,n}, 0)$, with f irreducible and $s \ge 2$, that have a resolution graph satisfying the semigroup and congruence conditions. For f irreducible, there is an explicit criterion given by Mendris and Némethi in [4], in terms of n and the topological pairs, that determines when the link of $(X_{f,n}, 0)$ is a $\mathbb{Q}$HS (see Proposition 3.2). One can see that there are plenty of $(X_{f,n}, 0)$ for which the link is a $\mathbb{Q}$HS but not a $\mathbb{Z}$HS. From now on, whenever we are not referring to topological pairs, the notation (m, n) denotes the greatest common divisor of the integers m and n. Our main result is the following theorem.

MAIN THEOREM. *Let f be irreducible with topological pairs $\{(p_i, a_i) \mid 1 \le i \le s\}$ with $s \ge 2$, and let n be an integer > 1. Then $(X_{f,n}, 0)$ has $\mathbb{Q}$HS link and a good resolution graph that satisfies the semigroup and congruence conditions if and only if either*:

(i) $(n, p_s) = 1$, $(n, p_i) = (n, a_i) = 1$ *for* $1 \le i \le s-1$, *and* $a_s/(n, a_s)$ *is in the semigroup generated by* $\{a_{s-1}, p_1 \cdots p_{s-1}, a_j p_{j+1} \cdots p_{s-1} \mid 1 \le j \le s-2\}$; *or*

(ii) $s = 2$, $p_2 = 2$, $(n, p_2) = 2$, *and* $(n, a_2) = \left(\frac{n}{2}, p_1\right) = \left(\frac{n}{2}, a_1\right) = 1$.

It is somewhat surprising that so few $(X_{f,n}, 0)$ satisfy the topological conditions, given the result in the $\mathbb{Z}$HS case. Aside from case (ii), which is rather restrictive, this result says that if any of the topological pairs other than a_s have factors in common with n, then $(X_{f,n}, 0)$ does not have the topological type of a splice quotient. One could say that if $(X_{f,n}, 0)$ gets “too far” from the $\mathbb{Z}$HS case (for which all *analytic* types are splice quotients) then it cannot even have the topology of a splice quotient.

If the resolution graph does satisfy the semigroup and congruence conditions, then we do not know a priori what the equations of the splice quotients produced from the Neumann–Wahl algorithm look like. Not only is it unclear whether $(X_{f,n}, 0)$ is itself a splice quotient, it is not even clear that there exist splice quotients defined by any equation of the form $z^n = g(x, y)$. It turns out that there do exist such splice quotients, but the length of the proof is such that it cannot be included here (see [16] for that result). In the case of weighted homogeneous splice quotients, it was shown in [15] that, in general, not every deformation with the same topological type is analytically isomorphic to a splice quotient. Therefore, we expect that there are few cases for which *every* $(X_{f,n}, 0)$ of a given topological type is a splice quotient.

Consider the following example.

EXAMPLE 1.1. Let $X_n := \{z^n = y^5 + (x^3 + y^2)^2\}$. The plane curve singularity defined by $y^5 + (x^3 + y^2)^2 = 0$ is irreducible with two topological pairs: $p_1 = 2$ and $a_1 = 3$; and $p_2 = 2$ and $a_2 = 15$. The link of $(X_n, 0)$ is a $\mathbb{Q}$HS if and only if $(n, 2) = 1$ or $(n, 15) = 1$. We can say the following about X_n.

- If n is relatively prime to 2, 3, and 5, then $(X_n, 0)$ has $\mathbb{Z}$HS link and hence is of splice type. In fact, we could replace $y^5 + (x^3 + y^2)^2$ by any curve with the same topological pairs and still have a singularity of splice type.
- If n is divisible by 3 then, according to the Main Theorem, $(X_n, 0)$ does not even have the topological type of a splice quotient.
- If $n = 5k$, where k is relatively prime to 2 and 3, then $(X_n, 0)$ has the topology of a splice quotient by case (i) of the Main Theorem; in fact, $(X_n, 0)$ is itself a splice quotient [16].
- If $n = 2k$, where k is relatively prime to 2, 3, and 5, then $(X_n, 0)$ has the topology of a splice quotient by case (ii) of the Main Theorem. It is unclear whether or not $(X_n, 0)$ is a splice quotient. However, if we replace $y^5 + (x^3 + y^2)^2$ by $(x^3 - y^2 - y^3)^2 - 4y^5$, which has the same topological pairs, then it is a splice quotient [16].

The rest of this paper is entirely devoted to proving the Main Theorem. In Section 2, we provide a brief summary of the work of Neumann and Wahl. Section 3 contains a description of the resolution graph and splice diagram for $(X_{f,n}, 0)$. Some of the computations that are necessary for the proof of the Main Theorem depend upon work done by Mendris and Némethi in [4]; Section 3.1 is a reiteration of this material. In Section 4, we analyze the semigroup conditions for the splice diagram associated to $(X_{f,n}, 0)$. Section 5 contains additional computations that are needed for checking the congruence conditions. Finally, in Section 6, we use the computations from the previous three sections to prove the Main Theorem.

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2. The Neumann–Wahl Algorithm

This section contains a summary of the method defined by Neumann and Wahl in [11] to produce equations for the splice quotients and their universal abelian covers; we refer to this method as the *Neumann–Wahl algorithm*. The algorithm begins with a negative-definite graph Γ that is a tree of smooth rational curves (equivalently, the dual resolution graph associated to a good resolution of a normal surface singularity with $\mathbb{Q}$HS link) and the *splice diagram* Δ associated to Γ. Splice diagrams were introduced by Eisenbud and Neumann [2] for plane curve singularities (building on work of Siebenmann) and were later generalized by Neumann and Wahl. If Δ satisfies the "semigroup conditions" (Definition 2.1), then the algorithm produces a set of equations that defines a family of isolated complete intersection surface singularities. The algorithm also produces an action of the finite abelian group $D(\Gamma)$, the discriminant group of Γ, on the coordinates used for the splice diagram equations. If Γ satisfies further combinatorial conditions, the "congruence conditions" (Definition 2.3), then one can choose a set of splice diagram equations such that the discriminant group acts on every singularity $(Y, 0)$ in the family. Furthermore, the quotient of $(Y, 0)$ by $D(\Gamma)$ is an isolated normal surface singularity with resolution graph Γ, and the covering given by the quotient map is the universal abelian covering (the maximal abelian covering that is unramified away from the singular point).

In a weighted graph, the *valency* of a vertex is the number of adjacent edges. A *node* is a vertex of valency at least 3, a *leaf* is a vertex of valency 1, and a *string* is a connected subgraph that does not include a node. The procedure for computing the splice diagram Δ associated to a resolution graph Γ is as follows. First, omit the self-intersection numbers of the vertices and contract all strings of valency-2 vertices in Γ. To each node v in the resulting diagram Δ, we then attach a weight d_{ve} in the direction of each adjacent edge e. Remove the vertex in Γ that corresponds to the node v and the edge that corresponds to e, and let Γ_{ve} be the remaining connected subgraph that was connected to v by e. Then the weight $d_{ve} = \det(-C_{ve})$, where C_{ve} is the intersection matrix of the graph Γ_{ve}. Figure 1 gives a simple example. Similarly, we define a subgraph Δ_{ve} of Δ as follows. Remove v and e, and let Δ_{ve} be the remaining connected subgraph that was connected to v by e. For any two vertices v and w in Δ, the *linking number* ℓ_{vw} is the product of the weights adjacent to but not on the shortest path from v to w. Let ℓ'_{vw} be the linking number of v and w, excluding the weights around v and w.

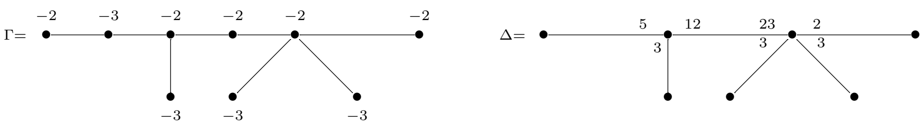


Figure 1 A resolution graph Γ and its associated splice diagram Δ

DEFINITION 2.1 (Semigroup conditions). The *semigroup condition* at v in the direction of e is

$$d_{ve} \in \mathbb{N}\langle \ell'_{vw} \mid w \text{ is a leaf in } \Delta_{ve}\rangle.$$

We say that Δ *satisfies the semigroup conditions* if the semigroup condition for every node v and every adjacent edge e is satisfied. Note that the condition is trivially satisfied for an edge leading to a leaf.

To each leaf w in Δ we associate a variable Z_w. If Δ satisfies the semigroup conditions, then for each v and e as before there exist $\alpha_{vw} \in \mathbb{N} \cup \{0\}$ such that

$$d_{ve} = \sum_{w \text{ a leaf in } \Delta_{ve}} \alpha_{vw} \ell'_{vw}.$$

Then a monomial $M_{ve} = \prod_w Z_w^{\alpha_{vw}}$, a product over leaves w in Δ_{ve} with α_{vw} as before, is called an *admissible monomial* for e at v. If one associates the weight ℓ_{vw} to Z_w then, for this weight system (the v-weighting), M_{ve} has weight $d_v = \prod_e d_{ve}$, where the product is taken over all edges e adjacent to v.

DEFINITION 2.2 (Splice diagram equations). Suppose Δ satisfies the semigroup conditions. For each node v and adjacent edge e, choose an admissible monomial M_{ve}. Let δ_v denote the valency of the vertex v. A set of *splice diagram equations* for Δ is a set of equations of the form

$$\left\{\sum_e a_{vie} M_{ve} = 0 \;\middle|\; 1 \le i \le \delta_v - 2,\ v \text{ a node in } \Delta\right\};$$

here, for each v, all maximal minors of the matrix (a_{vie}) have full rank. (One can also add to each equation a convergent power series in the Z_w for which all of the terms have v-weight greater than d_v; however, since this extension has no bearing upon the work herein, we omit it in further discussion.)

Each vertex $v \in \Gamma$ corresponds to an exceptional curve E_v. Let $\mathbb{E} := \bigoplus_{v\in\Gamma} \mathbb{Z}E_v$. The intersection pairing defines a natural injection $\mathbb{E} \hookrightarrow \mathbb{E}^* = \mathrm{Hom}(\mathbb{E}, \mathbb{Z})$, and the discriminant group is the finite abelian group $D(\Gamma) := \mathbb{E}^*/\mathbb{E}$. This group is isomorphic to $H_1(\Sigma, \mathbb{Z})$. The order of $D(\Gamma)$ is $\det(\Gamma) := \det(-C(\Gamma))$, where $C(\Gamma)\colon \mathbb{E} \times \mathbb{E} \to \mathbb{Z}$ is the intersection pairing. There are induced symmetric pairings of $\mathbb{E} \otimes \mathbb{Q}$ into $\mathbb{Q}$ and $D(\Gamma)$ into $\mathbb{Q}$.

Suppose Δ has t leaves, and let $Z_1, \dots, Z_t$ be the associated variables. Neumann and Wahl define a faithful diagonal representation of $D(\Gamma)$ on $\mathbb{C}[Z_1, \dots, Z_t]$. Let $E_1, \dots, E_t$ be the curves in Γ corresponding to the t leaves of Δ, and let $e_j \in \mathbb{E}^*$ be the dual basis element corresponding to E_j; that is, $e_j(E_k) = \delta_{jk}$. Finally, for $r \in \mathbb{Q}$, let $[r]$ denote the image of the equivalence class of r under the map $\mathbb{Q}/\mathbb{Z} \hookrightarrow \mathbb{C}^*$ defined by $r \mapsto \exp(2\pi i r)$. Then the action of the discriminant group on the polynomial ring $\mathbb{C}[Z_1, \dots, Z_t]$ is generated by the action of the e_j, $1 \le j \le t$, which is defined by $e_j \cdot Z_k = [-e_j \cdot e_k]Z_k$ for $1 \le j, k \le t$.

DEFINITION 2.3 (Congruence conditions). Let Γ be a graph for which the associated splice diagram Δ satisfies the semigroup conditions. Then we say that Γ satisfies the *congruence condition* at a node v if one can choose an admissible monomial for each adjacent edge e such that all of these monomials transform by the same character under the action of $D(\Gamma)$. If this condition is satisfied for every node v, then Γ *satisfies the congruence conditions.*

We should mention here that Okuma gives a single condition that is equivalent to the semigroup and congruence conditions together [13, Cond. 3.3]. That this condition is equivalent to the semigroup and congruence conditions is shown in [11]. We will often say "Γ satisfies the semigroup and congruence conditions" rather than "Δ satisfies the semigroup conditions and Γ satisfies the congruence conditions". Suppose a resolution graph Γ satisfies the semigroup and congruence conditions. Then by "a set of splice diagram equations for Γ" we mean equations as in Definition 2.1 such that, for each v, the admissible monomials M_{ve} transform equivariantly under $D(\Gamma)$. A resolution tree Γ is *quasi-minimal* if any string in Γ either contains no (-1)-weighted vertex or consists of a unique (-1)-weighted vertex.

THEOREM 2.4 [11]. *Suppose Γ is quasi-minimal and satisfies the semigroup and congruence conditions. Then a set of splice diagram equations for Γ defines an isolated complete intersection singularity $(Y,0)$, $D(\Gamma)$ acts freely on $Y-\{0\}$, and the quotient $X := Y/D(\Gamma)$ has an isolated normal surface singularity and a resolution with dual resolution graph Γ. Moreover, $(Y,0) \to (X,0)$ is the universal abelian cover.*

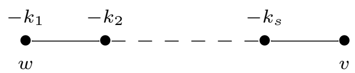
We will use the next two propositions to check the congruence conditions.

PROPOSITION 2.5 [11]. *Let Γ be a graph for which the associated splice diagram Δ satisfies the semigroup conditions. Then the congruence conditions are equivalent to the following: For every node v and adjacent edge e in Δ there is an admissible monomial $M_{ve} = \prod_w Z_w^{\alpha_w}$ such that, for every leaf w' in Δ_{ve},*

$$\left[\sum_{w \neq w'} \alpha_w \frac{\ell_{ww'}}{\det(\Gamma)} - \alpha_{w'} e_{w'} \cdot e_{w'}\right] = \left[\frac{\ell_{vw'}}{\det(\Gamma)}\right].$$

REMARK 2.6. It is easy to check, using the next proposition, that this condition is always satisfied for an edge leading directly to a leaf.

