

Topological Aspects of Poset Spaces

CARL MUMMERT & FRANK STEPHAN

1. Introduction

Recent work in mathematical logic [10; 11; 12] has led to an interest in certain topological spaces formed from filters on partially ordered sets. This paper describes the general topology of these poset spaces.

The results of the paper are divided as follows. In Section 2 we define two classes of spaces, MF spaces and UF spaces. Together these spaces form the class of poset spaces. We show that many familiar spaces are homeomorphic to poset spaces. In Section 3, we characterize the separation properties of poset spaces and show that any second-countable poset space is homeomorphic to a space of the same kind formed from a countable poset. In Section 4, we show that the class of MF spaces are closed under arbitrary topological products and that any G_δ subspace of an MF space is again an MF space. We show that UF spaces are closed under the action of taking G_δ subspaces but not closed under binary products. In Section 5, we establish that poset spaces are of the second Baire category and possess the strong Choquet property. We give a characterization of the class of countably based MF spaces as the class of second-countable T_1 spaces with the strong Choquet property. In Section 6, we apply the results of Section 5 to domain theory, giving a complete characterization of the second-countable topological spaces that have a domain representation. Section 7 contains results on the relationship between MF spaces (not necessarily countably based) and semi-topogenous orders. We use semi-topogenous orders to establish a sufficient condition for an arbitrary space to be homeomorphic to an MF space. In Section 8, we show that every second-countable poset space either is countable or contains a perfect closed set.

2. Poset Spaces

Our goal in this section is to define the class of poset spaces and show that this class includes all complete metric spaces and all locally compact Hausdorff spaces. We first review some basic definitions about partially ordered sets.

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A *poset* is a set P with an reflexive, antisymmetric, transitive relation \leq . That is, the following conditions hold for all p, q , and r in P :

- $p \leq p$;
- if $p \leq q$ and $q \leq p$ then $q = p$;
- if $p \leq q$ and $q \leq r$ then $p \leq r$.

We write $p < q$ if $p \leq q$ and $p \neq q$. If there is no r such that $r \leq p$ and $r \leq q$, we write $p \perp q$.

A *filter* is a subset F of a poset P that satisfies the following two conditions:

- (1) for every $p, q \in F$, there is an $r \in F$ such that $r \leq p$ and $r \leq q$;
- (2) for every $p \in F$ and $q \in P$, if $p \leq q$ then $q \in F$.

A filter F is *unbounded* if there is no $r \in P$ such that $r < q$ for every $q \in F$. Furthermore, F is *maximal* if there is no strictly larger filter containing F . Every maximal filter is unbounded, but in general not every unbounded filter is maximal.

For any poset P , we let $\text{UF}(P)$ denote the set of unbounded filters on P and let $\text{MF}(P)$ denote the set of maximal filters on P . We topologize $\text{UF}(P)$ with the basis $\{N_p \mid p \in P\}$, where

$$N_p = \{F \in \text{UF}(P) \mid p \in F\}.$$

We give $\text{MF}(P)$ the topology it inherits as a subset of $\text{UF}(P)$; when we work with spaces of maximal filters we may write N_p to denote the set of maximal filters containing p . To facilitate the exposition, we sometimes identify $p \in P$ with the open set N_p and identify a subset U of P with the open set $\bigcup_{p \in U} N_p$.

A *UF space* is a space of the form $\text{UF}(P)$, and an *MF space* is a space of the form $\text{MF}(P)$; UF spaces and MF spaces are collectively referred to as *poset spaces*. A poset space is *countably based* if it is formed from a countable poset. It is possible that P is uncountable but $\text{MF}(P)$ or $\text{UF}(P)$ is a second-countable space (an example is provided after Theorem 2.3). We shall demonstrate that every second-countable poset space is homeomorphic to a countably based poset space. This result justifies our terminology.

REMARK 2.1. It is sometimes convenient to work with strict partial orders instead of the nonstrict partial orders defined previously. A *strict* partial order is a set P with an irreflexive, transitive relation $<$. Every strict partial order $\langle P, < \rangle$ is canonically associated to nonstrict partial order $\langle P, \leq \rangle$ in which $p \leq q$ if and only if $p < q$ or $p = q$, and every nonstrict partial order arises in this way. A filter on a strict partial order $\langle P, < \rangle$ is a set $F \subseteq P$ that is upward closed and such that if $p, q \in F$ then there is an $r \in F$ with $r \leq p$ and $r \leq q$.

It follows immediately from these definitions that if $\langle P, < \rangle$ is a strict partial order, $\langle P, \leq \rangle$ is the corresponding nonstrict partial order, and $F \subseteq P$, then F is a filter in $\langle P, < \rangle$ if and only if F is a filter in $\langle P, \leq \rangle$, and vice versa. Moreover, F is a maximal (resp. unbounded) filter in either of these partial orders if and only if it is maximal (resp. unbounded) in the other partial order.

A topology on the set of maximal (resp. unbounded) filters of a strict partial order is defined in the same way as for a nonstrict partial order. Once this definition is made, it is immediate that, for any strict poset $\langle P, < \rangle$ and corresponding

nonstrict poset $\langle P, \preceq \rangle$, the identity map $P \rightarrow P$ induces a homeomorphism of the topological spaces of maximal (resp. unbounded) filters of these posets. For this reason, we may for convenience prove results using strict partial orders instead of nonstrict partial orders. This technique is sound because any example of a poset space obtained from a strict partial order can be converted to a homeomorphic example obtained from a nonstrict poset space, and vice versa.

We now present two examples showing that many familiar spaces are homeomorphic to poset spaces.

THEOREM 2.2. *Every locally compact Hausdorff space is homeomorphic to an MF space.*

Proof. Let X be a locally compact Hausdorff space, and let P be the set of all nonempty precompact open subsets of X . For $U, V \in P$ we put $U \preceq V$ if $U = V$ or if the closure of U is contained in V . If F is a filter and $U \in F$ then, because U is precompact,

$$\bigcap F = \bigcap \{\bar{V} \mid \bar{V} \subseteq U, V \in F\}$$

is the filtered intersection of nonempty compact sets and hence is nonempty and compact. Since X is Hausdorff, any two points of X have open neighborhoods whose closures are disjoint. If F is a maximal filter, then at most one of these neighborhoods can be in F , which implies that $\bigcap F$ is a singleton. Finally, the mapping $\phi: \text{MF}(P) \rightarrow X$ given by $F \mapsto \bigcap F$ has as its inverse the mapping

$$\phi^{-1}: x \mapsto \{p \in P \mid x \in N_p\}.$$

To prove that ϕ is continuous, fix $x \in \text{MF}(P)$ and let U be any open neighborhood of $\phi(x)$ in X . Because X is locally compact, we may assume without loss of generality that U is precompact, because the precompact sets form a basis for the topology. Thus we assume $U = N_p$ for some $p \in P$. Now, since $\phi(x) \in U$, we have $p \in x$ and so $x \in N_p$. Moreover, for any $F \in N_p$ in $\text{MF}(P)$ we have $\phi(F) = \bigcap F \in U$. This shows that ϕ is continuous.

To prove that ϕ^{-1} is continuous, let $y \in X$ be fixed and let V be any open neighborhood of $\phi^{-1}(y)$ in $\text{MF}(P)$. Without loss of generality, we may assume that $V = N_p$ for some $p \in P$. Now p itself is some precompact open subset U of X , and for any $y' \in U$ we have $p \in \phi^{-1}(y')$. Thus $\phi^{-1}(U) \subseteq V$, which shows that ϕ^{-1} is continuous. \square

Because there exist both non-locally compact complete separable metric spaces and locally compact Hausdorff nonmetrizable spaces, the next theorem is independent of Theorem 2.2. A construction similar to that in Theorem 2.3 was used by Lawson [7] to represent complete separable metric spaces in the context of domain theory (see Section 6).

