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On the Equation $\tau(\lambda(n)) = \omega(n) + k$

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1. Introduction

For every positive integer n, the function $\tau(n)$ counts the number of divisors of n, the function $\omega(n)$ counts the number of distinct prime divisors of n, and the Carmichael function $\lambda(n)$ is the exponent of the multiplicative group of the invertible congruence classes modulo n. The value of the function $\lambda(n)$ can be computed as follows:

$$\lambda(n) = \begin{cases} 1 & \text{if } n = 1; \\ 2^{\alpha-2} & \text{if } n = 2^\alpha,\ \alpha > 2; \\ p^{\alpha-1}(p-1) & \text{if } n = p^\alpha \text{ and } \begin{array}{l} p \geq 3 \text{ or} \\ p = 2,\ \alpha \leq 2; \end{array} \\ [\lambda(p_1^{\alpha_1}), \dots, \lambda(p_s^{\alpha_s})] & \text{if } n = p_1^{\alpha_1} \cdots p_s^{\alpha_s}. \end{cases}$$

In [6], Erdős, Pomerance, and Schmutz proved a number of fundamental properties of λ. In the process of proving the lower bound $\lambda(n) > (\log n)^{c_0 \log\log\log n}$ for all large n (provided $c_0 < 1/\log 2$), they proved the inequality

$$n \leq (4\lambda(n))^{3\tau(\lambda(n))}.$$

Numerical calculations suggest that the stronger inequality

$$n \leq \lambda(n)^{\tau(\lambda(n))} \tag{1}$$

holds except for $n = 2, 6, 8, 12, 24, 80, 120, 240$. This will be proved in Corollary 1. One of the tools for proving (1) is the inequality $\tau(\lambda(n)) > \omega(n)$, which holds except for $n = 2, 6, 12, 24, 30, 60, 120, 240$; we will prove this in Proposition 1 and Proposition 2.

This motivates us to compare $\tau(\lambda(n))$ with $\omega(n)$. Since $\tau(\lambda(n)) \geq \omega(n)$ holds for all positive integers n (see Proposition 1), we can write $\tau(\lambda(n)) = \omega(n) + k$, where k is some nonnegative integer depending on n. We then fix $k \geq 0$ and investigate the positive integers n such that $\tau(\lambda(n)) = \omega(n) + k$.

Throughout this paper, we use x to denote a positive real number. We also use the Landau symbols O and o and the Vinogradov symbols $\gg$ and $\ll$ with their

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usual meanings. We write $\log x$ for the maximum between 1 and the natural logarithm of x. For a set $\mathcal{A}$ of positive integers we write $\mathcal{A}(x) = \mathcal{A} \cap [1, x]$. We write p and q with or without subscripts for prime numbers.

Let us set

$$\mathcal{A}_k = \{n : \tau(\lambda(n)) = \omega(n) + k\}.$$

We will show in Theorem 3 that if k is a positive integer and

$$b_k = 2(k+1)^2 + 3 + \lfloor \log_2(2(k+1)^2 + k + 1) \rfloor,$$

then the upper bound

$$\#\mathcal{A}_k(x) \ll_k \frac{x(\log\log x)^{b_k}}{(\log x)^2}$$

holds as $x \to \infty$. Here, $\log_2 a$ stands for the base 2 logarithm of the positive number a. Furthermore, in Theorem 2, we will show that if $k > 4$ then the lower bound

$$\#\mathcal{A}_k(x) \gg_k \frac{x}{(\log x)^2}$$

holds as $x \to \infty$. We will also give complete descriptions of the sets $\mathcal{A}_0$, $\mathcal{A}_1$, and $\mathcal{A}_2$ (Proposition 2, Proposition 3, and Proposition 4). We will show that $\mathcal{A}_0$ contains eight integers and that the infiniteness of $\mathcal{A}_1$ and $\mathcal{A}_2$ would follow if it were known that there exist infinitely many primes of the form $2q+1$ with q also prime. Finally, in Proposition 5 we deal with the cases $k = 3, 4$ and prove that, if either $\mathcal{A}_3$ or $\mathcal{A}_4$ are infinite, then there exists an even positive integer c such that the set of primes of the form $p = cq^\beta + 1$ (with q prime and $\beta \le 4$) is infinite. This explains the difficulty of proving the infiniteness of $\mathcal{A}_k$ for $k = 1, 2, 3, 4$.