PROPOSITION 2.7 [11]. *Suppose we have a string from a leaf w to an adjacent node v in a resolution graph Γ, as in the following diagram, with associated continued fraction d/p.*



That is,

$$\frac{d}{p} = k_1 - \cfrac{1}{k_2 - \cfrac{1}{\ddots - \cfrac{1}{k_s}}}.$$

Then, if d_v is the product of weights at v, $e_w \cdot e_w = -d_v/(d^2 \det(\Gamma)) - p/d$.

3. The Resolution Graph and Splice Diagram

Let $\{f(x, y) = 0\} \subset \mathbb{C}^2$ define an analytically irreducible plane curve with a singularity at the origin, and let $X_{f,n} := \{z^n = f(x, y)\} \subset \mathbb{C}^3$. In [4], Mendris and Némethi prove that the link of $(X_{f,n}, 0)$ completely determines the Newton/topological pairs of f and the value of n, with two well-understood exceptions. In doing so, they give a presentation of the construction of the resolution graph of $(X_{f,n}, 0)$ that is useful for our purposes. Section 3.1 summarizes the results we need from Mendris and Némethi's work, and we use their notation whenever possible. In Section 3.2, we describe the associated splice diagram.

It turns out that, when $n = p_s = 2$, the resolution graph has a structure that differs significantly from the general case. It is referred to as the "pathological case" or "P-case" by Mendris and Némethi, and we use this terminology as well. Some of the computations must be done separately for the pathological case.

3.1. Resolution Graph

Suppose that f has *Newton pairs* $\{(p_k, q_k) \mid 1 \le k \le s\}$ (see [2, p. 49]). They satisfy the following properties: $q_1 > p_1$, $q_k \ge 1$, $p_k \ge 2$, and $\gcd(p_k, q_k) = 1$ for all k. Define integers a_k by $a_1 = q_1$ and

$$a_k = q_k + a_{k-1}p_{k-1}p_k, \quad 2 \le k \le s. \tag{1}$$

The pairs $\{(p_k, a_k) \mid 1 \le k \le s\}$, defined by Eisenbud and Neumann in [2], are referred to as the *topological pairs* of f. These are the integers that appear in the splice diagram of the link of the plane curve singularity defined by $f = 0$ in $\mathbb{C}^2$. Note that $a_1 > p_1$, $a_k > a_{k-1}p_{k-1}p_k$, and $\gcd(p_k, a_k) = 1$ for all k.

The topological pairs $\{(p_k, a_k) \mid 1 \le k \le s\}$ are related to the *Puiseux pairs* $\{(p_k, m_k) \mid 1 \le k \le s\}$ as follows: $a_1 = m_1$ and $a_k = m_k - m_{k-1}p_k + a_{k-1}p_{k-1}p_k$ for $2 \le k \le s$. Furthermore, let $\bar{\beta}_k$, $0 \le k \le s$, be the generators of the semigroup associated to the plane curve singularity defined by f (see [19]). Then we have $\bar{\beta}_0 = p_1p_2 \cdots p_s$, $\bar{\beta}_k = a_k p_{k+1} \cdots p_s$ for $1 \le k \le s - 1$, and $\bar{\beta}_s = a_s$.

By an "embedded" resolution of the germ of a function $g\colon (X, 0) \to (\mathbb{C}, 0)$ we mean a resolution of the singularity $\pi : \tilde{X} \to X$ such that $\pi^{-1}(\{g = 0\})$ is a divisor with only normal crossing singularities. We also assume that no irreducible component of the exceptional set $\pi^{-1}(0)$ intersects itself and that any two irreducible components have at most one intersection point. The minimal good embedded resolution graph of $f\colon (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ is a tree of rational curves, denoted $\Gamma(\mathbb{C}^2, f)$. The construction of the graph $\Gamma(\mathbb{C}^2, f)$ is well known (e.g., [1]). Reproducing the notation of Mendris and Nèmethi [4], we consider this graph in a convenient schematic form (Figure 2), where the dashed lines represent strings of rational curves (possibly empty) for which the self-intersection numbers are determined by the continued fraction expansions of p_k/q_k and q_k/p_k (see Section 5.2 for details).

There is an algorithm for constructing an embedded resolution graph (not necessarily minimal) of the function $z\colon (X_{f,n}, 0) \to (\mathbb{C}, 0)$ from the graph $\Gamma(\mathbb{C}^2, f)$. Here we follow the presentation in [4], reproducing only what is necessary for our

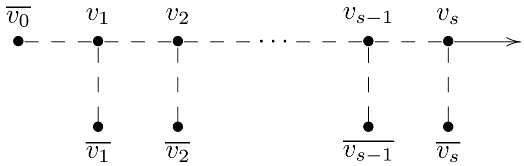


Figure 2 Schematic form of $\Gamma(\mathbb{C}^2, f)$ (reproduced from [4])

purposes. The output of this algorithm, without any modifications by blowing up or down, is referred to by Mendris and Némethi as the *canonical* embedded resolution graph of z in $(X_{f,n}, 0)$ and is denoted $\Gamma^{\text{can}}(X_{f,n}, z)$. The n-fold "covering" or "graph projection" produced in the algorithm is denoted $q\colon \Gamma^{\text{can}}(X_{f,n}, z) \to \Gamma(\mathbb{C}^2, f)$.

DEFINITION 3.1 [4]. Define positive integers d_k, h_k, $\widetilde{h_k}$, p'_k, and a'_k as follows:

- $d_k = (n, p_{k+1}p_{k+2}\cdots p_s)$ for $0 \le k \le s-1$,
- $d_s = 1$;

and, for $1 \le k \le s$,

- $h_k = (p_k, n/d_k)$,
- $\widetilde{h_k} = (a_k, n/d_k)$,
- $p'_k = p_k/h_k$,
- $a'_k = a_k/\widetilde{h_k}$.

If w is a vertex in $\Gamma(\mathbb{C}^2, f)$ then all vertices in $q^{-1}(w)$ have the same multiplicity and genus, which we denote by m_w and g_w (respectively).

PROPOSITION 3.2 [4]. *Let $q\colon \Gamma^{\text{can}}(X_{f,n}, z) \to \Gamma(\mathbb{C}^2, f)$ be the graph projection described just before Definition 3.1. Then $\Gamma^{\text{can}}(X_{f,n}, z)$ is a tree such that the following equalities hold*:

(a) $\#q^{-1}(v_s) = 1$ *and* $\#q^{-1}(v_k) = h_{k+1}\cdots h_s$ *for* $1 \le k \le s-1$,
$\#q^{-1}(\overline{v_s}) = \widetilde{h_s}$ *and* $\#q^{-1}(\overline{v_k}) = \widetilde{h_k}h_{k+1}\cdots h_s$ *for* $1 \le k \le s-1$,
$\#q^{-1}(\overline{v_0}) = h_1\cdots h_s$;

(b) $m_{v_k} = a'_k p'_k p'_{k+1}\cdots p'_s$ *for* $1 \le k \le s$,
$m_{\overline{v_0}} = p'_1 p'_2 \cdots p'_s$,
$m_{\overline{v_k}} = a'_k p'_{k+1}\cdots p'_s$ *for* $1 \le k \le s-1$,
$m_{\overline{v_s}} = a'_s$;

(c) $g_{v_k} = (h_k - 1)(\widetilde{h_k} - 1)/2$ *for* $1 \le k \le s$,
$g_{\overline{v_k}} = 0$ *for* $0 \le k \le s$.

In particular, the link of $(X_{f,n}, 0)$ is a $\mathbb{Q}HS$ if and only if $(h_k - 1)(\widetilde{h_k} - 1) = 0$ for all k, $1 \le k \le s$.

The schematic form of $\Gamma^{\text{can}}(X_{f,n}, z)$ is displayed in Figure 3. Abusing notation, we have labeled any vertex in $q^{-1}(v_k)$ (resp. $q^{-1}(\overline{v_k})$) with v_k (resp. $\overline{v_k}$). The dashed lines represent strings of vertices that are not necessarily minimal. By construction, each string must contain at least as many vertices as its image in $\Gamma(\mathbb{C}^2, f)$.

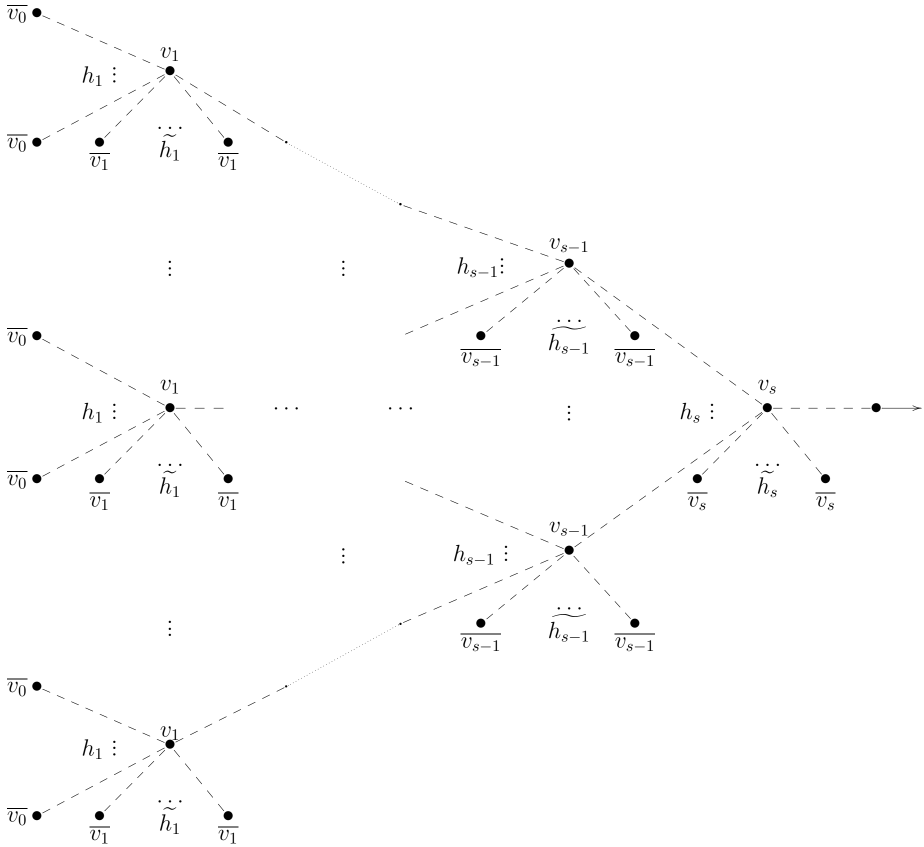


Figure 3 Schematic form of $\Gamma^{\mathrm{can}}(X_{f,n}, z)$ (reproduced from [4])

A vertex is called a *rupture vertex* if it has positive genus or is a node. Note that any rupture vertex of $\Gamma^{\mathrm{can}}(X_{f,n}, z)$ must be in $q^{-1}(v_k)$ for some k.

Certain subgraphs of $\Gamma^{\mathrm{can}}(X_{f,n}, z)$ *and their determinants*
Let w be a vertex in $\Gamma(\mathbb{C}^2, f)$, and let v' be any vertex in $q^{-1}(w)$. If $w = v_k$ for some k, $1 \leq k \leq s-1$, then the shortest path from v' to the arrowhead of $\Gamma^{\mathrm{can}}(X_{f,n}, z)$ contains at least one rupture vertex, and the rupture vertex along that path that is closest to v' is a vertex $v'' \in q^{-1}(v_{k+1})$. Define $\Gamma(v')$ to be the subgraph of $\Gamma^{\mathrm{can}}(X_{f,n}, z)$ consisting of the string of vertices between v' and v'', not including v' and v''. If $w = v_s$, then the shortest path from v' to the arrowhead is a string; let $\Gamma(v')$ be this string, not including v'. Finally, if $w = \overline{v_k}$, $0 \leq k \leq s$, let v'' be the rupture vertex that is closest to v' on the shortest path from v' to the arrowhead. Define $\Gamma(v')$ to be the subgraph consisting of the string of vertices from v' to v'', including v' but not v''. Up to isomorphism, none of these strings depends upon the choice of v' in $q^{-1}(w)$, so whenever the particular vertex v' does not matter, we will simply use $\Gamma(w)$ to denote these strings.

Fix an integer k, $1 \le k \le s$, and fix a vertex v' in $q^{-1}(v_k)$. Consider the collection of connected subgraphs that make up $\Gamma^{\mathrm{can}}(X_{f,n}, z) - \{v'\}$. There are $\widetilde{h_k}$ isomorphic components that are strings of isomorphism type $\Gamma(\overline{v_k})$. There is one connected subgraph that contains the arrowhead; denote this subgraph $\Gamma_A(v')$. The h_k remaining components are all isomorphic. Let $\Gamma_-(v')$ denote any of these isomorphic subgraphs. Again, whenever the particular choice of v' is unimportant, we use $\Gamma_-(v_k)$ instead of $\Gamma_-(v')$ and $\Gamma_A(v_k)$ instead of $\Gamma_A(v')$. Note that $\Gamma_-(v_1) = \Gamma(\overline{v_0})$ and $\Gamma_A(v_s) = \Gamma(v_s)$. We should also point out that the subgraphs $\Gamma_A(v_k)$ do not appear in [4]; in particular, $\Gamma_A(v_k)$ is not the same as their $\Gamma_+(v_k)$.

For any resolution graph Γ, let $\det(\Gamma) := \det(-C)$, where C is the intersection matrix of the exceptional curves in Γ. If Γ is empty, then we define $\det(\Gamma)$ to be 1. Nearly all of the determinants of the subgraphs just defined are explicitly computed by Mendris and Némethi in [4], and those that are not can be computed by the same method.