THEOREM 2.3. *For every complete metric space X there is a poset P such that $X \cong \text{UF}(P)$ and $\text{UF}(P) = \text{MF}(P)$. Moreover, if X is infinite then we may take the cardinality of P to be that of any dense subset of X .*

Proof. Let X be a complete metric space; we write $B(x, \varepsilon)$ for the open metric ball of radius $\varepsilon > 0$ around a point $x \in X$. Let A be a dense subset of X . The poset P is the set of all open balls $B(a, r)$, where r is a positive rational number and $a \in A$. For $p = B(a, r)$ and $p' = B(a', r')$ in P , we let $p < p'$ if and only if $d(a, a') + r < r'$. An argument similar to the one in the proof of Theorem 2.2 shows that any unbounded filter on P has a unique point in its intersection. The resulting mapping $\phi: F \mapsto \bigcap F$ from $\text{UF}(P)$ to X has as its inverse the mapping

$$x \mapsto \{B(a, r) \mid x \in B(a, r), a \in A, r \in \mathbb{Q}^+\}.$$

We can show that each of these mappings is continuous by using the same method as the proof of Theorem 2.2 and the fact that the open balls included in P form a basis for X . Finally, since X is a complete metric space, every unbounded filter is maximal (see Theorem 3.1 for details). \square

If Theorem 2.3 is applied to the real line using the line itself as the dense subset, then the resulting poset P will be uncountable but $\text{MF}(P) = \text{UF}(P)$ will be homeomorphic to the real line.

There are also second-countable nonmetrizable Hausdorff MF spaces. One example is the Gandy–Harrington space from modern descriptive set theory (see [11]).

3. Separation and Countability Properties

In this section, we determine the separation properties that a poset space must satisfy. We then show that every second countable poset space is homeomorphic to a poset space obtained from a countable poset. In Section 8 we will show that a countably based poset space either is countable or contains a perfect closed set.

THEOREM 3.1. (i) *Every UF space is T_0 .*

(ii) *Every MF space is T_1 .*

(iii) *If $\text{UF}(P)$ is T_1 , then every unbounded filter on P is maximal and thus $\text{UF}(P) = \text{MF}(P)$.*

Proof. Part (i) follows from the fact that distinct filters are distinct as subsets of P . Part (ii) follows from the fact that no maximal filter can properly contain another maximal filter. To prove (iii), suppose $\text{UF}(P)$ is T_1 and let F be an unbounded filter on P . Let G be a filter on P such that $F \subseteq G$. Clearly, G is unbounded. If $F \neq G$ then there must be a $p \in P$ such that $F \in N_p$ and $G \notin N_p$. This means $p \in (F \setminus G)$, which is impossible. Thus $F = G$, which shows that F is maximal. \square

THEOREM 3.2. *Suppose that P is a poset such that $\text{MF}(P)$ is second countable. Then there is a countable subposet R of P such that the map $F \mapsto R \cap F$ is a homeomorphism from $\text{MF}(P)$ to $\text{MF}(R)$.*

Proof. Suppose that $\text{MF}(P)$ is second countable; thus P contains a countable subset Q_0 such that $\{N_q : q \in Q_0\}$ is a basis for the topology, because every basis of a second-countable topology contains a countable subclass that is also a basis.

For $n = 0, 1, 2, \dots$, we construct inductively a set Q_{n+1} that satisfies the following conditions:

- Q_{n+1} is countable;
- $Q_n \subseteq Q_{n+1} \subseteq P$;
- for every $F \in \text{MF}(P)$ and every finite subset $D \subseteq Q_n \cap F$, there is a $q \in Q_{n+1}$ such that $q \preceq d$ for all $d \in D$.

In order to see that Q_{n+1} can be taken as countable, suppose D is a finite subset of Q_n with nonempty intersection. Let E_D be the set of all $p \in P$ such that $p \preceq d$ for every $d \in D$. For every filter $F \in \text{MF}(P)$ with $D \subseteq F$ there is an element $p \in E_D \cap F$; thus $\{N_e : e \in E_D\}$ is an open cover of the intersection of all open sets N_d with $d \in D$. Since the given space is second countable, there is a countable subset F_D of E_D covering the same set of maximal filters (if some finite subset D of Q_n is not contained in any filter, then let F_D be empty). Now take Q_{n+1} to be the union of all F_D , where $D \subseteq Q_n$ and D is finite; observe that Q_{n+1} is also at most countable.

Let $R = \bigcup_i Q_i$. Note that R is countable and $\{N_r \mid r \in R\}$ is a basis for $\text{MF}(P)$. For $F \subseteq P$ we write $\phi(F)$ for $F \cap R$. It is straightforward to verify, by the construction of R , that $\phi(F)$ is a filter for every $F \in \text{MF}(P)$. Because $R \subseteq P$, every $F \in \text{MF}(R)$ extends to some $F' \in \text{MF}(P)$; hence $\phi(F') = F$. This shows that ϕ determines a surjective map Φ from $\text{MF}(P)$ to $\text{MF}(R)$.

In order to prove that Φ is injective, it suffices to prove the following statement. For maximal filters V, W on P we have $V \subseteq W$ if and only if $\phi(V) \subseteq \phi(W)$. Suppose $p \in V \setminus W$. Then $W \not\subseteq N_p$ and thus $W \not\subseteq N_q$ for all q with $N_q \subseteq N_p$. On the other hand, R is a basis and N_p is the union of basic open sets. Since $V \in N_p$ there is an $r \in R$ with $N_r \subseteq N_p$ and $V \in N_r$. It follows that $r \in \phi(V) \setminus \phi(W)$. The other direction of the implication is trivial.

This shows that ϕ is a bijection from $\text{MF}(P)$ to $\text{MF}(R)$. To see that ϕ is continuous, let $x \in \text{MF}(P)$ be fixed and let U be an open neighborhood of $\phi(x) = x \cap R$ in $\text{MF}(R)$. Without loss of generality, we may assume that U is of the form N_r for some $r \in R$. Let $V = \{y \in \text{MF}(P) \mid r \in y\}$ be the basic open set determined by r in $\text{MF}(P)$. Now, because $r \in \phi(x) = x \cap R$, we see that $r \in x$ and thus $x \in V$. Moreover, for any $x' \in V$, we have $r \in x'$ and so $r \in x' \cap R$, which means that $\phi(x') \in U$. Thus ϕ is continuous.

To see that ϕ^{-1} is continuous, let V be any open subset of $\text{MF}(P)$ and let $\phi^{-1}(y)$ be in V . Because $\{N_r \subseteq \text{MF}(P) \mid r \in R\}$ is a basis for $\text{MF}(P)$, there exists an $r \in R$ with $\phi^{-1}(y) \in N_r \subseteq V$. Moreover, any $y' \in \text{MF}(R)$ with $r \in y'$ will satisfy $r \in \phi^{-1}(y')$. Thus, for $U = \{y \in \text{MF}(R) \mid r \in y\}$, we have $y \in U$ and $\phi^{-1}(U) \subseteq V$. This shows that ϕ^{-1} is continuous. \square

COROLLARY 3.3. *An MF space is homeomorphic to a countably based MF space if and only if it is second countable.*

COROLLARY 3.4. *A UF space is homeomorphic to a countably based UF space if and only if it is second countable.*

Proof. Let $X = \text{UF}(P)$ be second countable. Construct a poset R and a map ϕ in a manner analogous to the proof of Theorem 3.2. We show that ϕ is a homeomorphism from $\text{UF}(P)$ to $\text{UF}(R)$. It is clear that if $F \in \text{UF}(P)$ then $\phi(F) \in \text{UF}(R)$. Every $G \in \text{UF}(R)$ extends to some $G' \in \text{UF}(P)$, so $\phi(G') = G$. Thus ϕ is well-defined and surjective as a map from $\text{UF}(P)$ to $\text{UF}(R)$. To see that ϕ is injective, suppose that $F \neq G$ are unbounded filters on P . Without loss of generality we may assume there is some $p \in G \setminus F$. Hence there is an $r \in R \cap (G \setminus F)$, because R is a basis. But $r \in R \cap (G \setminus F)$ implies $r \in \phi(G) \setminus \phi(F)$, which shows that $\phi(G) \neq \phi(F)$. Thus ϕ is a bijection from $\text{UF}(P)$ to $\text{UF}(R)$. The proof that ϕ is a homeomorphism is the same as in the proof of Theorem 3.2. \square

4. Product and Subspace Properties

In this section, we show that the class of MF spaces is closed under taking G_δ subsets and arbitrary topological products. The class of UF spaces is closed under taking G_δ subspaces, but it is not closed under even finite products.