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2. Determining $\mathcal{A}_k$ for Small Values of k

PROPOSITION 1. *For any positive integer n, we have*

$$\tau(\lambda(n)) \ge \omega(n).$$

More precisely,

$$\tau(\lambda(n)) \ge \omega(n/(2^\infty, n)) + \tau(\lambda^o(n')),$$

where n' is the product of the primes dividing n and where $\lambda^o(m)$ denotes the odd part of $\lambda(m)$. That is, $\lambda^o(m) = \lambda(m)/(2^\infty, \lambda(m))$.

Proof. Let us first note that, if $n \mid m$, then $\lambda(n) \mid \lambda(m)$ and therefore $\tau(\lambda(n)) \le \tau(\lambda(m))$. Thus, we can assume that n is square-free (indeed, if n' is the product of the distinct primes dividing n, then $\omega(n) = \omega(n')$ and $\tau(\lambda(n)) \ge \tau(\lambda(n')))$.

Suppose that n is odd and $n = p_1 p_2 \cdots p_r$, where $p_1 < \cdots < p_r$ are primes. Let $2 < q_2 < \cdots < q_s$ be all the odd prime factors of $\lambda(n)$ and write

$$\begin{aligned} p_1 - 1 &= 2^{\alpha_{11}} q_2^{\alpha_{12}} \cdots q_s^{\alpha_{1s}}, \\ p_2 - 1 &= 2^{\alpha_{21}} q_2^{\alpha_{22}} \cdots q_s^{\alpha_{2s}}, \\ &\vdots \\ p_r - 1 &= 2^{\alpha_{r1}} q_2^{\alpha_{r2}} \cdots q_s^{\alpha_{rs}}. \end{aligned}$$

If $A_i = \max\{\alpha_{1i}, \dots, \alpha_{ri}\}$ for $i = 1, \dots, s$, then

$$\tau(\lambda(n)) = \tau([p_1 - 1, \dots, p_r - 1]) = (A_1 + 1)(A_2 + 1) \cdots (A_s + 1).$$

Consider now the matrix

$$\begin{pmatrix} \alpha_{11} & \dots & \alpha_{1s} \\ \vdots & & \vdots \\ \alpha_{r1} & \dots & \alpha_{rs} \end{pmatrix}.$$

We know that the entries of the matrix consist of nonnegative integers. The elements in the first column are positive and less than or equal to A_1. For each $i = 1, \dots, r$, the elements of the ith column are nonnegative integers less than or equal to A_i.

Furthermore, for each fixed natural number s, the number of rows r is less than or equal to the maximum number of distinct s-tuples $(a_1, \dots, a_s)$ with $a_1 \in [1, A_1]$ and $a_i \in [0, A_i]$ for $i = 2, \dots, s$. This follows because $\left(2^{\alpha_{i1}} \prod_{j=2}^{s} q_j^{\alpha_{ij}}\right)_{i=1,\dots,s}$ are distinct positive integers. Hence,

$$r \le A_1(A_2 + 1) \cdots (A_s + 1).$$

From the foregoing discussion we deduce that

$$\begin{aligned} \tau(\lambda(n)) &= (A_1 + 1)(A_2 + 1) \cdots (A_s + 1) \\ &\ge r + \tau(\lambda^o(n)) = \omega(n) + \tau(\lambda^o(n)), \end{aligned}$$

where $\lambda^o(n) = \lambda(n)/(2^\infty, \lambda(n))$ is the largest odd divisor of $\lambda(n)$. As a result, if n is square-free and odd then

$$\tau(\lambda(n)) \ge \omega(n) + 1,$$

but if n is square-free and even then

$$\tau(\lambda(n)) = \tau(\lambda(n/2)) \ge \omega(n/2) + 1 = \omega(n);$$

this concludes the proof. □

Lemma 1 is the main tool we use to determine the set $\mathcal{A}_k$ for $k \le 2$.

Proposition 2. $\mathcal{A}_0 = \{2, 6, 12, 24, 30, 60, 120, 240\}$.

Proof. Let $n \in \mathcal{A}_0$. Applying Lemma 1, we obtain that if n is odd then $\tau(\lambda(n)) > \omega(n)$, which is impossible.

If n is even then, by Lemma 1, the condition $\tau(\lambda(n)) = \omega(n)$ implies that

$$\tau(\lambda^o(n')) = 1.$$

This is possible only if $\lambda(n') = 2^\alpha$ for some $\alpha \in \mathbb{N}$. If $n = 2^\gamma$ and $\tau(\lambda(2^\gamma)) = 1$, then $\gamma = 1$ and so $n = 2$.