LEMMA 3.3 [4]. *For any w in $\Gamma(\mathbb{C}^2, f)$ as before, let $D(w) := \det(\Gamma(w))$. Then*

$$\begin{aligned} D(\overline{v_0}) &= a_1', \\ D(\overline{v_k}) &= p_k' \quad \text{for } 1 \le k \le s, \\ D(v_s) &= n/(h_s \widetilde{h_s}), \\ D(v_k) &= n q_{k+1}/(d_{k-1} \widetilde{h_k}\, \widetilde{h_{k+1}}) \quad \text{for } 1 \le k \le s-1. \end{aligned}$$

It follows from the construction of $\Gamma^{\mathrm{can}}(X_{f,n}, z)$ that if $D(v_s) = 1$, then $\Gamma(v_s)$ is empty; that is, the arrowhead in $\Gamma^{\mathrm{can}}(X_{f,n}, z)$ is connected directly to the unique vertex in $q^{-1}(v_s)$.

LEMMA 3.4 [4]. *Let $D_-(v_k) := \det(\Gamma_-(v_k))$, $1 \le k \le s$. If $s \ge 2$, then for $2 \le k \le s$ we have*

$$\frac{D_-(v_k)}{a_k'} = (a_{k-1}')^{h_{k-1}-1} (p_{k-1}')^{\widetilde{h_{k-1}}-1} \left[\frac{D_-(v_{k-1})}{a_{k-1}'} \right]^{h_{k-1}}.$$

The method used to prove Lemma 3.4 can be suitably modified to prove the next two lemmas. The computation is straightforward, so we omit the details.

LEMMA 3.5. *Assume $s \ge 2$, and let $D_A(v_k) := \det(\Gamma_A(v_k))$, $1 \le k \le s$. Let A_k be defined recursively by $A_{s-1} = a_{s-1} p_{s-1} p_s' + q_s$ and, for $1 \le k \le s-2$,*

$$A_k = a_k p_k p_{k+1}' A_{k+1} + q_{k+1} a_{k+2} \cdots a_s.$$

Then

$$D_A(v_k) = \frac{n A_k \left\{ \prod_{j=k+1}^{s} (p_j')^{\widetilde{h_j}-1} D_-(v_j)^{h_j - 1} \right\}}{h_k \widetilde{h_k} d_k a_{k+1} \cdots a_s} \quad \text{for } 1 \le k \le s-1.$$

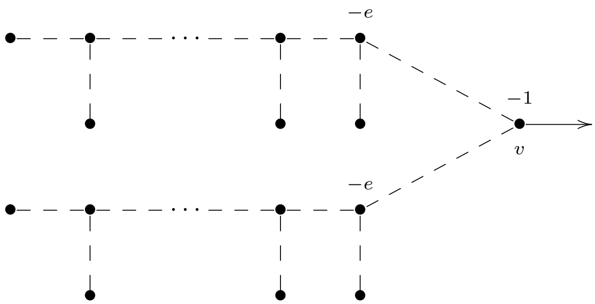
LEMMA 3.6. *The determinant of* $\Gamma^{\mathrm{can}}(X_{f,n}, z)$ *is given by*

$$\det(\Gamma^{\mathrm{can}}(X_{f,n}, z)) = (a_s')^{h_s-1}(p_s')^{\widetilde{h_s}-1}\left[\frac{D_-(v_s)}{a_s'}\right]^{h_s}.$$

A minimal good embedded resolution graph of z in $(X_{f,n}, 0)$, denoted $\Gamma^{\min}(X_{f,n}, z)$, is obtained from $\Gamma^{\mathrm{can}}(X_{f,n}, z)$ by repeatedly blowing down any rational (-1)-curves for which the corresponding vertex has valency 1 or 2. By dropping both the arrowhead and the multiplicities of $\Gamma^{\min}(X_{f,n}, z)$ and then blowing down any appropriate rational (-1)-curves, we obtain a minimal good resolution graph of $(X_{f,n}, 0)$, denoted $\Gamma^{\min}(X_{f,n})$.

PROPOSITION 3.7 [4]. *All of the rupture vertices in* $\Gamma^{\mathrm{can}}(X_{f,n}, z)$ *survive as rupture vertices in* $\Gamma^{\min}(X_{f,n}, z)$. *That is, they are not blown down in the minimalization process and, after minimalization, they are still rupture vertices.*

PROPOSITION 3.8 [4]. *Assume that deleting the arrowhead of* $\Gamma^{\min}(X_{f,n}, z)$ *yields a nonminimal graph. This can occur if and only if* $n = p_s = 2$. *Then the link is a* $\mathbb{Q}$*HS and* $\Gamma^{\min}(X_{f,n}, z)$ *has the following schematic form with* $e \geq 3$.



The minimal resolution graph $\Gamma^{\min}(X_{f,n})$ *is obtained from* $\Gamma^{\min}(X_{f,n}, z)$ *by deleting the arrowhead and blowing down* v.

Propositions 3.7 and 3.8 imply that all of the nodes in $\Gamma^{\mathrm{can}}(X_{f,n}, z)$ remain nodes in the minimal good resolution graph of $(X_{f,n}, 0)$ except in the case $n = p_s = 2$. We refer to $n = p_s = 2$ as the *pathological case,* which is treated separately in what follows.

3.2. *Splice Diagram*

From now on, we assume that the link of $(X_{f,n}, 0)$ is a $\mathbb{Q}$HS. That is, for each k ($1 \leq k \leq s$), either h_k or $\widetilde{h_k}$ is equal to 1. One complication that arises is that certain strings in $\Gamma^{\mathrm{can}}(X_{f,n}, z)$ may completely collapse upon minimalization. Therefore, if we used the minimal good resolution graph $\Gamma^{\min}(X_{f,n})$ in what follows, then we would constantly need to note that certain strings may be empty and, more importantly, that certain leaves in the splice diagram may not be present. We will avoid this by using the splice diagram associated to $\Gamma^{\mathrm{can}}(X_{f,n})$, the graph that results from deleting the arrowhead and multiplicities in $\Gamma^{\mathrm{can}}(X_{f,n}, z)$. We could easily use a quasi-minimal modification of $\Gamma^{\mathrm{can}}(X_{f,n})$, and the computation of the splice diagram would not change. Therefore, we can apply Theorem 2.4 to $\Gamma^{\mathrm{can}}(X_{f,n})$.

Splice diagram in the general case
Assume that we are not in the pathological case, and let $\Delta_{f,n}$ be the splice diagram associated to $\Gamma_{f,n} := \Gamma^{\mathrm{can}}(X_{f,n})$. If a vertex v in $\Gamma_{f,n}$ is in $q^{-1}(v_k)$ (resp. $q^{-1}(\overline{v_k})$), we say that v is "of type v_k" (resp. $\overline{v_k}$). We use the same terminology for the corresponding vertices of $\Delta_{f,n}$.

Consider a node v of type v_k, $1 \le k \le s$, in $\Gamma_{f,n}$. In general, there are $h_k + \widetilde{h_k} + 1$ edges adjacent to v: $\widetilde{h_k}$ edges that lead to strings of (isomorphism) type $\Gamma(\overline{v_k})$, h_k edges that lead to subgraphs of type $\Gamma_-(v_k)$, and one edge that leads toward a subgraph of type $\Gamma_A(v_k)$. The corresponding pieces of $\Delta_{f,n}$ associated to the subgraphs of type $\Gamma_-(v_k)$ and $\Gamma_A(v_k)$ are denoted $\Delta_-(v_k)$ and $\Delta_A(v_k)$, respectively. Recall that $\Gamma_-(v_1) = \Gamma(\overline{v_0})$ and $\Gamma_A(v_s) = \Gamma(v_s)$, and keep in mind that $\Gamma(v_s)$ may be empty.

The weights of the splice diagram $\Delta_{f,n}$ are given by Lemmas 3.3, 3.4, and 3.5. At a node of type v_k in $\Delta_{f,n}$, the weights on the $\widetilde{h_k}$ edges that lead to leaves of type $\overline{v_k}$ are $D(\overline{v_k}) = p'_k$; the weights on the h_k edges connected to subgraphs of type $\Delta_-(v_k)$ are $D_-(v_k)$; and the weight on the single edge connected to the subgraph of type $\Delta_A(v_k)$ is $D_A(v_k)$ (see Figure 4).

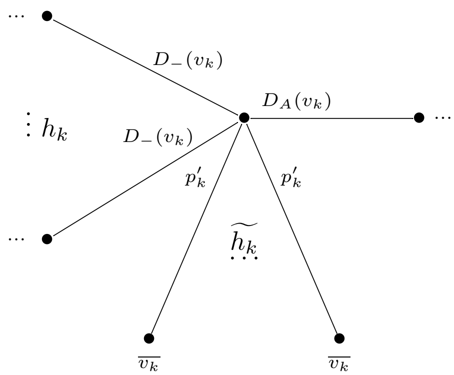


Figure 4 Splice diagram at a node of type v_k, $2 \le k \le s-1$

The pathological case
For this case ($n = p_s = 2$), it is more convenient to use the splice diagram associated to the minimal resolution graph $\Gamma^{\min}(X_{f,n})$ (see Figure 5). Here $h_s = 2$ and hence $n/h_s = n/d_{s-1} = 1$. Then, by definition, $h_k = \widetilde{h_k} = 1$ for $1 \le k \le s-1$, and $\widetilde{h_s} = 1$ because $\gcd(p_s, a_s) = 1$. The link is a $\mathbb{Q}$HS, and the only string of type $\Gamma(\overline{v_k})$ that collapses completely in $\Gamma^{\min}(X_{f,n}, z)$ is $\Gamma(\overline{v_s})$ (Proposition 3.8).

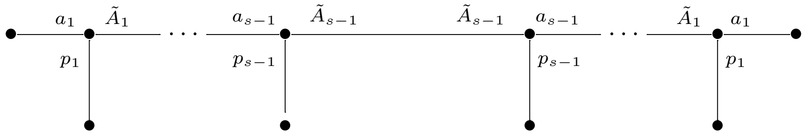


Figure 5 Splice diagram for the pathological case

The graph $\Gamma^{\min}(X_{f,n})$ has a total of $2(s-1)$ nodes: two of type v_k for each k, $1 \le k \le s-1$. Each of these nodes has valency 3.

Because the determinant of a resolution tree remains constant throughout the minimalization process, the weights of the splice diagram associated to $\Gamma^{\min}(X_{f,n})$ can be determined from Lemmas 3.3, 3.4, and 3.5. Since $h_k\widetilde{h_k} = 1$ for $1 \le k \le s-1$, we have $D_-(v_k) = a_k$ for $2 \le k \le s$. Define integers $\tilde{A}_k$ as follows:

$$\tilde{A}_k := a_s - a_k p_k p_{k+1}^2 \cdots p_{s-1}^2 \quad \text{for } 1 \le k \le s-2,$$

$$\tilde{A}_{s-1} := a_s - a_{s-1}p_{s-1}.$$

It is easy to check that $D_A(v_k) = \tilde{A}_k$ for $1 \le k \le s-1$.

4. The Semigroup Conditions

In this section we discuss the semigroup conditions for the splice diagram $\Delta_{f,n}$. Throughout this section, we assume that we are not in the pathological case. For a node v of type v_k in $\Delta_{f,n}$, $1 \le k \le s$, there are at most two inequivalent semigroup conditions to check: one for an edge that leads to a subdiagram of type $\Delta_-(v_k)$ (nontrivial for $2 \le k \le s$) and one for an edge that leads to a subdiagram of type $\Delta_A(v_k)$ (nontrivial for $1 \le k \le s-1$). Clearly, for a fixed k, the semigroup conditions are equivalent for any node v of type v_k.

4.1. Semigroup Conditions in the Direction of $\Delta_-(v_k)$

LEMMA 4.1. *Let v be a node of type v_k, $2 \le k \le s$, and let w_j be a leaf of type $\overline{v_j}$ in $\Delta_-(v)$, $0 \le j \le k-1$. Then*

$$\ell'_{vw_j} = \begin{cases} (D_-(v_k)/a'_k)p'_1 \cdots p'_{k-1} & \textit{for } j = 0, \\ (D_-(v_k)/a'_k)a'_j p'_{j+1} \cdots p'_{k-1} & \textit{for } 1 \le j \le k-2, \\ (D_-(v_k)/a'_k)a'_{k-1} & \textit{for } j = k-1. \end{cases}$$

Proof. We prove this by induction on k. For $k = 2$, the lemma is true because, if v is a node of type v_2, then

$$\ell'_{vw_0} = (a'_1)^{h_1-1}(p'_1)^{\widetilde{h_1}},$$

$$\ell'_{vw_1} = (a'_1)^{h_1}(p'_1)^{\widetilde{h_1}-1},$$

$$D_-(v_2)/a'_2 = (a'_1)^{h_1-1}(p'_1)^{\widetilde{h_1}-1}.$$

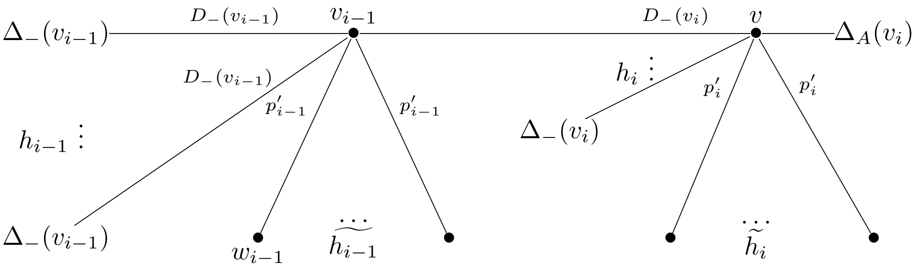


Figure 6 Relevant portion of $\Delta_{f,n}$ at a node v of type v_i

Now assume the lemma is true for $k = i - 1$; we show that it is true for $k = i$. Fix a node v of type v_i, and (abusing notation) let v_{i-1} denote the unique node of type v_{i-1} in $\Delta_-(v)$. For $0 \le j \le i-2$, any leaf of type $\overline{v_j}$ in $\Delta_-(v)$ is in one of the subdiagrams of type $\Delta_-(v_{i-1})$. Thus (refer to Figure 6)

$$\ell'_{vw_j} = \begin{cases} D_-(v_{i-1})^{h_{i-1}-1}(p'_{i-1})^{\widetilde{h_{i-1}}}\ell'_{v_{i-1}w_j} & \text{for } 0 \le j \le i-2, \\ D_-(v_{i-1})^{h_{i-1}}(p'_{i-1})^{\widetilde{h_{i-1}}-1} & \text{for } j = i-1. \end{cases}$$

By Lemma 3.4, we have

$$\frac{D_-(v_i)}{a'_i} = (p'_{i-1})^{\widetilde{h_{i-1}}-1}D_-(v_{i-1})^{h_{i-1}-1}\cdot\frac{D_-(v_{i-1})}{a'_{i-1}}.$$

Applying this fact and the induction hypothesis yields the desired result. □

PROPOSITION 4.2. *At a node of type v_k, $2 \le k \le s$, the semigroup condition in the direction of any of the h_k edges that lead to a subdiagram of type $\Delta_-(v_k)$ is equivalent to*

$$a'_k \in \mathbb{N}\langle a'_{k-1}, p'_1p'_2\cdots p'_{k-1}, a'_jp'_{j+1}\cdots p'_{k-1}, 1 \le j \le k-2\rangle. \tag{2}$$

Furthermore, if $\widetilde{h_k} = 1$ then this condition is automatically satisfied.