THEOREM 4.1. *The class of MF spaces is closed under arbitrary topological products.*

Proof. Let $\langle\langle P_i, \leq_i \rangle \mid i \in I \rangle$ be a collection of posets. We may assume without loss of generality that each poset has a greatest element, which we denote by p_i . We form a poset P consisting of those functions f from I to $\bigcup_{i \in I} P_i$ such that $f(i) \in P_i$ for all i and $f(i) = p_i$ for all but finitely many i . For $f, g \in P$ we write $f \leq g$ if $f(i) \leq_i g(i)$ for all i .

We define a map ϕ from $\prod_i \text{MF}(P_i)$ to $\text{MF}(P)$ by sending $\prod_i F_i$ to the set of all functions $f \in P$ such that $f(i) \in F_i$ for all i . The inverse of ϕ takes $x \in \text{MF}(P)$ and returns $\prod_i x_i$, where

$$x_i = \{p \in P_i \mid q(i) = p \text{ for some } q \in x\}.$$

To see that ϕ is continuous, let $x \in \prod_i \text{MF}(P_i)$ be fixed and let U be a basic open neighborhood of $\phi(x)$, so that U is of the form N_p for some $p \in P$. Now p is represented by a function $f: I \rightarrow \bigcup_i P_i$ that returns the maximal element of P_i for all but finitely many $i \in I$. Thus f determines a basic open set V in the product topology $\prod_i \text{MF}(P_i)$ such that V is equal, in each coordinate $i \in I$, to the open set determined by $f(i)$. Then $x \in V$. Suppose $x' = \prod_i x'_i$ is any point of $\prod_i \text{MF}(P_i)$ that is in V , which means that $f(i) \in x'_i$ for all $i \in I$. Then $\phi(x')$ will have the property that $p_i \in x'_i$ for each $i \in I$, which means $\phi(x') \in N_p$. Thus ϕ is continuous.

To see that ϕ^{-1} is continuous, let $y \in \text{MF}(P)$ be fixed and let V be any neighborhood of $\phi^{-1}(y)$ in $\prod_i \text{MF}(P_i)$. By the definition of the product topology, there is a basic open neighborhood of $\phi^{-1}(y)$, obtained as a product $\prod_i V_i$ of open sets $V_i \subseteq \text{MF}(P_i)$, such that $V_i = \text{MF}(P_i)$ for all but finitely many $i \in I$. Moreover, in the finitely many coordinates where V_i is a proper subset of $\text{MF}(P_i)$, we can find a basic open subset $N_{r(i)} \subseteq V_i$ such that the projection of $\phi^{-1}(y)$ to coordinate i is in $N_{r(i)}$. For all i where $V_i = \text{MF}(P_i)$ we let $r(i)$ be the greatest element of P_i .

Now let f be the element of P such that $f(i) = r(i)$ for all $i \in I$. Then $y \in N_f$ (in $\text{MF}(P)$), and any $y' \in N_f$ will satisfy $\phi^{-1}(y) \in V$. Thus ϕ^{-1} is continuous. \square

COROLLARY 4.2. *Every topological product of countably many countably based MF spaces is homeomorphic to a countably based MF space.*

Proof. Under these hypotheses, the poset constructed in Theorem 4.1 is countable. \square

THEOREM 4.3. *The class of MF spaces is closed under taking G_δ subspaces.*

Proof. Suppose that $(U_i \mid i \in \mathbb{N})$ is a sequence of open subsets of $\text{MF}(P)$ and that $U = \bigcap_i U_i$ is nonempty. We form a poset Q of pairs $\langle n, p \rangle$ such that $n \in \mathbb{N}$ and $N_p \subseteq \bigcap_{i < n} U_i$, declaring $\langle n, p \rangle < \langle n', p' \rangle$ if $n > n'$ and $p \leq p'$. We define a map ϕ from $\bigcap_i U_i$ to $\text{MF}(Q)$ by sending a maximal filter F to the set of all $\langle n, p \rangle$ in Q such that $F \in N_p$. The inverse ψ of ϕ takes a maximal filter $G \in \text{MF}(Q)$ and returns the set

$$\psi(G) = \{p \in P \mid \langle n, p \rangle \in G \text{ for some } n \in \mathbb{N}\}.$$

To see that $\psi(G)$ is a filter, note that if $\langle n, p \rangle \in G$ and $\langle m, q \rangle \in G$ then there is some common extension $\langle o, r \rangle \in G$, and thus r is a common extension of p and q in $\psi(G)$.

To see that $\psi(G)$ is maximal, note that if $\bigcap \{p \mid p \in \psi(G)\}$ contained more than one point of U , then at least one of the points of the intersection would have a basic open neighborhood N_q that does not contain one other point of the intersection. It would then be possible to adjoin N_q to G and extend this to a filter, contradicting the maximality of G .

To see that ϕ is continuous, note that if $\phi(F) \in \langle n, p \rangle$ then for every $F' \in \text{MF}(P) \cap N_p$ we have $\phi(F) \in \langle n, p \rangle$. Conversely, if $\psi(G) \in N_p \cap U$ then $\psi(G) \in U_1$ and thus every $G' \in N_{(1,p)} \subseteq \text{MF}(Q)$ will have $\psi(G') \in N_p$. \square

Theorem 4.3 gives an optimal result. We will show in what follows that all poset spaces have the property of Baire. The real line is homeomorphic to a UF space, but the F_σ subset of rational numbers does not have the property of Baire and therefore is not homeomorphic to a poset space.

The class of UF spaces does not possess the closure properties enjoyed by the class of MF spaces. We now give an example showing that the class of UF spaces is not closed under finite products.

EXAMPLE 4.4. *There are two posets P, Q such that $\text{MF}(P) = \text{UF}(P)$ and $\text{MF}(Q) = \text{UF}(Q)$ yet the topological product $\text{MF}(P) \times \text{MF}(Q)$ is not homeomorphic to any UF space.*

Proof. Let ω denote the least infinite countable ordinal and let ω_1 denote the least uncountable ordinal. We define P as the set of functions from finite initial segments of ω to $\{0, 1\}$ and Q as the set of functions from countable initial segments of ω_1 to $\{0, 1\}$. For both posets, the relation \leq is given by extension: $p \leq q$ if, for all α in the domain of q , $p(\alpha)$ is defined and takes the value $q(\alpha)$.

We first show that $\text{MF}(P) = \text{UF}(P)$ and $\text{MF}(Q) = \text{UF}(Q)$. Assume that F is an unbounded filter on P (the argument for Q is parallel). Then all functions in F are compatible; that is, they do not contradict each other on any value in the intersection of their domains. There is thus a total limit function f , because otherwise there would be a first ordinal α where f is undefined and the function extending f that maps α to 0 would define an element of P that would be a lower bound for the filter F . Since f is total, all functions mapping the ordinals up to some α in the domain of f to the corresponding value of f are in the filter. One can see that this filter is already maximal, because any element outside it but still in P is incompatible with this function and adding it would destroy the filter property.

Assume now, by way of contradiction, that $\text{UF}(P) \times \text{UF}(Q)$ is homeomorphic to a space $\text{UF}(R)$. We denote by π_P and π_Q the continuous open projection maps from $\text{UF}(R)$ to its factor spaces. There is a filter F in $\text{UF}(R)$ such that $\pi_P(F)$ and $\pi_Q(F)$ are the filters generated by the set of all functions in P and Q , respectively, that map all inputs to 0. Now one can select an infinite sequence r_0, r_1, \dots in F such that, for each n : (a) the projection $\pi_P(N_{r_n})$ consists only of functions mapping the first n numbers to 0 and (b) $r_{n+1} \leq r_n$. The sequence $\langle r_i \rangle$ generates a subfilter $G \subseteq F$. There is no lower bound r for G , because otherwise $\pi_P(N_r)$ would be an open set containing some basic open set N_p such that $N_p \subseteq \pi_P(N_{r_n})$ for all n ; by construction, such a p cannot exist.