Assume now that n is not a power of 2 and write

$$n = 2^{\gamma_0}(2^{2^{\alpha_1}} + 1)^{\gamma_1} \cdots (2^{2^{\alpha_r}} + 1)^{\gamma_r},$$

where $\gamma_j \geq 1$ ($j = 0, \dots, r$), $0 \leq \alpha_1 < \cdots < \alpha_r$, and the numbers $2^{2^{\alpha_i}} + 1$ are primes for each $i = 1, \dots, r$. Plugging our expression for n into the identity $\tau(\lambda(n)) = \omega(n)$ yields

$$\max\{\tau(\lambda(2^{\gamma_0})), 2^{\alpha_r} + 1\} \cdot \gamma_1 \cdots \gamma_r = r + 1,$$

which is satisfied only for $r = 1$ or $r = 2$ (because we can now gather that $r + 1 \geq 2^{\alpha_r} + 1 \geq 2^{r-1} + 1$).

If $r = 2$ then necessarily $\alpha_2 = 1$. This forces $\alpha_1 = 0$, $\gamma_1 = \gamma_2 = 1$, and $1 \leq \gamma_0 \leq 4$, which correspond to the four values for n of 30, 60, 120, and 240. Finally, if $r = 1$ then $\alpha_1 = 0$; this forces $\gamma_1 = 1$ and $1 \leq \gamma_0 \leq 3$, which correspond to the three values for n of 6, 12, and 24. □

We are now ready to prove the motivating inequality (1).

COROLLARY 1. *Let φ denote the Euler function. Excepting only $n = 2, 6, 8, 12, 24, 80, 120, 240$, we have*

$$n \leq \lambda(n)^{\tau(\lambda(n))}.$$

Furthermore, $\varphi(n) \leq \lambda(n)^{\tau(\lambda(n))}$ except for $n = 24$. Finally, the inequality $\varphi(n) \leq \lambda(n)^{\omega(n)}$ holds unless n is a power of 2 times a product of distinct Fermat primes.

Proof. Let $v_p(m)$ be the exponent of the prime p in the factorization of the positive integer m. We know that $\lambda(n)$ divides $\varphi(n)$. We also know that if p odd then

$$\begin{aligned} v_p(\varphi(n)) &= \sum_{l^\beta \| n} v_p(l^{\beta-1}(l-1)) \\ &\leq \omega(n)\Big(\max_{l^\beta \| n}\{v_p(l^{\beta-1}(l-1))\}\Big) \leq v_p(\lambda(n)^{\omega(n)}), \end{aligned}$$

while $v_2(\varphi(n)) = v_2(n) - 1 + \sum_{l|n} v_2(l-1) \leq 1 + \omega(n)v_2(\lambda(n))$.

Necessarily, then, $\varphi(n) \mid 2\lambda(n)^{\omega(n)}$. Furthermore, the only circumstances in which $\varphi(n) = 2\lambda(n)^{\omega(n)}$ is when $\varphi(n)$ is a power of 2. If this happens, then n is necessarily a power of 2 times a product of distinct Fermat primes. In all other cases we have $\varphi(n) \leq \lambda(n)^{\omega(n)}$, and this proves the third inequality.

In order to prove the second, it is enough to notice that $\tau(\lambda(n)) \geq \omega(n)$ by Proposition 1; hence we need only show that $\varphi(n) \leq \lambda(n)^{\tau(\lambda(n))}$ when $\varphi(n) = 2^a$ and $n \neq 24$. Observe that the latter is certainly true when n is a power of 2, since $\varphi(2^\alpha) = 2^{\alpha-1} \leq 2^{(\alpha-2)(\alpha-1)} = \lambda(2^\alpha)^{\tau(\lambda(2^\alpha))}$ for $\alpha > 2$. In the other cases, if we write

$$n = 2^{\alpha_0} \cdot (2^{2^{\alpha_1}} + 1) \cdots (2^{2^{\alpha_r}} + 1)$$

with $\alpha_1 < \cdots < \alpha_r$, then

$$\begin{aligned}\varphi(n) &= 2^{2^{\alpha_1}+\cdots+2^{\alpha_r}+\max\{\alpha_0-1,0\}} \\ &\leq 2^{2^{\alpha_r}(1+1/2+\cdots+1/2^{r-1})+\max\{\alpha_0-1,0\}} \leq 2^{3M+1},\end{aligned}$$