Proof. Fix a node v of type v_k in $\Delta_{f,n}$. By Definition 2.1, the condition is

$$D_-(v_k) \in \mathbb{N}\langle \ell'_{vw} \mid w \text{ is a leaf in } \Delta_-(v)\rangle.$$

The leaves in $\Delta_-(v)$ are of type $\overline{v_j}$ for $0 \le j \le k-1$. Hence there are k generators for the semigroup in question—namely, ℓ'_{vw_j} for $0 \le j \le k-1$, where w_j denotes any leaf in $\Delta_-(v)$ of type $\overline{v_j}$. The first statement of the proposition follows from Lemmas 3.4 and 4.1, since $D_-(v_k)$ and all generators of the semigroup are divisible by $D_-(v_k)/a'_k$.

The second statement follows from [12, Prop. 8.1]. □

4.2. Semigroup Conditions in the Direction of $\Delta_A(v_k)$

Fix an integer k, $1 \le k \le s-1$, and fix a node v of type v_k. By definition, the semigroup condition is $D_A(v_k) \in \mathrm{R}_k$, where

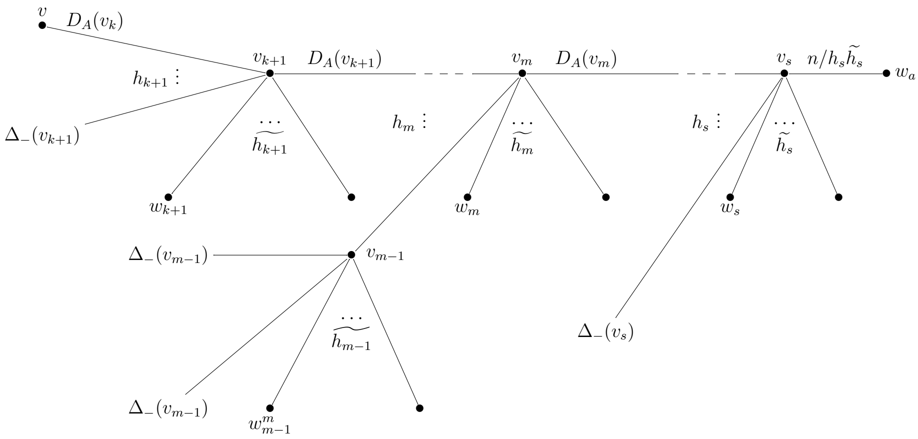


Figure 7 Relevant portion of $\Delta_{f,n}$ at a node v of type v_k

$$\mathrm{R}_k := \mathbb{N}\langle \ell'_{vw} \mid w \text{ is a leaf in } \Delta_A(v)\rangle.$$

Refer to Figure 7 for what follows. There is at least one leaf w_s in $\Delta_A(v)$ of type $\overline{v_s}$ connected to v_s (the unique node of type v_s), and if $n/h_s\widetilde{h_s} \neq 1$ then there is a leaf w_a resulting from the string $\Gamma(v_s)$ in $\Gamma_{f,n}$. These contribute ℓ'_{vw_s} and ℓ'_{vw_a} as generators of R_k.

Next, travel along the shortest path from v to v_s. If $k < s-1$, then this path contains one node of type v_m for each m such that $k+1 \leq m \leq s-1$. Since there can be no confusion here, we will simply refer to the nodes along this path as v_m. Each of these nodes is directly connected to at least one leaf w_m of type $\overline{v_m}$, and each such leaf contributes the generator ℓ'_{vw_m} to R_k. If $h_i = 1$ for $k+1 \leq i \leq s$, then there are no other types of leaves in $\Delta_A(v)$ and so we have listed all the generators of R_k.

For each m such that $h_m \neq 1$, $k+1 \leq m \leq s$, there are more generators for R_k—namely, ℓ'_{vw} for each type of leaf w in $\Delta_-(v_m)$. There are m different types of such leaves: type $\overline{v_j}$ for j where $0 \leq j \leq m-1$. Let w^m_j be a leaf of type $\overline{v_j}$ in $\Delta_-(v_m)$. Then the generators of the semigroup R_k are:

$$\begin{aligned} &\ell'_{vw_m}, && k+1 \leq m \leq s;\\ &\ell'_{vw^m_j}, && 0 \leq j \leq m-1 \text{ for all } m \text{ such that } k+1 \leq m \leq s \text{ and } h_m \neq 1;\\ &\ell'_{vw_a} && (\text{absent if } n/h_s\widetilde{h_s} = 1). \end{aligned}$$

Proposition 4.3. *Suppose $h_s > 1$. Then the semigroup conditions imply that $h_s = p_s$ and $h_{s-1}\widetilde{h_{s-1}} = 1$.*

Proof. Observe that, since the link is a $\mathbb{Q}$HS, $h_s > 1$ implies $\widetilde{h_s} = 1$. Let v be a node of type v_{s-1}, and consider the semigroup condition at v in the direction of $\Delta_A(v)$: $D_A(v_{s-1})$ is in the semigroup R_{s-1}. The generators of R_{s-1} are ℓ'_{vw_s} and

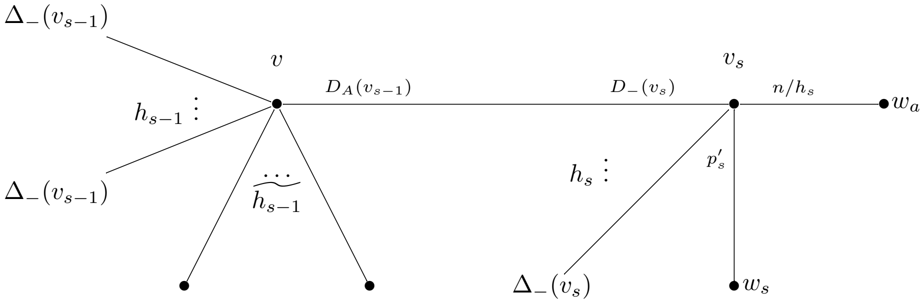


Figure 8 Splice diagram $\Delta_{f,n}$ for $\widetilde{h_s} = 1$

$\ell'_{vw_j^s}$ $(0 \le j \le s-1)$ and ℓ'_{vw_a} (absent if $n/h_s = 1$). It is easy to check (see Figure 8) that

$$\begin{aligned} \ell'_{vw_s} &= (n/h_s) D_-(v_s)^{h_s-1}, \\ \ell'_{vw_a} &= p'_s D_-(v_s)^{h_s-1}, \\ \ell'_{vw_j^s} &= (n/h_s) p'_s D_-(v_s)^{h_s-2} \ell'_{v_s w_j^s}. \end{aligned}$$

Since $d_{s-1} = h_s$ and $a'_s = a_s$, it follows from Lemma 3.5 that

$$D_A(v_{s-1}) = \frac{n A_{s-1} D_-(v_s)^{h_s-1}}{h_{s-1}\widetilde{h_{s-1}} h_s a_s},$$

where

$$A_{s-1} = a_{s-1}p_{s-1}p'_s + q_s = a_s - a_{s-1}p_{s-1}(p_s - p'_s).$$

Note that $n/(h_{s-1}\widetilde{h_{s-1}}h_s)$ and $D_-(v_s)^{h_s-1}/a_s$ are both integers in this case. Since $p_s > p'_s = p_s/h_s$, we have

$$\frac{n}{h_{s-1}\widetilde{h_{s-1}}h_s}[a_s - a_{s-1}p_{s-1}(p_s - p'_s)] < \frac{n}{h_{s-1}\widetilde{h_{s-1}}h_s} a_s \le \frac{n}{h_s} a_s$$

and therefore $D_A(v_{s-1}) < \ell'_{vw_s}$. Hence we can forget about the generator ℓ'_{vw_s}, since it is too large.

By Lemma 4.1,

$$\ell'_{v_s w_j^s} = \begin{cases} p'_1 \cdots p'_{s-1} \cdot D_-(v_s)/a_s & \text{for } j = 0, \\ a'_j p'_{j+1} \cdots p'_{s-1} \cdot D_-(v_s)/a_s & \text{for } 1 \le j \le s-2, \\ a'_{s-1} \cdot D_-(v_s)/a_s & \text{for } j = s-1. \end{cases}$$

Thus, all generators of R_{s-1} and $D_A(v_{s-1})$ are divisible by $D_-(v_s)^{h_s-1}/a_s$, and the semigroup condition is equivalent to the following: $n/(h_{s-1}\widetilde{h_{s-1}}h_s)A_{s-1}$ is in the semigroup generated by

$$\left\{ \frac{n}{h_s} p'_1 \cdots p'_s, \quad \frac{n}{h_s} a'_j p'_{j+1} \cdots p'_s \;\middle|\; 1 \le j \le s-1,\ a_s p'_s \left(\text{absent if } \frac{n}{h_s} = 1\right) \right\}. \tag{3}$$

All of the generators of this semigroup are divisible by p'_s. Therefore, the semigroup condition implies that p'_s divides $n/(h_{s-1}\widetilde{h_{s-1}}h_s)[a_s - a_{s-1}p_{s-1}(p_s - p'_s)]$. Suppose $p'_s > 1$. Since p'_s divides $p_s - p'_s$ and since $(a_s, p_s) = 1$, it follows that p'_s divides $n/(h_{s-1}\widetilde{h_{s-1}}h_s)$. This is impossible, since by definition $p'_s = p_s/(n, p_s)$ and thus $(p'_s, n) = 1$. Therefore we must have $p'_s = 1$. Since $p'_s = p_s/h_s$, we have shown that the semigroup conditions imply $h_s = p_s$.

Now we show that the semigroup conditions imply $h_{s-1}\widetilde{h_{s-1}} = 1$. Note that if $n/h_s = 1$ then this is automatically true by definition of h_i and $\widetilde{h_i}$, so assume that $n/h_s \neq 1$. Observe that all of the generators in (3) are divisible by n/h_s except for a_s. Therefore, if the semigroup condition is satisfied, then there exist M and N in $\mathbb{N} \cup \{0\}$ such that

$$n/(h_{s-1}\widetilde{h_{s-1}}h_s)[a_s - a_{s-1}p_{s-1}(p_s - 1)] = Ma_s + Nn/h_s.$$

Hence,

$$\begin{aligned}(n/(h_{s-1}\widetilde{h_{s-1}}h_s) - M)a_s &= Nn/h_s + n/(h_{s-1}\widetilde{h_{s-1}}h_s)a_{s-1}p_{s-1}(p_s - 1)\\ &= n/h_s(N + a'_{s-1}p'_{s-1}(p_s - 1)).\end{aligned}$$

Since $(n, a_s) = 1$ by assumption, this implies that $n/h_s \neq 1$ divides $\frac{n}{h_{s-1}\widetilde{h_{s-1}}h_s} - M$. But we have

$$0 < \frac{n}{h_{s-1}\widetilde{h_{s-1}}h_s} - M \leq \frac{n}{h_{s-1}\widetilde{h_{s-1}}h_s} \leq \frac{n}{h_s}.$$

Therefore, the only possibility is $n/h_s = n/(h_{s-1}\widetilde{h_{s-1}}h_s) - M$; that is, $M = 0$ and $h_{s-1}\widetilde{h_{s-1}} = 1$. □

LEMMA 4.4. *Assume that $s \geq 3$ and that $h_{s-1}\widetilde{h_{s-1}} = 1$. Then the semigroup conditions imply that $h_k\widetilde{h_k} = 1$ for $1 \leq k \leq s-2$.*

Proof. We prove this by strong downward induction on k. First we show that the semigroup conditions imply that $h_{s-2}\widetilde{h_{s-2}} = 1$. By Proposition 3.2(a), there are h_s nodes of type v_{s-2}; let v be any such node. We will show that the semigroup condition for v in the direction of $\Delta_A(v)$ cannot be satisfied if $h_{s-2}\widetilde{h_{s-2}} \neq 1$.