On the other hand, there is a function f contained in all the open sets $\pi_Q(N_{r_n})$ and there are basic open neighborhoods of f generated by q_0, q_1, \dots such that $N_{q_n} \subseteq \pi_Q(N_{r_n})$ for each n . The basic open sets N_{q_0}, N_{q_1}, \dots fix f only on countably many ordinals and thus their intersection is also a basic open set. Hence $\pi_Q(G)$ is bounded while $\pi_Q(F)$ is not, so $G \subset F$. It follows that $\text{UF}(R)$ is not a T_1 space—contradicting the assumption that $\text{UF}(R)$ is homeomorphic to $\text{MF}(P) \times \text{MF}(Q)$. \square

We note that the preceding example is not second countable and that the failure of second countability was important to the proof.

QUESTION 4.5. Is the class of countably based UF spaces closed under taking finite (or arbitrary) topological products?

We end the section by showing that the class of UF spaces is closed under taking G_δ subspaces. As with the class of MF spaces, this result cannot be extended to include F_σ subspaces. We first show the result for open subspaces, which has a much simpler proof.

THEOREM 4.6. *The class of UF spaces is closed under taking open subspaces.*

Proof. Let P be a poset, and let U be an open subset of $\text{UF}(P)$. Let R be the set of all $r \in P$ such that $N_r \subseteq U$; we regard R as a subset of P . Then any $x \in U$ has a neighborhood $N_r \subseteq U$, where $r \in R$. Thus the restriction map $\phi: x \mapsto x \cap R$ sends each element of U to a filter on R . Note that if this filter were not unbounded as a subset of R then it would have a lower bound in R and therefore would not be unbounded in P .

The inverse map of ϕ sends each maximal filter on R to its upward closure in P . If $\phi(G)$ were bounded below by $p \in P$, then in particular $p \leq r$ for some $r \in R$. Thus $N_p \subseteq N_r \subseteq R$, which means that $p \in R$ and $G \notin \text{UF}(R)$.

To see that ϕ and its inverse are continuous, note that

$$\{N_r^P = \{F \in \text{MF}(P) \mid r \in F\} \mid r \in R\}$$

is a basis for the restriction of $\text{MF}(P)$ to the subspace U , that

$$\{N_r^R = \{F \in \text{MF}(R) \mid r \in F\} \mid r \in R\}$$

is likewise a basis for $\text{MF}(R)$, and that a point $x \in U \subseteq \text{MF}(P)$ is in N_r^P if and only if $\phi(x)$ is in N_r^R . \square

THEOREM 4.7. *The class of UF spaces is closed under taking G_δ subspaces.*

Proof. Let G_0 be the space $\text{UF}(P)$ for some poset P with order $<_P$, and let G be a G_δ subset of G_0 . Thus there is a descending sequence G_1, G_2, \dots of open subsets of G_0 such that $G_0 \supseteq G_1 \supseteq G_2 \supseteq \dots$ and $G = \bigcap_n G_n$. Define

$$R = \{p \in P \mid p \in F \text{ for some } F \in \text{UF}(P) \cap G\}.$$

For each $p \in R$ let $g(p) = \sup\{n \in \mathbb{N} \mid N_p \subseteq G_n\}$, where $g(p) = \infty$ if $N_p \subseteq G$. Define an order relation $<_R$ on R by putting $p <_R q$ if $p <_P q$ and either $g(q) < g(p) \leq \infty$ or $g(q) = g(p) = \infty$. We will show that the unbounded filters on (R, \leq_R) are precisely the unbounded filters on P that are in G . We proceed by demonstrating four claims as follows.

Claim 1: *Let $F \in G \subseteq \text{UF}(P)$; then F is an unbounded filter in R under \leq_R .* By definition of R , $F \subseteq R$. To show that F is a filter on R , fix $p, q \in F$. If either $g(p)$ or $g(q)$ is infinite, then p and q have a common extension r under \leq_P with $g(r) = \infty$. Thus r is a common extension of p and q under \leq_R . Otherwise, because $F \in G$, there is an $r \in F$ with $r \leq_P p$, $r \leq_P q$, and $N_r \subseteq G_{g(p)+g(q)+1}$. Then $g(r) > g(p) + g(q)$, $r <_R p$, and $r <_R q$. Since \leq_R is a restriction of \leq_P , it follows that F is upward closed under \leq_R and that F is a filter in R . Furthermore, F must be unbounded in R because a bound in R would also be in a bound P .

Claim 2: *Let $F \subseteq R$ be a filter in R ; then either $\sup\{g(p) \mid p \in F\} = \infty$ or F is bounded.* Suppose the supremum is $n < \infty$ instead. There can only be one $r \in F$ with $g(r) = n$, because F is a filter on R . Since $r \in R$, there is an $F' \in \text{UF}(P)$ with $r \in F'$ and $F' \in G$. Hence there is an $r' \in F'$ with $g(r') > g(r)$ and $r' <_P r$; this means $r' <_R r$, which shows that F is bounded in R .

Claim 3: *Let F be a bounded filter of P that is also a filter in R ; then F is bounded in R .* Let $r \in P$ be a lower bound for F . If $N_r \not\subseteq G_n$ for some n , then $\sup\{g(p) \mid p \in F\} < n$ and F is bounded in R by Claim 2. Otherwise $N_r \subseteq G$, in which case $r \in R$ and F is again bounded as a subset of R .

Claim 4: *Let F be an unbounded filter in R ; then F is also an unbounded filter in P .* To see this, consider the upward closure F' of F in P . By Claim 3, F' is unbounded in P . Claim 2 shows that $F' \in G$; thus $F' \subseteq R$. By the definition of F'

we have $F \subseteq F'$. Fix $r \in F'$; then there must be a $p \in F$ with $p \leq_P r$. If $g(p) = \infty$ then $p \leq_R r$ and so $r \in F$. Otherwise there must be a $q \in F$ with $q \leq_P p$ and $g(q) > g(r)$. Then it follows from transitivity of \leq_P and the definition of $<_R$ that $q <_P r$, $q <_R r$, and $r \in F$. This shows $F' = F$.

Claims 1 and 4 show that the unbounded filters on R are exactly those unbounded filters on P that are in G . Hence, by Claim 4, the identity map $\phi: \text{UF}(P) \cap G \rightarrow \text{UF}(R)$ is surjective. This map is invertible because it is (trivially) injective. To see that ϕ and ϕ^{-1} are continuous, let $x \in \text{UF}(P) \cap G$ be fixed. For any $r \in R$ we have $r \in x$ if and only if $r \in \phi(x)$, because ϕ is the identity map on filters. Thus $\phi(x)$ is in the basic open neighborhood of $\text{UF}(R)$ determined by r if and only if x is in the basic open neighborhood of $\text{UF}(P) \cap G$ determined by r . \square

5. Completeness Properties

In this section, we establish that every poset space has the a completeness property known as the strong Choquet property. We then characterize the class of countably based MF spaces as precisely the class of second-countable T_1 spaces with the strong Choquet property. We first establish a weaker property.

THEOREM 5.1. *Every poset space has the property of Baire.*

Proof. Let X be $\text{MF}(P)$ or $\text{UF}(P)$. Suppose that $\langle U_i \mid i \in \mathbb{N} \rangle$ is a sequence of dense open subsets of the space X and that V is a fixed open set. We construct a sequence $\langle p_i \mid i \in \mathbb{N} \rangle$ of elements of P . Let p_0 be such that $N_{p_0} \subseteq V \cap U_0$. Given p_i , there is an unbounded or maximal filter in $N_{p_i} \cap U_{i+1}$. Choose p_{i+1} such that $N_{p_{i+1}} \subseteq U_{i+1} \cap N_{p_i}$ and $p_{i+1} \leq p_i$. In the end, $F = \langle p_i \rangle$ is a linearly ordered subset of P . Thus F extends to an element of X . Clearly this element is in $V \cap \bigcap_i U_i$. \square

We will now show that every poset space has the strong Choquet property, which is defined using a certain game first introduced by Choquet [1]. Let X be an arbitrary topological space. The strong Choquet game is the Gale–Stewart game (see [4; 6]) defined as follows. The stages of play are numbered $0, 1, 2, \dots$, and both players make a move in each stage. In stage i , player I plays an open set U_i and a point x_i such that $x_i \in U_i$ and if $i > 0$ then $U_i \subseteq V_{i-1}$. Next, player II plays an open set V_i such that $x_i \in V_i$ and $V_i \subseteq U_i$. At the end of the game, player I wins if $\bigcap_i U_i$ is empty (or, equivalently, if $\bigcap_i V_i$ is empty); player II wins if $\bigcap_i U_i$ is nonempty. A *position* in the game is a finite (possibly empty) sequence

$$\langle \langle U_0, x_0 \rangle, V_0, \langle U_1, x_1 \rangle, \dots \rangle,$$

which is an initial segment of an infinite play of the game following the rules just described.

A space X has the *strong Choquet property* if player II has a winning strategy for the strong Choquet game on X . A “winning strategy” is a function that takes as input a position of the game just after player I has played, and returns an open set for player II to play, such that if player II follows the strategy then player II will win the game regardless of what moves are made by player I.

The strong Choquet property is strictly stronger than the property of Baire. Moreover, the class of topological spaces with the strong Choquet property is closed under taking G_δ subspaces and arbitrary topological products. It is known that the class of topological spaces with the property of Baire is not closed under binary products (an example is provided in [3]).

THEOREM 5.2. *Every poset space has the strong Choquet property.*

Proof. We describe informally the strategy for player II. At the start of the game, player I plays an open set U_0 and a point x_0 . Player II translates the point x_0 into a filter on P before finding a basic neighborhood q_0 of x such that $N_{q_0} \subseteq U_0$. Player II then plays N_{q_0} . Now given $\langle x_1, U_1 \rangle$ with $x_1 \in N_{q_0}$, Player II translates x_1 to a filter on P before finding a neighborhood q_1 of x_1 such that $q_1 \leq_P q_0$ and $N_{q_1} \subseteq U_1$. Player II then plays N_{q_1} . Player II continues this strategy, always choosing $q_{i+1} \leq_P q_i$. At the end of the game, player II has determined $\{q_i \mid i \in \mathbb{N}\}$, a descending sequence of elements of P . This sequence extends to an element of X that is in $\bigcap N_{q_i}$, so player II has won the game. \square

We use the strong Choquet property to obtain the following characterization of countably based MF spaces.

THEOREM 5.3. *A topological space is homeomorphic to a countably based MF space if and only if it is second countable, is T_1 , and has the strong Choquet property.*

We postpone the proof of this theorem in order to comment on the hypotheses involved. Clearly, any space X that is homeomorphic to a countably based MF space must be T_1 and second countable. We have already shown that X must also have the strong Choquet property. Thus the new content of Theorem 5.3 is that the strong Choquet property is sufficient for a T_1 second-countable space to be homeomorphic to a countably based MF space. In the non-second-countable setting, the strong Choquet property is not sufficient for a T_1 space to be homeomorphic to an MF space.

EXAMPLE 5.4. *There is a Hausdorff strong Choquet space that is not homeomorphic to any MF space.*

Proof. The space X consists of certain functions from ω_1 to $\{0, 1\}$. We put a function f in X if and only if there is an ordinal $\alpha < \omega_1$ such that $f(\beta) = 0$ for all $\beta > \alpha$. For each $f \in X$ and each $\alpha < \omega_1$, the set

$$\{g \in X \mid f(\beta) = g(\beta) \text{ for all } \beta < \alpha\}$$

is declared to be an open set. The topology on X is the one generated by these open sets. It is clear that X is a Hausdorff space.

It is easy to show that X has the strong Choquet property, as follows. All that player II has to do is play any basic open subset of the open set played by player I that also contains the point given by player I. In the end, the open sets played by player I in each round will fix countably many coordinates of a function in X . In

the limit, countably many coordinates are fixed and we can find a point in the intersection of the sets played by I by forcing the remaining coordinates to map to 0.

We now show that X is not homeomorphic to any MF space. Suppose, by way of contradiction, that $X \cong \text{MF}(P)$. We construct inductively a transfinite sequence $\langle p_\alpha \mid \alpha < \omega_1 \rangle$ inductively. Let p_0 be any basic open neighborhood of the constant 0 function. Given $\langle p_\alpha \mid \alpha < \beta \rangle$, there is a first coordinate $\gamma < \omega_1$ that is not fixed by any p_α ; let f be the function that is 0 except at γ , and let $f(\gamma) = 1$. Observe that any intersection of countably many open sets in X is open. Thus we may choose $p_\beta \in P$ such that $p_\beta \leq p_\alpha$ for all $\alpha < \beta$ and $f \in N_{p_\beta}$. Choose any such p_β . At the end of this construction, $\langle p_\alpha \mid \alpha < \omega_1 \rangle$ is linearly ordered and thus extends to a maximal filter F . Now the element of X corresponding to F sends uncountably many ordinals to 1, which is impossible. \square

We now return to the proof of Theorem 5.3, which will occupy the remainder of this section. Let X be a fixed T_1 space with a fixed countable basis and a fixed winning strategy for player II in the strong Choquet game. Our first step is to define a poset P . The elements of P are called *conditions*. A condition is a finite list of the form

$$\langle A, \pi_1, \pi_2, \dots, \pi_k \rangle$$

that satisfies the following requirements.

- (1) The set A is a nonempty basic open set from the fixed countable basis. For each condition c we let $S(c)$ denote the basic open set A appearing in c .
- (2) Each π_i is a finite (i.e., partial) play of the strong Choquet game on X following the fixed winning strategy s_{II} for player II. We require each π_i to be of the form

$$\langle V_1, x_1, s_{\text{II}}(V_1, x_1), V_2, x_2, s_{\text{II}}(V_1, x_1, V_2, x_2), \dots, \\ V_r, x_r, s_{\text{II}}(V_1, x_1, V_2, x_2, \dots, V_r, x_r) \rangle.$$

Thus each π_i ends with an open set, which we will denote by $U(\pi_i)$. A play π may consist of the empty sequence $\langle \rangle$, in which case $U(\pi) = X$.

- (3) If a play π is an element of a condition then so is every initial segment of π that ends with a move by player II.
- (4) $A \subseteq U(\pi_i)$ for each $i \leq k$.

We define the order $<$ on P as follows. Let $c = \langle A, \pi_1, \pi_2, \dots, \pi_k \rangle$ and $c' = \langle A', \pi'_1, \pi'_2, \dots, \pi'_l \rangle$ be any two conditions. We let $c' < c$ if and only if the following statements hold.

- (5) For each finite play π_i in c there is a point $x_n \in S(c)$ such that the longer play

$$\pi_i \frown \langle A, x_n, s_{\text{II}}(\pi_i \frown \langle A, x_n \rangle) \rangle$$

is in c' —that is, equals π'_j for some $j \leq l$.

- (6) $A' \subseteq A$ (this is actually a consequence of requirement (5)).

Requirement (3) in our definition of a condition allows us to prove that the order on P is transitive. Because each condition is finite, requirement (5) in our definition of the order relation ensures that $c \not< c$ for all $c \in P$. Thus $<$ is a partial order on P .

LEMMA 5.5. *For any filter F on P , the intersection $\bigcap_{c \in F} S(c)$ is nonempty.*

Proof. Let $\langle A_i \mid i \in \mathbb{N} \rangle$ be an enumeration of all of the basic open sets appearing as $S(c)$ for some $c \in F$; here we use that X is second countable and that each $S(c)$ is drawn from a fixed countable basis of X . It is immediate that $\bigcap_{c \in F} S(c)$ equals $\bigcap_{i \in \mathbb{N}} A_i$. We will show that the latter intersection is nonempty.

We inductively construct a descending sequence of conditions $\langle c_i \mid i \in \mathbb{N} \rangle$ and a sequence of finite plays $\langle \pi_i \mid i \in \mathbb{N} \rangle$ such that π_{i+1} is an immediate extension of π_i for each $i \in \mathbb{N}$. At stage 0, let c_0 be any condition in F such that $S(c_0) = A_0$ and let π_0 be any finite play in c_0 . At stage $i + 1$, let c be any condition in F such that $S(c) = A_i$ and let c_{i+1} be a common extension of c and c_i in F . It is clear that $S(c_{i+1}) \subseteq S(c) = A_i$. Choose π_{i+1} to be any play in c_{i+1} that is an immediate extension of π_i .