Let $\tilde{A}_i = a_s - a_i p_i p_{i+1}^2 \cdots p_{s-1}^2(p_s - p'_s)$, $1 \leq i \leq s-2$. By Lemma 3.5,

$$D_A(v_{s-1}) = \begin{cases} np_s^{\widetilde{h_s}-1} & \text{for } h_s = 1,\\ \dfrac{nA_{s-1}D_-(v_s)^{h_s-1}}{h_s a_s} & \text{for } h_s > 1;\end{cases}$$

$$D_A(v_{s-2}) = \begin{cases} \dfrac{n}{h_{s-2}\widetilde{h_{s-2}}}p_s^{\widetilde{h_s}-1} & \text{for } h_s = 1,\\ \dfrac{n\tilde{A}_{s-2}D_-(v_s)^{h_s-1}}{h_{s-2}\widetilde{h_{s-2}}h_s a_s} & \text{for } h_s > 1.\end{cases}$$

The generators of R_{s-2} are

$$\begin{aligned}
\ell'_{vw_{s-1}} &= D_A(v_{s-1}),\\
\ell'_{vw_s} &= n/(h_s\widetilde{h_s})p_{s-1}D_-(v_s)^{h_s-1}(p'_s)^{\widetilde{h_s}-1},\\
\ell'_{vw^s_j} &= n/(h_s\widetilde{h_s})p_{s-1}D_-(v_s)^{h_s-2}(p'_s)^{\widetilde{h_s}}\ell'_{v_sw^s_j}, \quad 0 \le j \le s-1,\\
\ell'_{vw_a} &= p_{s-1}D_-(v_s)^{h_s-1}(p'_s)^{\widetilde{h_s}},
\end{aligned}$$

although the $\{\ell'_{vw^s_j}\}_{j=0}^{s-1}$ are absent if $h_s = 1$ and ℓ'_{vw_a} is absent if $n/h_s\widetilde{h_s} = 1$.

We will consider two separate cases: (i) $h_s = 1$, and (ii) $h_s > 1$.

Case (i). If $h_s = 1$, it is easy to see that if $h_{s-2}\widetilde{h_{s-2}} \neq 1$ then $\ell'_{vw_{s-1}} > D_A(v_{s-2})$. Then, since $D_A(v_{s-2})$ and every generator of the semigroup are divisible by $p_s^{\widetilde{h_s}-1}$, the semigroup condition is equivalent to: $n/(h_{s-2}\widetilde{h_{s-2}})$ is in the semigroup generated by $p_{s-1}n/\widetilde{h_s}$ and $p_{s-1}p_s$ (absent if $n/\widetilde{h_s} = 1$). Thus the semigroup condition implies that $n/(h_{s-2}\widetilde{h_{s-2}})$ is divisible by p_{s-1}, which is impossible because $h_{s-1} = (n, p_{s-1}) = 1$. Therefore, we must have $h_{s-2}\widetilde{h_{s-2}} = 1$. (Note that the argument is valid even if $n/\widetilde{h_s} = 1$ or $\widetilde{h_s} = 1$.)

Case (ii). For $h_s > 1$, the proof that $h_{s-2}\widetilde{h_{s-2}} = 1$ is nearly identical to the proof of Proposition 4.3, so we just give the outline here. Recall that the semigroup conditions imply that $p'_s = 1$ in this case. We will assume that $n/h_s \neq 1$, for otherwise the lemma is trivially true by definition of h_i and $\widetilde{h_i}$.

Dividing $D_A(v_{s-2})$ and all the generators of R_{s-2} by $D_-(v_s)^{h_s-1}/a_s$, we see that the semigroup condition for v in the direction of $\Delta_A(v)$ implies that $n/(h_{s-2}\widetilde{h_{s-2}}h_s)\tilde{A}_{s-2}$ is in the semigroup generated by a_sp_{s-1} and a collection of positive integers divisible by n/h_s. The semigroup condition implies that there exist M and N in $\mathbb{N} \cup \{0\}$ such that

$$n/(h_{s-2}\widetilde{h_{s-2}}h_s)\tilde{A}_{s-2} = Ma_sp_{s-1} + Nn/h_s.$$

Just as in the proof of Proposition 4.3, we must have $M = 0$ and $h_{s-2}\widetilde{h_{s-2}} = 1$. Thus, we have taken care of both cases in the basis step.

For the inductive step, assume that $h_i\widetilde{h_i} = 1$ for all i such that $k+1 \le i \le s-1$. Now let v be one of the h_s nodes of type v_k. One can show that the semigroup condition for v in the direction of $\Delta_A(v)$ cannot be satisfied if $h_k\widetilde{h_k} \neq 1$. In both cases ($h_s = 1$ and $h_s > 1$) the proof is essentially the same as that of the basis step, so we omit the details. □

Proposition 4.3 and Lemma 4.4 together imply the following result.

COROLLARY 4.5. *Suppose that $h_s > 1$. Then the semigroup conditions imply that $h_k\widetilde{h_k} = 1$ for $1 \le k \le s-1$.*

In Section 6 we will see that, for the case $h_s = 1$, the semigroup conditions *and* congruence conditions together imply that $h_k\widetilde{h_k} = 1$ for $1 \le k \le s-1$.

5. Action of the Discriminant Group

In order to use Proposition 2.5 to check the congruence conditions for the resolution graph $\Gamma_{f,n}$, we must compute $e_w \cdot e_w$ for all leaves w. By Proposition 2.7, this amounts to computing the continued fraction expansions of the strings from leaves to nodes. This is essentially done in Mendris and Némethi's paper [4, proof of Prop. 3.5], but we need a bit more detail than they included.

5.1. Background

We begin with a summary of facts that we need, which can be found in [5]. Let a, Q, and P be strictly positive integers with $\gcd(a, Q, P) = 1$. Let $(X(a, Q, P), 0)$ be the isolated surface singularity lying over the origin in the normalization of $(\{U^a V^Q = W^P\}, 0)$. Let λ be the unique integer such that $0 \le \lambda < P/(a, P)$ and

$$Q + \lambda \cdot \frac{a}{(a, P)} = m \cdot \frac{P}{(a, P)}$$

for some positive integer m. If $\lambda \neq 0$, then let $k_1, \ldots, k_t \ge 2$ be the integers in the continued fraction expansion of $\frac{P/(a,P)}{\lambda}$.

The minimal embedded resolution graph of the germ induced by the coordinate function V on $(X(a, Q, P), 0)$ is given by the string in Figure 9 (omitting the multiplicities of the vertices). If $\lambda = 0$ then the string is empty. One can similarly describe the embedded resolution graphs of the functions U and W, but we do not need them here.

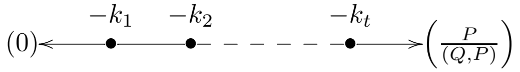


Figure 9 The embedded resolution graph $\Gamma(X(a, Q, P), V)$

Lemma 5.1. *Let N, M, P, and Q be positive integers such that $(Q, P) = 1$ and $(N, M) = 1$. Let Γ be the resolution graph of the singularity in the normalization of $(\{UV^Q = W^P, T^N = V^M\}, 0) \subseteq (\mathbb{C}^4, 0)$. Let λ be the unique integer such that $0 \le \lambda < P/(N, P)$ and*

$$Q \frac{N}{(N, P)} + \lambda = m \cdot \frac{P}{(N, P)}$$

for some positive integer m. Then, if $\lambda \neq 0$, Γ is a string of vertices with continued fraction expansion $\frac{P/(N,P)}{\lambda}$.

Proof. We may assume $M = 1$, since it easy to check that the singularity in question has the same normalization as $\{UV^Q = W^P, T^N = V\} \subseteq \mathbb{C}^4$. Therefore, Γ is the resolution graph of the singularity in the normalization of $\{UV^{QN} = W^P\}$, which is the same as the resolution graph of

$$X\left(1, Q\frac{N}{(N,P)}, \frac{P}{(N,P)}\right) = \{UV^{QN/(N,P)} = W^{P/(N,P)}\}. \qquad \square$$

5.2. *Strings in* $\Gamma_{f,n}$

We need the continued fraction expansion of the strings in $\Gamma_{f,n}$ from leaves of type $\overline{v_k}$, $0 \le k \le s$, to the corresponding node of type v_k (from type $\overline{v_0}$ to type v_1). First we recall the construction of $\Gamma(\mathbb{C}^2, f)$, the minimal good embedded resolution graph of f in $\mathbb{C}^2$, as in [4]. Let f have Newton pairs $\{(p_k, q_k) \mid 1 \le k \le s\}$. Determine the continued fraction expansions

$$\frac{p_k}{q_k} = \mu_k^0 - \cfrac{1}{\mu_k^1 - \cfrac{1}{\ddots - \cfrac{1}{\mu_k^{t_k}}}} \quad \text{and} \quad \frac{q_k}{p_k} = \nu_k^0 - \cfrac{1}{\nu_k^1 - \cfrac{1}{\ddots - \cfrac{1}{\nu_k^{r_k}}}},$$

where $\mu_k^0, \nu_k^0 \ge 1$ and $\mu_k^j, \nu_k^j \ge 2$ for $j > 0$. Then $\Gamma(\mathbb{C}^2, f)$ has the schematic form given in Figure 2. The strings from $\overline{v_0}$ to v_1 and from $\overline{v_k}$ to v_k, $1 \le k \le s$, are given in Figure 10. The multiplicities of the vertices v_k are $m_{v_k} = a_k p_k p_{k+1} \cdots p_s$ for $1 \le k \le s$.

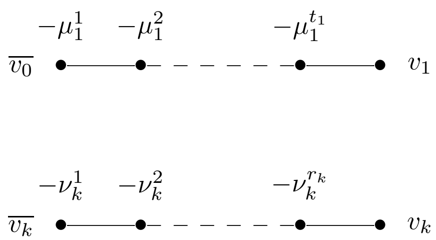


Figure 10 Strings in $\Gamma(\mathbb{C}^2, f)$

Consider the string in Figure 11. The continued fraction expansion $[\nu_k^1, \dots, \nu_k^{r_k}]$ corresponds to p_k/η_k, where $q_k + \eta_k = \nu_k^0 p_k$. Let $X := X(1, q_k, p_k)$. Then this string is the embedded resolution graph of $V^{a_k p_{k+1} \cdots p_s}$ in X. It follows from the construction of $\Gamma_{f,n}$ that the collection of strings that lies above this one in $\Gamma_{f,n}$ is the (possibly nonconnected) resolution graph of the singularity in the normalization of $\{UV^{q_k} = W^{p_k}, T^n = V^{a_k p_{k+1} \cdots p_s}\}$. There are $(n, a_k p_{k+1} \cdots p_s) = \widetilde{h_k} d_k = \widetilde{h_k} h_{k+1} \cdots h_s$ connected components (see Definition 3.1), and each is the resolution graph of the normalization of

$$\{UV^{q_k} = W^{p_k},\ T^{n/(\widetilde{h_k} d_k)} = V^{a_k' p_{k+1}' \cdots p_s'}\}.$$

Now we are in the situation of Lemma 5.1, with $Q = q_k$, $P = p_k$, and $N = n/(\widetilde{h_k} d_k)$. We have $(N, P) = (n/(\widetilde{h_k} d_k), p_k) = h_k$ by definition of h_k, so in this

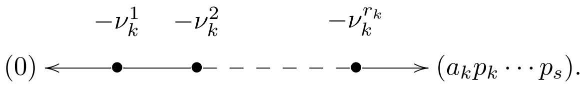


Figure 11 String from $\Gamma(\mathbb{C}^2, f)$

case $P/(N, P) = p'_k$ (as expected from Proposition 3.3). If $p'_k = 1$ then, upon minimalization, the string of type $\overline{v_k}$ would completely collapse.

Suppose $p'_k \neq 1$. By Lemma 5.1, the continued fraction expansion of the string(s) from a leaf of type $\overline{v_k}$ to the corresponding node of type v_k in the minimalization of the resolution graph $\Gamma_{f,n}$ is given by p'_k/η'_k, where η'_k is the unique integer such that $0 < \eta'_k < p'_k$ and

$$q_k \frac{n}{h_k \widetilde{h_k} d_k} + \eta'_k = m p'_k$$

for some positive integer m. Since $a_k = q_k + a_{k-1} p_{k-1} p_k$, we have

$$\eta'_k \equiv -a_k \cdot \frac{n}{h_k \widetilde{h_k} d_k} \pmod{p'_k}. \tag{4}$$

Knowing the congruence class of η'_k modulo p'_k is enough for our purposes.

The continued fraction expansion from $\overline{v_0}$ to v_1 in $\Gamma(\mathbb{C}^2, f)$ is given by $q_1/\eta_0 = a_1/\eta_0$, where $p_1 + \eta_0 = \mu_1^0 a_1$. Using an argument analogous to the previous one, we have that if $a'_1 \neq 1$ then the continued fraction expansion of the string(s) from a leaf of type $\overline{v_0}$ to the corresponding node of type v_1 in the minimalization of $\Gamma_{f,n}$ is a'_1/η'_0, where

$$\eta'_0 \equiv -p_1 \cdot \frac{n}{h_1 \widetilde{h_1} d_1} \pmod{a'_1}.$$

Recall the notation defined in Section 2: for $r \in \mathbb{Q}$, $[r] = \exp(2\pi i r)$, and for a leaf $w \in \Gamma_{f,n}$, e_w denotes the image in the discriminant group of the dual basis element in $\mathbb{E}^*$ corresponding to w.

Corollary 5.2. *Let w_k be any leaf of type $\overline{v_k}$ in $\Gamma_{f,n}$, $0 \leq k \leq s$, and assume that $p'_k \neq 1$ (assume $a'_1 \neq 1$ for $k = 0$). Then*

$$[e_{w_k} \cdot e_{w_k}] = \begin{cases} \left[\dfrac{(n/h_1 \widetilde{h_1} d_1)(p_1 a_2 \cdots a_s - A_1 p'_1)}{a'_1 a_2 \cdots a_s}\right] & \text{for } k = 0, \\ \left[\dfrac{(n/h_k \widetilde{h_k} d_k)(a_k a_{k+1} \cdots a_s - A_k a'_k)}{p'_k a_{k+1} \cdots a_s}\right] & \text{for } 1 \leq k \leq s-1, \\ \left[\dfrac{(n/h_s \widetilde{h_s})(a_s - a'_s)}{p'_s}\right] & \text{for } k = s. \end{cases}$$

Proof. Proposition 2.7 states that, for a leaf w connected by a string of vertices to a node v,

$$e_w \cdot e_w = -d_v/(d^2 \det(\Gamma)) - p/d,$$

where d_v is the product of weights at the node v and d/p is the fraction corresponding to the string from w to v. Let d_{v_k} be the product of the weights at any node of type v_k, $1 \le k \le s$ (refer to Figure 4). Then $d_{v_k} = D_A(v_k)D_-(v_k)^{h_k}(p'_k)^{\widetilde{h_k}}$.