Now suppose the entire sequence $\langle \pi_i \rangle$ has been constructed. These partial plays determine an infinite play γ of the strong Choquet game following the strategy for player II. Hence the intersection of the open sets played by player I in γ is nonempty. By construction, each set A_i has a subset played by player I at some stage of γ . Thus $\bigcap_i A_i$ is nonempty. \square

LEMMA 5.6. *Let c_1 and c_2 be two conditions, and let $x \in S(c_1) \cap S(c_2)$. Then there is a condition c such that $c < c_1$, $c < c_2$, and $x \in S(c)$.*

Proof. Begin by letting c be empty. For each π in c_1 we put the longer play

$$\pi \frown \langle S(c_1), x, s_{\text{II}}(\pi \frown \langle S(c_1), x \rangle) \rangle$$

into c ; likewise, for each π in c_2 we put

$$\pi \frown \langle S(c_2), x, s_{\text{II}}(\pi \frown \langle S(c_2), x \rangle) \rangle$$

into c . For each π that has been added to c we add all initial segments of π ending with a move by player II. We then let $S(c)$ be a basic open neighborhood of x that is a subset of the open set $\bigcap_{\pi \in c} U(\pi)$. This construction ensures that c is a condition satisfying the conclusions of the lemma. \square

LEMMA 5.7. *Let F be a maximal filter on P . Then the intersection $\bigcap_{c \in F} S(c)$ contains a single point.*

Proof. By Lemma 5.5 we know that $\bigcap_{c \in F} S(c)$ is nonempty. Suppose that x and y are distinct points in $\bigcap_{c \in F} S(c)$. Let A be a basic open neighborhood of x such that $y \notin A$. We construct a filter G inductively. At stage n we construct $G_n \subseteq P$ and in the end we let G be the upward closure of $\bigcup_n G_n$. To begin, let $G_0 = F \cup \{A, \langle \rangle\}$. At stage $i + 1$, we know by induction that $x \in S(c)$ for every $c \in G_i$.

Thus we can apply Lemma 5.6 repeatedly so that $G_i \subseteq G_{i+1}$, every pair of conditions in G_i has a common extension in G_{i+1} , and $x \in S(c)$ for every $c \in G_{i+1}$.

It is immediate from the construction that $G = \bigcup_i G_i$ is a filter that properly extends F . This shows that F was not maximal. \square

Proof of Theorem 5.3. For each $F \in \text{MF}(P)$ we denote the single point in $\bigcap_{c \in F} S(c)$ by $\phi(F)$. We show that ϕ is a homeomorphism from $\text{MF}(P)$ to X .

We first show that ϕ is an injective map. Suppose that F and F' are maximal filters on P such that $x \in \bigcap_{c \in F} S(c)$ and $x \in \bigcap_{c \in F'} S(c)$. By following a procedure similar to the proof of Lemma 5.7, we may find a filter G such that $F \subseteq G$ and $F' \subseteq G$. Thus, by maximality, we have $F = F' = G$.

Next we show that ϕ is a surjective map. Let $x \in X$ be fixed, and let $\langle A_i \mid i \in \mathbb{N} \rangle$ be a sequence of basic open sets such that $\bigcap_i A_i = \{x\}$. The existence of this sequence requires that X be T_1 and first countable. For each $i \in \mathbb{N}$ let $c_i = \langle A_i, \langle \rangle \rangle$. Following a method similar to the proof of Lemma 5.7, we can construct a filter F such that $c_i \in F$ for each $i \in \mathbb{N}$. Let G be an extension of F to a maximal filter. Now $S = \bigcap_{c \in G} S(c)$ is nonempty by Lemma 5.5 and $S \subseteq \bigcap_i A_i = \{x\}$ by construction, so $\phi(G) = x$.

It remains to show that ϕ is open and continuous. This follows from Lemma 5.6; for each $x \in X$ and each condition c , we have $c \in \phi^{-1}(x)$ if and only if $x \in S(c)$. Therefore, X is homeomorphic to $\text{MF}(P)$. By Theorem 3.2, we may find a countable subposet R of P such that X is homeomorphic to $\text{MF}(R)$. This completes the proof. \square

6. An Application to Domain Theory

In this section, we apply the characterization of countably based MF spaces to characterize those second-countable spaces with a domain representation. Our result gives a complete solution to the so-called model problem for second-countable spaces in domain theory.

A domain is a certain type of poset (to be defined shortly), and every domain is a topological space with a topology known as the Scott topology. A domain representation of a topological space X is a domain D such that X is homeomorphic to the topological space consisting of the maximal elements of D with the relative Scott topology. The history of such representations is thoroughly described by Martin [8]. It is known that every complete separable metric space has a domain representation (see Lawson [7]) and that every space with a domain representation is T_1 and has the strong Choquet property (Martin [8]). We now show that the strong Choquet property is sufficient for a T_1 second-countable space to have a domain representation.

We summarize the definitions from domain theory that we require; these definitions are explored fully in [5]. A nonempty subset I of a poset $\langle P, \preceq \rangle$ is *directed* if every pair of elements in I has an upper bound in I . A poset P is said to be a *dcpo* (for “directed-complete partial ordering”) if every directed subset of P has a least upper bound. Any dcpo D has a second order relation \ll , known as the *way*

below relation, under which $q \ll p$ if and only if, whenever $I \subseteq D$ is a directed set with $p \ll \sup I$, there is some $r \in I$ with $q \leq r$. For each $p \in D$ we put $\downarrow p = \{q \in D \mid q \ll p\}$ and $\uparrow p = \{p \in D \mid q \ll p\}$. A dcpo D is *continuous* if $\downarrow p$ is directed and the equality $p = \sup \downarrow p$ holds for every $p \in D$. A *domain* is a continuous dcpo. A subset B of a domain D is a *basis* if $B \cap \downarrow p$ is directed and also $p = \sup(B \cap \downarrow p)$ for every $p \in D$. A domain is ω -*continuous* if it has a countable basis. An element p of a dcpo is *compact* if $p \ll p$. A dcpo D is ω -*algebraic* if there is a countable basis for D consisting of compact elements. The *Scott topology* on a dcpo D is generated by the basis $\{\uparrow p \mid p \in D\}$. A *domain representation* of a space X is a homeomorphism between X and the maximal elements of a domain with the Scott topology.

THEOREM 6.1. *A topological space has a domain representation via an ω -algebraic dcpo if and only if the space is second countable, is T_1 , and has the strong Choquet property.*

Proof. It can be seen that any space with a domain representation satisfies the T_1 separation property, and a result of Martin [8] shows that any space with a domain representation has the strong Choquet property. Therefore, we need only prove that a second-countable T_1 strong Choquet space has a domain representation via an ω -algebraic dcpo. We use the following lemma, which follows easily from the definitions.

LEMMA 6.2. *Suppose that P is a countable poset. Then the set of all filters on P , ordered by inclusion, is an ω -algebraic dcpo D . The maximal filters on P are precisely the maximal elements of D , and the compact elements of D are precisely the principal filters on P . Moreover, the poset topology on $\text{MF}(P)$ corresponds exactly to the Scott topology on the maximal elements of D .*

We showed in Theorem 5.3 that any second-countable T_1 strong Choquet space is homeomorphic to $\text{MF}(P)$ for a countable poset P . It follows immediately from Lemma 6.2 that such a space also has a domain representation via an ω -algebraic dcpo. \square

The next corollary follows from the fact that any space with a domain representation is T_1 and has the strong Choquet property. Although this corollary is already known, the proof here is new.

COROLLARY 6.3. *If a second-countable space has a domain representation then it has a representation via an ω -algebraic dcpo.*

We end this section with several remarks on the relationship between domain-representable spaces and MF spaces.

A proof of Lemma 6.2 can be modified to show that the collection of all ideals on a poset (sometimes called the *ideal completion* of the poset) forms a domain whose maximal elements in the Scott topology correspond to the maximal ideals of the poset under the Stone topology. By duality, all results we have proved for

MF spaces hold also for these spaces of maximal ideals. The relationship between ideal completions and domain representations has been investigated by Martin [9].

A *Scott domain* is a domain in which every pair of elements with an upper bound has a least upper bound. Lawson [7] has shown that any space with a domain representation via a countably based Scott domain is a complete separable metric space. It can be seen that posets as constructed in Theorem 5.3 do not, in general, give Scott domains, even when the posets are constructed from formal balls in complete separable metric spaces.

The proof of Example 5.4 can be modified to obtain the following.

EXAMPLE 6.4. *There is a Hausdorff strong Choquet space that does not have a domain representation.*

7. Semi-Topogenous Orders

In this section, we prove results that give a partial solution to the question of which arbitrary (not necessarily second-countable) topological spaces are homeomorphic to MF spaces.

Suppose that a topological space X is homeomorphic to $\text{MF}(P)$, for some poset P , via a fixed homeomorphism ϕ . If each element of $p \in P$ is replaced by the corresponding open subset $\phi(N_p) \subseteq X$, then the poset order on P will determine a corresponding order relation on these subsets of X . Moreover, the collection of all these open subsets forms a basis for the topology on X . It is thus natural to ask whether the existence of a basis with a suitable order relation is sufficient for a topological space to be homeomorphic to an MF space.

Császár [2] considered many different types of orders and their connections to topology. The basic concept is that of a semi-topogenous order.

DEFINITION 7.1. *A semi-topogenous order is a binary relation \sqsubset on the power-set of a topological space X satisfying the following axioms for all $u, v, w \subseteq X$ [2, Chap. 2]:*

- $\emptyset \sqsubset \emptyset$ and $X \sqsubset X$;
- $v \sqsubset w \Rightarrow v \subseteq w$;
- $u \subseteq v \sqsubset w \Rightarrow u \sqsubset w$;
- $u \sqsubset v \subseteq w \Rightarrow u \sqsubset w$.

Császár considered only those orders linked to topology, such as the order which says that w is a neighborhood of v . It might happen that some but not all open supersets w of a given set v satisfy $v \sqsubset w$. Nevertheless, although this is not made explicit by Császár, it is quite convenient to postulate also a connection between the topology and the open spaces.

Recall that the *open kernel* of a set is the union of all its open subsets. We say that the topological space X is *generated* by the order \sqsubset if, for each $u \subseteq X$, the set $\bigcup\{o \subseteq X \mid o \sqsubset u\}$ is the open kernel of u . In this case, a set w is open if and only if it is the union of all v such that $v \sqsubset w$. It follows that if $v \sqsubset w$ then there is an open o with $v \subseteq o \subseteq w$; however, the converse of this last implication does

not always hold. Every topological space is generated by some semi-topogenous order, for one can define $v \sqsubset w$ to hold if and only if there is an open set o with $v \subseteq o \subseteq w$.

REMARK 7.2. There is a close relationship between semi-topogenous orders and the way below relation \ll on a continuous dcpo, which was discussed in Section 6. The following properties of the way below relation are obtained by dualizing the second, third, and fourth properties in the definition of a semi-topogenous order:

$$\begin{aligned} v \ll w &\implies v \leq w; \\ u \leq v \ll w &\implies u \ll w; \\ u \ll v \leq w &\implies u \ll w. \end{aligned}$$

That these are dual forms follows because points in a topological space are minimal as nonempty subsets under \subseteq but are maximal elements of a domain representing the topological space; for this reason, we write \leq for \supseteq and \ll for \sqsubset . The requirement that $\bigcup\{o \mid o \sqsubset u\}$ be the open kernel of u corresponds exactly to the fact that $\{x \mid y \ll x\}$ is the open kernel of an element y of a continuous dcpo with the Scott topology.

Thus, if a space X has a representation via a continuous dcpo D , then the dual of the way below relation on D is a semi-topogenous order (except that it is defined only on a subset of the powerset of X) that generates the topology on X . Semi-topogenous orders can be viewed as a generalization of the way below relation applicable to the case when the dcpo is the full powerset of a topological space. It appears that semi-topogenous orders are related to auxilliary relations as defined in [5], although a formal relationship seems difficult to state.

A *filter* in a topological space X is a collection of nonempty subsets that is closed under finite intersection and under superset. A filter *has an open basis* if for every w there is an open v in the filter with $v \subseteq w$. In general, there need not be a point contained in the intersections of the sets in a filter, so we are interested in a condition on filters that requires their sets to contain a common point. Our condition that a filter meets a semi-topogenous order will imply that this filter also has an open basis; a completeness condition will ensure that each filter meeting the order has a nonempty intersection.

DEFINITION 7.3. Let X be a space with a semi-topogenous order \sqsubset generating its topology. A filter U on X *meets* \sqsubset if for every $w \in U$ there is a $v \in U$ with $v \sqsubset w$. A space X is *complete for* \sqsubset if, for every filter U in X that meets \sqsubset , there is a point x with $x \in u$ for all $u \in U$.

THEOREM 7.4. Let X be a T_1 space with a semi-topogenous order \sqsubset generating its topology such that X is complete for \sqsubset . Then X is homeomorphic to an MF space.

Proof. Let P consist of the nonempty open subsets of X , and let $p \prec q$ if and only if $p \neq q$ and $p \sqsubset q$. The relation \prec is obviously transitive and antireflexive, so it makes P into a poset.

For each $x \in X$, let U_x be the set of all $p \in P$ with $\{x\} \sqsubset p$. If $p, q \in U_x$ then the open kernel u of $p \sqcap q$ contains x and thus there is an open $r \sqsubset u$ with $x \in r$. Since the open kernel of r again contains x , it follows that $\{x\} \sqsubset r$. Hence $r \in U_x$, $r \sqsubset p$, and $r \sqsubset q$. Thus U_x is a filter on P .

If V is a maximal filter on $\text{MF}(P)$, then V also meets \sqsubset . If v generates V , then v is open (by definition) and not empty. For every $x \in v$ there is a $w \sqsubset v$ with $x \in w$; by maximality, $w = v$. Therefore, $v \sqsubset v$ and every $w \subseteq X$ with $v \subseteq w$ satisfies $v \sqsubset w$ and $w \in V$. If there is no single element generating V then there is, for every $v, w \in V$, some $u \in V$ with $u \prec v$ and $u \prec w$. Then it follows that $u \sqsubset v$ and $u \sqsubset w$. Furthermore, there is a $t \prec u$ with $t \in V$; hence $t \sqsubset v \cap w$. Thus V contains all supersets of $v \cap w$ and so V is a filter. Furthermore, V meets \sqsubset .

This means, by assumption, that there is a point x contained in all sets of V . Thus $V \subseteq U_x$ and, by the maximality of V , we have $V = U_x$. As a result, every filter U_y is contained in a filter U_x that is maximal. Because of the T_1 property, $y = x$; otherwise U_y would contain a p with $x \notin p$, in contradiction to the fact that $U_y \subseteq U_x$.

This shows that the mapping $\phi: x \mapsto U_x$ is a bijection from X to the maximal filters on P . To see that ϕ is open and continuous, first note that if $y \in X$ and U is an open set, then $y \in U$ if and only if $\{y\} \sqsubset U$. To see this, fix $y \in X$ and any open U such that $y \in U$, which means $\{y\} \subseteq U$. Then, because \sqsubset generates the topology and y is trivially in the open kernel of U , there is some $W \sqsubset U$ with $y \in W$. Hence $\{y\} \subseteq W \sqsubset U$, which means that $\{y\} \sqsubset U$ by the definition of semi-topogenous orders. The converse direction of the equivalence follows directly from the definition of a semi-topogenous order.

Now, to see that ϕ is open and continuous, note that for any point $x \in X$ and any open set U we have

$$x \in U \iff \{x\} \sqsubset U \iff U \in \phi(x) \iff \phi(x) \in N_p,$$

where N_p is the basic open subset of $\text{MF}(P)$ corresponding to U . □

We do not know whether every MF space has a semi-topogenous order satisfying the hypotheses of Theorem 7.4. We have established the following partial result.

THEOREM 7.5. *If $X = \text{MF}(P)$ and if P satisfies*

$$\forall p, q, r [p \prec q \wedge N_q \subseteq N_r \Rightarrow p \prec r], \quad (*)$$

then there is a semi-topogenous order \sqsubset generating the topology of X such that X is complete for \sqsubset .

Proof. For any $v, w \subset X$, let $v \sqsubset w$ if either $v = \emptyset$, $w = X$, there is an open atom u with $v \subseteq u \subseteq w$, or there are $p, q \in P$ with $v \subseteq N_p$, $p \prec q$, and $N_q \subseteq w$. Note that $N_p \subseteq N_q$ in the last case.