We need the following fact, which is a consequence of Lemmas 3.4 and 3.6. For any k such that $1 \le k \le s$,

$$\det(\Gamma_{f,n}) = \frac{D_-(v_k)}{a'_k} \prod_{j=k}^{s} (p'_j)^{\widetilde{h_j}-1} D_-(v_j)^{h_j-1}.$$

Now, for $1 \le k \le s-1$,

$$\begin{aligned}
e_{w_k} \cdot e_{w_k} &= -\frac{D_A(v_k)D_-(v_k)^{h_k}(p'_k)^{\widetilde{h_k}}}{(p'_k)^2 \det(\Gamma)} - \frac{\eta'_k}{p'_k} \\
&= -\frac{\left(\dfrac{nA_k \cdot \prod_{j=k+1}^{s}(p'_j)^{\widetilde{h_j}-1}D_-(v_j)^{h_j-1}}{h_k\widetilde{h_k}d_k a_{k+1}\cdots a_s}\right) D_-(v_k)^{h_k}(p'_k)^{\widetilde{h_k}}}{(p'_k)^2 \dfrac{D_-(v_k)}{a'_k}\left(\displaystyle\prod_{j=k}^{s}(p'_j)^{\widetilde{h_j}-1}D_-(v_j)^{h_j-1}\right)} - \frac{\eta'_k}{p'_k} \\
&= -\frac{n/(h_k\widetilde{h_k}d_k)A_k a'_k}{p'_k a_{k+1}\cdots a_s} - \frac{\eta'_k}{p'_k}.
\end{aligned}$$

Applying the congruence (4), we have

$$[e_{w_k} \cdot e_{w_k}] = \left[\frac{(n/h_k\widetilde{h_k}d_k)a_k}{p'_k} - \frac{(n/h_k\widetilde{h_k}d_k)A_k a'_k}{p'_k a_{k+1}\cdots a_s}\right],$$

and from this it is clear that the corollary is true. In the same way, it is easy to check that $[e_{w_0} \cdot e_{w_0}]$ and $[e_{w_s} \cdot e_{w_s}]$ are as stated. □

6. Proof of the Main Theorem

In this section we prove the Main Theorem, which determines precisely which $(X_{f,n}, 0)$, with f irreducible, have a resolution graph $\Gamma_{f,n}$ and associated splice diagram $\Delta_{f,n}$ that satisfy both the semigroup and congruence conditions.

REMARK 6.1. 1. The link is a $\mathbb{Z}$HS if and only if n is relatively prime to all p_i and a_i (see [12]). This is equivalent to all h_i and $\widetilde{h_i}$ being equal to 1. Hence this case belongs to (i) of the Main Theorem.

2. For the so-called pathological case $n = p_s = 2$, both semigroup and congruence conditions are satisfied only for $s = 2$.

3. There are classes of $(X_{f,n}, 0)$ for which the semigroup conditions are satisfied but the congruence conditions are not, but we do not write up a complete list of these types. An example with this property is given by $n = 2$, $s = 2$, $p_1 = 2$, $a_1 = 3$, $p_2 = 3$, and $a_2 = 20$. The minimal good resolution graph and splice diagram for this example are given in Figure 12.

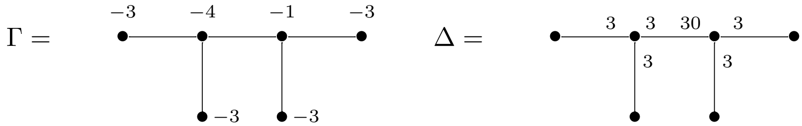


Figure 12 Example for which the semigroup conditions are satisfied but the congruence conditions are not

We must treat the cases $h_s = 1$ and $h_s > 1$ separately. The second case takes much more work than the first.

6.1. Case (i), $h_s = (n, p_s) = 1$

First of all, we have the following statement.

PROPOSITION 6.2. *Suppose $h_s = 1$. If $\Gamma_{f,n}$ satisfies the semigroup and congruence conditions, then $h_i\widetilde{h_i} = 1$ for $1 \le i \le s-1$.*

Proof. In light of Lemma 4.4, it suffices to show that the semigroup and congruence conditions imply $h_{s-1}\widetilde{h_{s-1}} = 1$. We claim that the congruence condition at the unique node v of type v_{s-1} cannot be satisfied if $h_{s-1}\widetilde{h_{s-1}} \neq 1$. Let u_j, $1 \le j \le \widetilde{h_s}$, denote the leaves of type $\overline{v_s}$ in $\Delta_{f,n}$, and let y denote the leaf that arises from the string $\Gamma(v_s)$ in $\Gamma^{\mathrm{can}}(X_{f,n}, z)$, as in Figure 13. If $n/\widetilde{h_s} = 1$, then the leaf y does not exist, but one can see that the argument holds regardless.

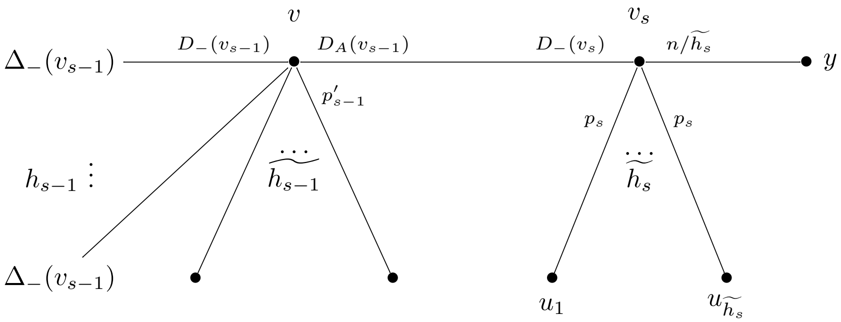


Figure 13 Splice diagram for $h_s = 1$

The semigroup condition at v in the direction of $\Delta_A(v)$ says that there exist β and α_i, $1 \le i \le \widetilde{h_s}$, in $\mathbb{N} \cup \{0\}$ such that

$$D_A(v_{s-1}) = \left(\sum_{i=1}^{\widetilde{h_s}} \alpha_i\right) p_s^{\widetilde{h_s}-1} \frac{n}{\widetilde{h_s}} + \beta p_s^{\widetilde{h_s}}.$$

It follows from Lemma 3.5 that $D_A(v_{s-1}) = n/(h_{s-1}\widetilde{h_{s-1}})(p_s)^{\widetilde{h_s}-1}$. Therefore,

$$\frac{n}{h_{s-1}\widetilde{h_{s-1}}} = \left(\sum_{i=1}^{\widetilde{h_s}} \alpha_i\right)\frac{n}{\widetilde{h_s}} + \beta p_s. \tag{5}$$

If $\widetilde{h_s} = 1$ then it is clear that $h_{s-1}\widetilde{h_{s-1}}$ must be 1, for otherwise $\alpha_1 = 0$, implying that p_s divides $n/(h_{s-1}\widetilde{h_{s-1}})$. But this contradicts the assumption that $h_s = 1$. Furthermore, note that if all $\alpha_i \geq 1$ then this implies that all $\alpha_i = 1$, $\beta = 0$, and $h_{s-1}\widetilde{h_{s-1}} = 1$. If we assume $h_{s-1}\widetilde{h_{s-1}} \neq 1$, then there exists a j such that $\alpha_j = 0$.

Let U_j be the variable associated to the leaf u_j (resp., Y associated to y). By Proposition 2.5, the congruence condition at v in the direction of $\Delta_A(v)$ implies in particular that there exists an admissible monomial $H = U_1^{\alpha_1} \cdots U_{\widetilde{h_s}}^{\alpha_{\widetilde{h_s}}} Y^\beta$ such that, for every leaf u_j $(1 \leq j \leq \widetilde{h_s})$,

$$\left[\beta\frac{\ell_{yu_j}}{\det(\Gamma_{f,n})} + \sum_{i\neq j}\alpha_i\frac{\ell_{u_iu_j}}{\det(\Gamma_{f,n})} - \alpha_j e_{u_j}\cdot e_{u_j}\right] = \left[\frac{\ell_{vu_j}}{\det(\Gamma_{f,n})}\right].$$

For the particular j such that $\alpha_j = 0$, this condition reduces to

$$\left[\beta\frac{\ell_{yu_j}}{\det(\Gamma_{f,n})} + \sum_{i\neq j}\alpha_i\frac{\ell_{u_iu_j}}{\det(\Gamma_{f,n})}\right] = \left[\frac{\ell_{vu_j}}{\det(\Gamma_{f,n})}\right]. \tag{6}$$

By Lemmas 3.4 and 3.6,

$$\det(\Gamma_{f,n}) = p_s^{\widetilde{h_s}-1}\left(\frac{D_-(v_s)}{a'_s}\right) = p_s^{\widetilde{h_s}-1}(p'_{s-1})^{\widetilde{h_{s-1}}-1}\frac{D_-(v_{s-1})^{h_{s-1}}}{a'_{s-1}}.$$

One can easily see that $[\ell_{vu_j}/\det(\Gamma_{f,n})] = [0]$, $[\ell_{yu_j}/\det(\Gamma_{f,n})] = [0]$, and $[\ell_{u_iu_j}/\det(\Gamma_{f,n})] = [(a'_s n/\widetilde{h_s})/p_s]$ for $i \neq j$. Thus the congruence condition (6) for the leaf u_j is $\left[\left(\sum_{i\neq j}\alpha_i\right)\frac{a'_s n/\widetilde{h_s}}{p_s}\right] = [0]$; that is, $\left(\sum_{i\neq j}\alpha_i\right)a'_s n/\widetilde{h_s} \in \mathbb{Z}p_s$. Since a'_s and $n/\widetilde{h_s}$ are relatively prime to p_s, this implies that $\sum_{i\neq j}\alpha_i \in \mathbb{Z}p_s$. But then (5) implies that $n/(h_{s-1}\widetilde{h_{s-1}})$ is divisible by p_s, which is a contradiction. Therefore, we must have $h_{s-1}\widetilde{h_{s-1}} = 1$. □

This leads us to the next result.

PROPOSITION 6.3. *Suppose $h_s = 1$. Then $\Gamma_{f,n}$ satisfies the semigroup and congruence conditions if and only if both of the following statements hold*:

(I) $h_i\widetilde{h_i} = 1$ *for* $1 \leq i \leq s-1$;
(II) $a'_s = a_s/\widetilde{h_s} \in \mathbb{N}\langle a_{s-1}, p_1\cdots p_{s-1}, a_j p_{j+1}\cdots p_{s-1} \mid 1 \leq j \leq s-2\rangle$.

REMARK 6.4. Condition (II) is clearly not always satisfied. For example, take n divisible by a_s.

Proof of Proposition 6.3. We have already shown (Propositions 6.2 and 4.2) that if the semigroup and congruence conditions are satisfied then (I) and (II) must hold. So assume that (I) and (II) are satisfied. When $\widetilde{h_s} = 1$, the link is a $\mathbb{Z}$HS and the semigroup conditions are satisfied [12]. (There are no congruence conditions when the link is a $\mathbb{Z}$HS.)

Assume $\widetilde{h_s} \neq 1$. By Lemma 3.4, $D_-(v_k) = a_k$ for $2 \le k \le s-1$ and $D_-(v_s) = a'_s$; it follows from Lemma 3.5 that $D_A(v_k) = np_s^{\widetilde{h_s}-1}$ for $1 \le k \le s-1$. There is exactly one node of type v_k in $\Delta_{f,n}$ for $1 \le k \le s$, which we simply denote v_k. We denote the leaves $z_0, \ldots, z_{s-1}, u_1, \ldots, u_{\widetilde{h_s}}$, and y, as in Figure 14.

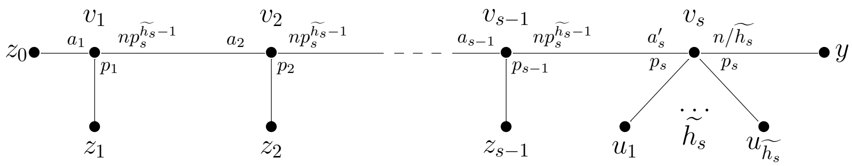


Figure 14 Splice diagram for $\widetilde{h_s} \neq 1$ and $h_i\widetilde{h_i} = 1, 1 \le i \le s-1$

It is clear from Proposition 4.2 that (a) the semigroup condition at the node v_k in the direction of $\Delta_-(v_k)$ is satisfied for $2 \le k \le s-1$ and (b) at the node v_s, this semigroup condition is equivalent to (II). Furthermore, one can see by examination of the splice diagram that the semigroup condition at each v_k in the direction of $\Delta_A(v_k)$ is always satisfied (including in the case $n = \widetilde{h_s}$).

It remains to show that $\Delta_{f,n}$ satisfies the congruence conditions. Lemma 3.6 implies that $\det(\Gamma_{f,n}) = p_s^{\widetilde{h_s}-1}$. In Figure 14 it is easy to see that, for any node v and any leaf w in $\Delta_{f,n}$, ℓ_{vw} is always divisible by $p_s^{\widetilde{h_s}-1}$. Therefore, $[\ell_{vw}/\det(\Gamma_{f,n})] = [0]$ for any node v and any leaf w. For each node, there are at most two conditions to check: one for each adjacent edge that does not lead directly to a leaf. By Proposition 2.5, we must show that for every node v and adjacent edge e there is an admissible monomial $M_{ve} = \prod_{w \in \Delta_{ve}} Z_w^{\alpha_w}$ such that, for every leaf w' in Δ_{ve},

$$\left[\sum_{w \neq w'} \alpha_w \frac{\ell_{ww'}}{\det(\Gamma)} - \alpha_{w'} e_{w'} \cdot e_{w'}\right] = [0]. \tag{7}$$

In this case, we have $A_i = a_{i+1} \cdots a_s$ for $1 \le i \le s-1$. Since $A_1 p'_1 = a_2 \cdots a_s p_1$ and $A_j a'_j = a_{j+1} \cdots a_s a_j$, it follows from Corollary 5.2 that $[e_{z_j} \cdot e_{z_j}] = [0]$ for $0 \le j \le s-1$. For any leaf z_j, $0 \le j \le s-1$, it is easy to see that $\ell_{z_j w'}$ is divisible by $p_s^{\widetilde{h_s}-1}$ for all leaves $w' \neq z_j$ in $\Delta_{f,n}$. Because the only leaves in the subgraph $\Delta_-(v_k)$ are of the form z_j, $0 \le j \le k-1$, equation (7) holds for all leaves in $\Delta_-(v_k)$ for any choice of admissible monomial. (In fact, we have shown that the action of the discriminant group element e_{z_j} is trivial for $0 \le j \le s-1$.)

Let Z_j be the variable associated to the leaf z_j, $0 \le j \le s-1$. It is easy to check that, for $1 \le k \le s-2$, the congruence condition at v_k in the direction of $\Delta_A(v_k)$ is satisfied for the admissible monomial Z_{k+1}. The only remaining condition is for the node v_{s-1} in the direction of v_s. Let U_j be the variable associated to the leaf

$u_j, 1 \le j \le \widetilde{h_s}$. We claim that the monomial $U_1 \cdots U_{\widetilde{h_s}}$ (which is easily seen to be an admissible monomial) satisfies the congruence condition. It is clear from the splice diagram that $[\ell_{u_i u_j}/\det(\Gamma_{f,n})] = [(n/\widetilde{h_s})a'_s/p_s]$ for $i \neq j$; since each u_j is a leaf of type $\overline{v_s}$, by Corollary 5.2 we have $[e_{u_j} \cdot e_{u_j}] = [(n/\widetilde{h_s})(a_s - a'_s)/p_s]$ for all j. Hence, for each u_j, equation (7) for the monomial $U_1 \cdots U_{\widetilde{h_s}}$ is

$$[(\widetilde{h_s} - 1)(n/\widetilde{h_s})a'_s/p_s - (n/\widetilde{h_s})(a_s - a'_s)/p_s] = [0].$$

This is clearly true, since $\widetilde{h_s} a'_s = a_s$. Finally, for the leaf y, equation (7) for $U_1 \cdots U_{\widetilde{h_s}}$ is $[\widetilde{h_s}\ell_{yu_j}/\det(\Gamma_{f,n})] = [0]$ (for any choice of j). Since ℓ_{yu_j} is divisible by $p_s^{\widetilde{h_s}-1}$, the condition is satisfied. □

6.2. Case (ii), $h_s = (n, p_s) > 1$

The pathological case $n = p_s = 2$ is treated separately at the end of the section. Our main goal is to prove the following.

PROPOSITION 6.5. *Suppose $h_s > 1$ and $n > 2$. Then $\Gamma_{f,n}$ satisfies the semigroup and congruence conditions if and only if*

$$s = 2, \quad p_2 = 2, \quad (n, p_2) = 2, \quad (n, a_2) = (n/2, p_1) = (n/2, a_1) = 1. \qquad (*)$$

Let us first assume that $\Gamma_{f,n}$ satisfies the semigroup and congruence conditions. We have already shown in Section 4 that the semigroup conditions imply $h_s = (n, p_s) = p_s$ and $h_i\widetilde{h_i} = 1$ for $1 \le i \le s-1$. Recall that, since the link is a $\mathbb{Q}$HS, $\widetilde{h_s} = 1$ and $a'_s = a_s$. We prove that $(*)$ must hold in two steps as follows.

Step 1. The congruence conditions imply that $p_s = 2$.
Step 2. The congruence conditions imply that $s = 2$.

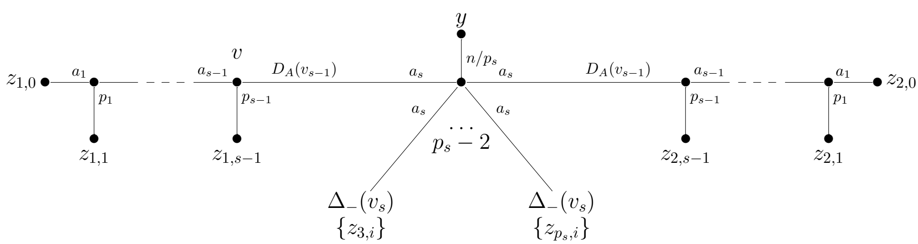


Figure 15 Splice diagram for $h_s = p_s$ and $h_i\widetilde{h_i} = 1$ for $1 \le i \le s-1$

Proof of Step 1. For maximum convenience, we will use the splice diagram Δ associated to the *minimal* good resolution graph $\Gamma^{\min}(X_{f,n})$ (see Figure 15). Recall that $p'_s = 1$ implies that there is no leaf of type $\overline{v_s}$, since that string completely collapses in the minimal resolution graph. We show that the congruence condition as in Proposition 2.5 for a node v of type v_{s-1} in the direction of $\Delta_A(v)$ cannot hold unless $p_s = 2$. The only difficulty is in notation.

By Lemmas 3.4 and 3.5, $D_-(v_k) = a_k$ for $2 \le k \le s$ and

$$D_A(v_k) = \frac{n}{p_s} \tilde{A}_k a_s^{p_s-2} \quad \text{for } 1 \le k \le s-1, \tag{8}$$

where $\tilde{A}_{s-1} = a_s - a_{s-1}p_{s-1}(p_s - 1)$ and $\tilde{A}_k = a_s - a_k p_k p_{k+1}^2 \cdots p_{s-1}^2(p_s - 1)$ for $1 \le k \le s-2$. Suppose that $p_s > 2$. For each i, $0 \le i \le s-1$, there are $h_s = p_s$ leaves of type $\overline{v_i}$. We label these leaves $\{z_{j,i} \mid 1 \le j \le p_s\}$, as indicated in Figure 15. The leaf on the edge with weight n/p_s is denoted y, and it is absent if $n/p_s = 1$. Let the variables corresponding to those in the Neumann–Wahl algorithm be $\{Z_{j,i}\}$ and Y, respectively. Let G be an admissible monomial for v in the direction of $\Delta_A(v)$ (i.e., in the direction of the central node). By the proof of Proposition 4.3 ($M = 0$), we know that the variable Y cannot appear in any admissible monomial G. Therefore, $G = \prod_{j=2}^{p_s}(Z_{j,0})^{\alpha_{j,0}} \cdots (Z_{j,s-1})^{\alpha_{j,s-1}}$, with $\alpha_{j,k} \in \mathbb{N} \cup \{0\}$ such that

$$D_A(v_{s-1}) = \sum_{k=0}^{s-1} \sum_{j=2}^{p_s} \ell'_{vz_{j,k}} \alpha_{j,k}. \tag{9}$$

For convenience of notation, we define integers M_i as follows:

$$M_i := \begin{cases} p_1 \cdots p_{s-1} & \text{for } i = 0, \\ a_i p_{i+1} \cdots p_{s-1} & \text{for } 1 \le i \le s-2, \\ a_{s-1} & \text{for } i = s-1. \end{cases}$$

(Note that $M_i = \bar{\beta}_i / p_s$.) Let v_s denote the unique node of type v_s (the central node). By Lemma 4.1, $\ell'_{v_s z_{j,i}} = M_i$ for all j. Therefore, $\ell_{vz_{j,i}} = M_i a_{s-1} p_{s-1} a_s^{p_s-2} n/p_s$ and $\ell'_{vz_{j,i}} = M_i a_s^{p_s-2} n/p_s$ for $1 \le i \le s-1$. Applying equation (8) and cancelling $a_s^{p_s-2} n/p_s$ from both sides of equation (9) yields

$$\tilde{A}_{s-1} = \sum_{k=0}^{s-1} \sum_{j=2}^{p_s} M_k \alpha_{j,k}. \tag{10}$$

Consider the congruence condition in Proposition 2.5 for the node v in the direction of $\Delta_A(v)$ for each of the leaves $z_{2,i}$, $0 \le i \le s-1$. By Lemma 3.6, $\det(\Gamma_{f,n}) = a_s^{p_s-1}$. For any admissible monomial G, the condition for $w' = z_{2,i}$ is equivalent to

$$\left[\sum_{k=0}^{s-1} \sum_{j=3}^{p_s} \alpha_{j,k} \frac{\ell_{z_{j,k} z_{2,i}}}{a_s^{p_s-1}} + \sum_{k \ne i} \alpha_{2,k} \frac{\ell_{z_{2,k} z_{2,i}}}{a_s^{p_s-1}} - \alpha_{2,i} e_{z_{2,i}} \cdot e_{z_{2,i}} \right] = \left[\frac{\ell_{vz_{2,i}}}{a_s^{p_s-1}} \right]. \tag{11}$$

For $0 \le i \le s-1$,

$$\frac{\ell_{vz_{2,i}}}{a_s^{p_s-1}} = \frac{(n/p_s) M_i a_{s-1} p_{s-1}}{a_s}. \tag{12}$$

Furthermore, for any $j \ne 2$ and for $0 \le k, i \le s-1$,

$$\frac{\ell_{z_{j,k} z_{2,i}}}{a_s^{p_s-1}} = \frac{(n/p_s) M_i M_k}{a_s}. \tag{13}$$

CLAIM 6.6. *Fix i such that $0 \le i \le s-1$. Then*

(a) $[e_{z_{2,i}} \cdot e_{z_{2,i}}] = \left[\dfrac{(n/p_s)M_i^2(p_s-1)}{a_s}\right]$ *and*

(b) *for $k \neq i$,* $\left[\dfrac{\ell_{z_{2,k}z_{2,i}}}{a_s^{p_s-1}}\right] = \left[\dfrac{-(n/p_s)M_iM_k(p_s-1)}{a_s}\right]$, $0 \le k \le s-1$.

Let us assume for now that Claim 6.6 is true and finish the proof of Step 1. By equation (13) and the claim, we have the following:

$$\begin{aligned}
\text{LHS of (11)} &= \left[\sum_{k=0}^{s-1}\sum_{j=3}^{p_s}\alpha_{j,k}\frac{(n/p_s)M_iM_k}{a_s} - \sum_{k=0}^{s-1}\alpha_{2,k}\frac{(n/p_s)M_iM_k(p_s-1)}{a_s}\right]\\
&= \left[\frac{(n/p_s)M_i}{a_s}\left\{\sum_{k=0}^{s-1}\sum_{j=2}^{p_s}\alpha_{j,k}M_k - p_s\sum_{k=0}^{s-1}\alpha_{2,k}M_k\right\}\right]\\
&= \left[\frac{(n/p_s)M_i}{a_s}\left\{\tilde{A}_{s-1} - p_s\sum_{k=0}^{s-1}\alpha_{2,k}M_k\right\}\right] \quad \text{(by (10))}\\
&= \left[\frac{(n/p_s)M_i}{a_s}\left\{a_s - a_{s-1}p_{s-1}(p_s-1) - p_s\sum_{k=0}^{s-1}\alpha_{2,k}M_k\right\}\right]\\
&= \left[\frac{(n/p_s)M_i}{a_s}\left\{-a_{s-1}p_{s-1}(p_s-1) - p_s\sum_{k=0}^{s-1}\alpha_{2,k}M_k\right\}\right].
\end{aligned}$$

Therefore, by (12), the congruence condition (11) is equivalent to

$$\left[\frac{(n/p_s)M_i}{a_s}\left\{-a_{s-1}p_{s-1}(p_s-1) - p_s\sum_{k=0}^{s-1}\alpha_{2,k}M_k\right\}\right] = \left[\frac{(n/p_s)M_ia_{s-1}p_{s-1}}{a_s}\right],$$

which is clearly equivalent to

$$\left[-\frac{(n/p_s)M_ip_s}{a_s}\left(a_{s-1}p_{s-1} + \sum_{k=0}^{s-1}\alpha_{2,k}M_k\right)\right] = [0].$$

Since $(a_s, n) = 1$ and $(a_s, p_s) = 1$, this is equivalent to

$$M_i\left(a_{s-1}p_{s-1} + \sum_{k=0}^{s-1}\alpha_{2,k}M_k\right) \in \mathbb{Z}a_s. \tag{14}$$

Therefore, if the congruence conditions are satisfied then this implies, in particular, that (14) holds for all i such that $0 \le i \le s-1$.

We claim that if (14) holds for all i then this implies that a_s divides

$$\mathrm{S} := a_{s-1}p_{s-1} + \sum_{k=0}^{s-1}\alpha_{2,k}M_k.$$

Let $a_s = q_1^{e_1} \cdots q_l^{e_l}$ be the prime power factorization of a_s. Suppose there is a j such that $q_j^{e_j}$ does not divide S. Then at least one power of q_j must divide M_i for $0 \le i \le s-1$. In particular, q_j divides $M_{s-1} = a_{s-1}$ and, since $(a_{s-1}, p_{s-1}) = 1$, this implies that q_j divides a_{s-2} because $M_{s-2} = a_{s-2}p_{s-1}$. This, in turn, implies q_j divides a_{s-3} and so forth, down to a_1. But $M_0 = p_1 \cdots p_{s-1}$, which cannot possibly be divisible by q_j. We have derived a contradiction, so a_s divides S.