It follows directly from definitions and the present assumptions that \sqsubset is a semi-topogenous order. We must show that that \sqsubset generates the topology of X . Let w be

an open set and let x be a point in w . There is an open set N_q with $\{x\} \subseteq N_q \subseteq w$. If $\{x\} = N_q$ then $N_q \sqsubset w$. If $\{x\} \neq N_q$, then there is a further $p \prec q$ with $x \in N_p$. The reason is that, given any $y \in N_q \setminus \{x\}$, the maximal filter U_x belonging to x must contain a $p \prec q$, which does not contain y by the T_1 axiom. Then $\{x\} \subseteq N_p \sqsubset w$, so w is the union of all v with $v \sqsubset w$.

Now let W be a filter in the topological space X that meets \sqsubset . If W contains an r such that N_r is atomic (i.e., a singleton $\{x\}$), then every $u \in W$ contains x because otherwise $N_r \cap u = \emptyset$, in contradiction to W being a filter.

If W does not contain an r such that N_r is atomic, then let V be the set of all $p \in P$ such that $N_p \in W$. Given any $p, q \in V$, there is a u such that $u \sqsubset N_p \cap N_q$. Hence there exist r, t with $u \subseteq N_r, N_t \subseteq N_p \cap N_q$, and $r \prec t$. It follows that $r \prec p$ and $r \prec q$. Thus V is the basis of a filter on P ; this filter is contained in a maximal filter on P that is of the form U_x for some point $x \in X$. This x is then in N_p for all $p \in V$. Let $u \in W$. Since W meets \sqsubset , there is a $p \in V$ with $N_p \subseteq u$. It follows that $x \in N_p$ and $x \in u$. Therefore, x is a common point of the sets in W . □

The posets constructed in Theorems 2.2 and 2.3 satisfy condition (*) and so are examples of a poset space that is complete for a semi-topogenous order generating its topology.

EXAMPLE 7.6. *For every complete metric space and every locally compact Hausdorff space, there exists a semi-topogenous order \sqsubset that generates the topology of X and for which X is complete.*

REMARK 7.7. Assume that X is a space that is complete for a semi-topogenous order generating its topology. Then one can show not only that X is homeomorphic to an MF space but also that the winning strategy for player II is quite easy to obtain. Given any open set u and any point $x \in u$ by player I, player II need only choose an open v with $\{x\} \subseteq v \sqsubset u$. It does not matter which v with this condition is chosen, and the history of the game can be ignored. The result of the construction will be, at the end of the game, a basis for a filter that meets \sqsubset ; hence this filter has a common point.

This shows that the “neighborhood spaces” that we consider here satisfy a restricted version of the strong Choquet property. The intuition behind this restriction is that one wishes to study non-second-countable spaces by considering “transfinite games”. The role of player I is replaced by considering filters instead of descending sequences; and the winning strategy of player II is reduced to a neighborhood relation \sqsubset , which could be interpreted as saying that if $\{x\} \subseteq v \sqsubset u$ then v is a good move for player II.

Indeed, the notion of completeness of spaces with respect to a semi-topogenous order \sqsubset is based on this idea. Let the strategy of player II be simply to follow \sqsubset , and let player I build a filter U such that for every $w \in U$ there is a $v \in U$ that player II might have chosen as a response to w (i.e., a $v \sqsubset w$). Then the intersection of all $u \in U$ is not empty.

8. Cardinality of Poset Spaces

In this section, we establish perfect set theorems for countably based Hausdorff poset spaces. The theorems show that these spaces either are countable or have the cardinality 2^{\aleph_0} of the continuum.

THEOREM 8.1. *Any countably based Hausdorff poset space either has countably many points or has cardinality 2^{\aleph_0} .*

COROLLARY 8.2. *Any countably based Hausdorff poset space either has countably many points or contains a perfect closed set.*

Proof. Any second-countable Hausdorff space of cardinality 2^{\aleph_0} contains a perfect closed set. The complement of the perfect closed set is the union of all the basic open sets from a fixed countable basis that contain fewer than 2^{\aleph_0} points. \square

To prove Theorem 8.1, we introduce a class of Gale–Stewart games. These games are inspired by the $*$ -games in descriptive set theory (as described in [6]). For each poset P we define a game that we call the *poset star game on P* . There are two players. The play proceeds in stages numbered $0, 1, 2, \dots$. At stage t , player I plays a pair $\langle p_1^t, p_2^t \rangle \in P \times P$. Then player II plays a number $n_t \in \{1, 2\}$. Player I wins the game if the following conditions hold for all t :

- $p_1^t \perp p_2^t$;
- $p_1^{t+1} \leq p_{n_t}^t$ and $p_2^{t+1} \leq p_{n_t}^t$.

Player II wins if player I does not win; there are no ties.

A *strategy* for a player is a function that tells the player what to do at any possible move of the game. The strategy is a *winning strategy* if the player will win any play of the game in which the player uses the strategy to choose every move. It is impossible for both players to have a winning strategy for the same game.

LEMMA 8.3. *Let P be a poset. Then either player I or player II has a winning strategy for the poset star game on P .*

Proof. The set of infinite plays of the poset star game on P that are winning for player I is closed in the space of all possible plays of the game. (This space is the space of infinite sequences of moves; the set of moves is assigned the discrete topology and the space of infinite plays carries the product topology.) The proof follows from a theorem of Gale and Stewart known as closed determinacy. \square

LEMMA 8.4. *Suppose that X is a Hausdorff poset space based on a countable poset P and that player I has a winning strategy for the poset star game on P . Then X has cardinality 2^{\aleph_0} .*

Proof. It suffices to prove the result for $\text{MF}(P)$, which is a subset of $\text{UF}(P)$. Let s_1 be a winning strategy for player I and let $f \in \{1, 2\}^{\mathbb{N}}$. Consider the play in which player I follows s_1 while player II uses f as a guide; that is, player II plays $f(n)$ at stage $2n$. Because s_1 is a winning strategy for player I, this play determines a

descending sequence $F(f)$ of elements of P . This sequence extends to a maximal filter. For distinct $f, g \in \{0, 1\}^{\mathbb{N}}$, the sequences $F(f)$ and $F(g)$ contain incompatible elements and thus cannot extend to the same filter. Hence the space $\text{MF}(P)$ has cardinality 2^{\aleph_0} . \square

LEMMA 8.5. *Let X be a countably based Hausdorff poset space based on the poset P . If player II has a winning strategy for the poset star game on P , then X is countable.*

Proof. Let s_{II} be a winning strategy for player II. We say that a finite play σ of length $2k$ is *compatible* with s_{II} if $s_{\text{II}}(\sigma[2i + 1]) = \sigma(2i + 2)$ whenever $2i + 2 \leq k$. We say that a play σ of even length is a *good play* for a point x if σ is compatible with s_{II} and if x is in the open set chosen by player II in the last move of σ . A good play for x is a *maximal play* if it cannot be extended to a longer good play for x ; this means that, no matter what pair of disjoint open sets player I plays, s_{II} will direct player II to choose an open set not containing x .

If player II has a winning strategy then every point x has a maximal play. Observe that the empty play is trivially a good play for x . If every good play for x could be extended to a larger good play for x , then it would be possible for player I to win the game by always leaving the game in a position that is good for x . This play of the game would follow s_{II} , a winning strategy for player II, which is a contradiction.

If σ is a good play for two points x and y , then σ is not a maximal play for both x and y . This follows because player I could play (U_1, U_2) in response to σ , where $x \in U_1$, $y \in U_2$, and $U_1 \cap U_2 = \emptyset$. Here we assume that the topology of X is Hausdorff.

We have now shown that every point in the space X has a maximal play and that no play is maximal for two points. Since the set of maximal plays is countable, this implies that the set of points in X is countable. \square

We remark that the statement ‘‘Every closed subset of a countably based Hausdorff MF space either is countable or has a perfect closed subset’’ is independent of ZFC set theory; this result is established in [11].

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C. Mummert
 Department of Mathematics
 Marshall University
 Huntington, WV 25755
 mummertc@marshall.edu

F. Stephan
 Department of Mathematics and
 School of Computing
 National University of Singapore
 Singapore 119076
 Republic of Singapore
 fstephan@comp.nus.edu.sg