Finally, we claim that for $p_s > 2$ it is impossible for a_s to divide S. Equation (10), which is equivalent to $a_s - a_{s-1}p_{s-1}(p_s - 1) = \sum_{k=0}^{s-1}\sum_{j=2}^{p_s} \alpha_{j,k}M_k$, implies that $\sum_{k=0}^{s-1} \alpha_{2,k}M_k \le a_s - a_{s-1}p_{s-1}(p_s-1)$, and hence

$$\mathrm{S} = a_{s-1}p_{s-1} + \sum_{k=0}^{s-1} \alpha_{2,k}M_k \le a_s - a_{s-1}p_{s-1}(p_s - 2).$$

If $p_s > 2$ then $a_s - a_{s-1}p_{s-1}(p_s-2) < a_s$, which implies that $\mathrm{S} < a_s$; hence S cannot be divisible by a_s, which is a contradiction. Therefore, we must have $p_s = 2$ for the congruence conditions to be satisfied. This completes the proof of Step 1. □

Proof of Claim 6.6. Since $z_{2,i}$ is a leaf of type $\overline{v_i}$, part (a) of the claim follows from Corollary 5.2. For part (b), without loss of generality we assume $i < k$. For $1 \le i < k \le s-2$ with $i \ne k-1$, we have $\ell_{z_{2,k}z_{2,i}} = D_A(v_k)a_ip_{i+1}\cdots p_{k-1}$ and hence

$$\begin{aligned}
\left[\frac{\ell_{z_{2,k}z_{2,i}}}{\det(\Gamma_{f,n})}\right] &= \left[\frac{(n/p_s)\tilde{A}_k a_i p_{i+1}\cdots p_{k-1}}{a_s}\right] \\
&= \left[\frac{(n/p_s)(a_s - a_k p_k p_{k+1}^2 \cdots p_{s-1}^2(p_s-1))a_ip_{i+1}\cdots p_{k-1}}{a_s}\right] \\
&= \left[\frac{-(n/p_s)(p_s-1)a_kp_kp_{k+1}^2\cdots p_{s-1}^2 \cdot a_ip_{i+1}\cdots p_{k-1}}{a_s}\right] \\
&= \left[\frac{-(n/p_s)(p_s-1)M_kM_i}{a_s}\right].
\end{aligned}$$

The remaining cases are all similar and easy to check. □

Proof of Step 2. So far, we have that the semigroup and congruence conditions imply that $h_s = p_s = 2$ and $h_i\widetilde{h_i} = 1$ for $1 \le i \le s-1$. Write $n = 2n'$ with $n' > 1$. We will show that, for $s \ge 3$, the congruence conditions at a node v of type v_{s-2} in the direction of $\Delta_A(v)$ cannot be satisfied. We should note that the congruence condition at a node of type v_{s-1} that we studied in Step 1 *can* be satisfied for $s \ge 3$. For example, take

$$\begin{aligned}
&a_1 = 3, \quad a_2 = 19, \quad a_3 = 117, \\
&p_1 = 2, \quad p_2 = 3, \quad\ \ p_3 = 2,
\end{aligned}$$

and any $n = 2n'$ such that n' is relatively prime to 2, 3, 13, and 19.

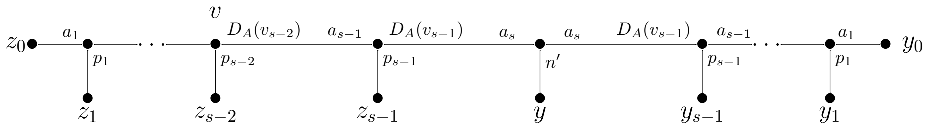


Figure 16 Splice diagram for $n > 2$, $h_s = p_s = 2$, and $h_i \widetilde{h_i} = 1$ for $1 \le i \le s-1$

Figure 16 depicts the splice diagram in the general situation. The semigroup condition at v in the direction of $\Delta_A(v)$ is

$$D_A(v_{s-2}) \in \mathbb{N}\langle D_A(v_{s-1}), a_s p_{s-1}, n' p_{s-1} M_i \mid 0 \le i \le s-1\rangle.$$

Recall that $D_A(v_{s-1}) = n'(a_s - a_{s-1}p_{s-1})$ and $D_A(v_{s-2}) = n'(a_s - a_{s-2}p_{s-2}p_{s-1}^2)$. The semigroup condition implies that there exist α, β, and $\gamma_i \in \mathbb{N} \cup \{0\}$ such that

$$n'(a_s - a_{s-2}p_{s-2}p_{s-1}^2) = \alpha n'(a_s - a_{s-1}p_{s-1}) + \beta a_s p_{s-1} + \sum_{i=0}^{s-1} \gamma_i n' M_i p_{s-1}.$$

If $\beta \neq 0$, then $\beta a_s p_{s-1}$ must be divisible by $n' > 1$. By assumption, $(a_s, n') = \widetilde{h_s} = 1$ and $(p_{s-1}, n') = h_{s-1} = 1$; therefore, n' must divide β. But then $\beta a_s p_{s-1} \ge n' a_s p_{s-1} > n' a_s > D_A(v_{s-2})$, and this is impossible. Hence $\beta = 0$.

We can thus cancel n' from the previous equation, leaving

$$a_s - a_{s-2}p_{s-2}p_{s-1}^2 = \alpha(a_s - a_{s-1}p_{s-1}) + \sum_{i=0}^{s-1} \gamma_i M_i p_{s-1}.$$

Since $M_{s-1} = a_{s-1}$, we have

$$(\alpha - \gamma_{s-1}) a_{s-1} p_{s-1} = (\alpha - 1) a_s + \sum_{i=0}^{s-2} \gamma_i M_i p_{s-1} + a_{s-2}p_{s-2}p_{s-1}^2, \tag{15}$$

which implies $(\alpha - \gamma_{s-1}) a_{s-1} p_{s-1} > (\alpha - 1) a_s$. Suppose $\alpha > 1$. Then, since $a_s = q_s + a_{s-1}p_{s-1}p_s$ and $p_s = 2$, it follows that

$$(\alpha - \gamma_{s-1}) a_{s-1} p_{s-1} > (\alpha - 1) a_s > (\alpha - 1) 2 a_{s-1} p_{s-1},$$

which implies $(\alpha - \gamma_{s-1}) - 2(\alpha - 1) > 0$ (i.e., $2 > \alpha + \gamma_{s-1}$). But this is impossible for $\alpha > 1$.

Now suppose $\alpha = 1$. It is clear from equation (15) that $\gamma_{s-1} = 0$, so

$$a_{s-1}p_{s-1} = \sum_{i=0}^{s-2} \gamma_i M_i p_{s-1} + a_{s-2}p_{s-2}p_{s-1}^2;$$

that is, $a_{s-1} = \sum_{i=0}^{s-2} \gamma_i M_i + a_{s-2}p_{s-2}p_{s-1}$. But, since M_i is divisible by p_{s-1} for $0 \le i \le s-2$, this would imply a_{s-1} is divisible by p_{s-1}, which is impossible. Therefore, $\alpha = 0$ and we have

$$a_s - a_{s-2}p_{s-2}p_{s-1}^2 = \sum_{i=0}^{s-1} \gamma_i M_i p_{s-1}. \tag{16}$$

(Note that this semigroup condition is already quite restrictive, because it requires a_s to be divisible by p_{s-1}.)

Now let us return to the congruence conditions for the node v in the direction of $\Delta_A(v)$. An admissible monomial for v in that direction must be of the form $H = Y_0^{\gamma_0} \cdots Y_{s-1}^{\gamma_{s-1}}$, with $\gamma_i \in \mathbb{N} \cup \{0\}$. The congruence condition for the leaf y_{s-1} is

$$\left[\frac{\ell_{v y_{s-1}}}{\det(\Gamma_{f,n})}\right] = \left[\sum_{i=0}^{s-2} \gamma_i \frac{\ell_{y_{s-1} y_i}}{\det(\Gamma_{f,n})} - \gamma_{s-1} e_{y_{s-1}} \cdot e_{y_{s-1}}\right].$$

Applying Claim 6.6, we see that this condition is equivalent to

$$\left[\frac{n' a_{s-2} p_{s-2} a_{s-1} p_{s-1}}{a_s}\right] = \left[-\frac{n' a_{s-1}}{a_s}\left(\sum_{i=0}^{s-1} \gamma_i M_i\right)\right];$$

that is, $n' a_{s-1}\big(a_{s-2} p_{s-2} p_{s-1} + \sum_{i=0}^{s-1} \gamma_i M_i\big) \in \mathbb{Z} a_s$. Since $(a_s, n') = 1$, we must have $a_{s-1}\big(a_{s-2} p_{s-2} p_{s-1} + \sum_{i=0}^{s-1} \gamma_i M_i\big) = N a_s$ for some N in $\mathbb{Z}$. If we multiply both sides of this equation by p_{s-1} and apply equation (16), the result is

$$a_{s-1} a_{s-2} p_{s-2} p_{s-1}^2 + a_{s-1}(a_s - a_{s-2} p_{s-2} p_{s-1}^2) = N a_s p_{s-1}$$

(i.e., $a_{s-1} = N p_{s-1}$). This implies that p_{s-1} divides a_{s-1}, which is a contradiction.

Thus we have shown that, if $s \geq 3$, then the congruence condition for the node v of type v_{s-2} in the direction of $\Delta_A(v)$ cannot be satisfied for the leaf y_{s-1}. Hence, the congruence conditions imply that $s = 2$. $\square$

We have finished Steps 1 and 2, proving one direction of Proposition 6.5.

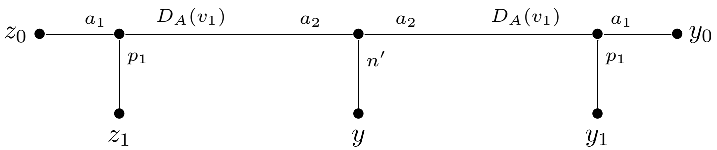


Figure 17 Splice diagram for $(*)$, $n > 2$

For the other direction, we must check that $(*)$ implies that the semigroup and congruence conditions are satisfied. The splice diagram in this situation is shown in Figure 17. The only semigroup condition that needs to be checked is

$$D_A(v_1) \in \mathbb{N}\langle a_2, n' a_1, n' p_1\rangle,$$

where $D_A(v_1) = n'(a_2 - a_1 p_1) = n'(q_2 + a_1 p_1)$. Since a_1 and p_1 are relatively prime, the conductor of the semigroup generated by a_1 and p_1 is less than $a_1 p_1$; hence $a_1 p_1 + q_2$ is in the semigroup generated by a_1 and p_1 and so this semigroup condition is satisfied.

There are only two congruence conditions to check. One is equivalent to the following: there exist α_0 and α_1 in $\mathbb{N} \cup \{0\}$ such that $a_2 = \alpha_0 p_1 + \alpha_1 a_1$,

$$\left[\alpha_1\frac{-n'a_1p_1}{a_2}-\alpha_0\frac{n'p_1^2}{a_2}\right]=[0],\quad\text{and}\quad\left[\alpha_0\frac{-n'a_1p_1}{a_2}-\alpha_1\frac{n'a_1^2}{a_2}\right]=[0].$$

But these conditions are obviously both satisfied for any α_0 and α_1 such that $a_2 = \alpha_0 p_1 + \alpha_1 a_1$. The other congruence condition is equivalent to the following: there exist γ_0 and γ_1 in $\mathbb{N} \cup \{0\}$ such that $a_2 - a_1 p_1 = \gamma_0 p_1 + \gamma_1 a_1$,

$$\left[\gamma_1\frac{-n'a_1p_1}{a_2}-\gamma_0\frac{n'p_1^2}{a_2}\right]=\left[\frac{n'a_1p_1^2}{a_2}\right],\quad\text{and}$$
$$\left[\gamma_0\frac{-n'a_1p_1}{a_2}-\gamma_1\frac{n'a_1^2}{a_2}\right]=\left[\frac{n'a_1^2p_1}{a_2}\right].$$

But obviously these conditions are also both satisfied for any γ_0 and γ_1 such that $a_2 - a_1 p_1 = \gamma_0 p_1 + \gamma_1 a_1$. This concludes the proof of Proposition 6.5.

The pathological case

If $h_s > 1$ and $n = 2$, then the semigroup conditions imply that $p_s = 2$ by Proposition 4.3. Therefore, all that remains in our proof of the Main Theorem is the pathological case. Let $\Gamma_{f,n}$ be the graph associated to the minimal good resolution (see Section 3).

PROPOSITION 6.7. *Suppose $n = p_s = 2$. Then $\Gamma_{f,n}$ satisfies the semigroup and congruence conditions if and only if $s = 2$.*

Proof. We begin by assuming that $\Gamma_{f,n}$ satisfies the semigroup and congruence conditions. It is automatically true that $h_i\widetilde{h_i} = 1$ for $1 \le i \le s-1$ and that $h_s = 2$. We must show that s must be 2. The splice diagram is pictured in Figure 18. We can use essentially the same argument as in Step 2 of the proof of Proposition 6.5 to show that, for $s \ge 3$, the congruence conditions at the node v of type v_{s-2} in the direction of $\Delta_A(v)$ cannot possibly be satisfied for the leaf y_{s-1}.

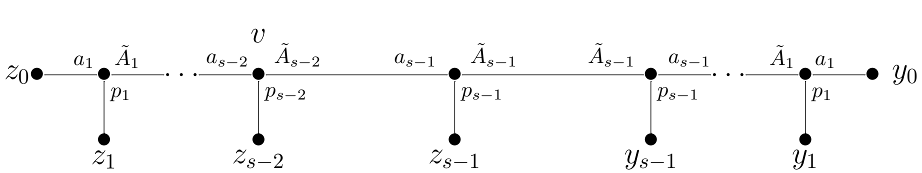


Figure 18 Splice diagram for the pathological case, $s > 2$

The semigroup condition at v in the direction of $\Delta_A(v)$ is

$$\tilde{A}_{s-2} \in \mathbb{N}\langle \tilde{A}_{s-1}, p_{s-1}M_i \mid 0 \le i \le s-1\rangle.$$

Precisely the same argument as in Step 2 shows that $\tilde{A}_{s-1}$ cannot appear in the expression for $\tilde{A}_{s-2}$ that comes from the semigroup condition. Hence there exist γ_i in $\mathbb{N} \cup \{0\}$ such that $a_s - a_{s-2}p_{s-2}p_{s-1}^2 = \sum_{i=0}^{s-1} \gamma_i M_i p_{s-1}$.

Let $H = Y_0^{\gamma_0} \cdots Y_{s-1}^{\gamma_{s-1}}$ be an admissible monomial for v in the direction of $\Delta_A(v)$. Then the congruence condition for the leaf y_{s-1} is equivalent to

$$\left[\frac{a_{s-2}p_{s-2}a_{s-1}p_{s-1}}{a_s}\right] = \left[-\frac{a_{s-1}}{a_s}\left(\sum_{i=0}^{s-1}\gamma_i M_i\right)\right].$$

Just as in Step 2, this implies that p_{s-1} divides a_{s-1} and hence that the congruence conditions cannot be satisfied for $s > 2$.

Finally, for $s = 2$, it is easy to check that the semigroup and congruence conditions are satisfied. □

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