where $M = \max\{\log_2(\lambda(2^{\alpha_0}), 2^{\alpha_r}\}$. Similarly,

$$\lambda(n)^{\tau(\lambda(n))} = 2^{M(M+1)}.$$

Finally, $3M + 1 \leq M(M+1)$ for $M > 2$ while for $M \leq 2$ we have $r \leq 2$ and so $n \in \{3, 6, 12, 24, 48, 5, 10, 20, 40, 80, 15, 30, 60, 120, 240\}$; the only value of n from this set that does not satisfy the inequality $\varphi(n) \leq \lambda(n)^{\tau(\lambda(n))}$ is $n = 24$. This completes the proof of the second statement.

Observe that for $n \in \mathcal{A}_0$ the first statement holds if and only if $n \in \{30, 60\}$. So we can assume that $n \notin \mathcal{A}_0$ and thus $\tau(\lambda(n)) \geq \omega(n) + 1$. This implies that

$$\lambda(n)^{\tau(\lambda(n))} \geq \lambda(n)\varphi(n)$$

unless $\varphi(n)$ is a power of 2. In order to conclude the proof we must verify that the statement holds when $\varphi(n)$ is a power of 2 and $n \neq 2, 8$, and we must also show that

$$\lambda(n)\varphi(n) \geq n.$$

We claim that this inequality holds unless $n \in \{2, 3, 6, 12, 24\}$ (values for which the statement is verified directly). Indeed, let p be the greatest prime divisor of n. If $p \geq 5$, then

$$\frac{n}{\varphi(n)} = \prod_{l \mid n} \frac{l}{l-1} \leq \frac{3}{4}p \leq p - 1 \leq \lambda(n).$$

Similarly, if $p = 3$, then $n/\varphi(n) \leq 3 \leq \lambda(n)$ unless $n \in \{3, 6, 12, 24\}$. Finally, if $p = 2$, then $n/\varphi(n) = 2 \leq \lambda(n)$ unless $n = 2$.

If $\varphi(n)$ is a power of 2, we proceed as in the proof of the second inequality. Observe that if $n = 2^{\alpha_0}$ then $n \leq \lambda(n)^{\tau(\lambda(n))}$ unless $\alpha_0 = 1, 3$. If $n = 2^{\alpha_0} \cdot (2^{2^{\alpha_1}}+1) \cdots (2^{2^{\alpha_r}}+1)$ with $\alpha_1 < \cdots < \alpha_r$ and if $M = \max\{\log_2(\lambda(2^{\alpha_0}), 2^{\alpha_r}\}$ so that $2^{M(M+1)} = \lambda(n)^{\tau(\lambda(n))}$, then

$$n \leq 2^{2(2^{\alpha_1}+\cdots+2^{\alpha_r})+\alpha_0} \leq 2^{5M+2}.$$

Since $5M + 2 \leq M(M+1)$ for $M > 5$, we are left with checking the statement for integers that divide $2^7 \cdot 3 \cdot 5 \cdot 17$, and this is done by a short calculation. □

PROPOSITION 3.

$$\begin{aligned}\mathcal{A}_1 = \{&1, 3, 4, 8, 10, 15, 20, 40, 48, 80, 126, 252, 480, 504, \\ &510, 1020, 2040, 2730, 4080, 5460, 8160, 8190, 10920, \\ &16320, 16380, 21840, 32760, 65520, 6q, 12q, 24q\},\end{aligned}$$

where $q = 2p + 1$ *is prime with* $p > 2$ *also prime.*

Proof. We follow the same method as in the proof of Proposition 2.

If $n > 1$ is odd then, by Lemma 1, $\lambda^o(n') = 1$. This implies that $\lambda(n') = 2^\alpha$ for some $\alpha \geq 0$. Thus,

$$n = (2^{2^{\alpha_1}} + 1)^{\gamma_1} \cdots (2^{2^{\alpha_r}} + 1)^{\gamma_r},$$

where $\gamma_j \geq 1$ ($j = 1, \dots, r$), $0 \leq \alpha_1 < \cdots < \alpha_r$, and again $2^{2^{\alpha_i}} + 1$ is prime for $i = 1, \dots, r$.

The equation $\tau(\lambda(n)) = \omega(n) + 1$ is equivalent to

$$(2^{\alpha_r} + 1)\gamma_1 \cdots \gamma_r = r + 1.$$

Since $\alpha_r \geq r - 1$, the preceding equality is satisfied only if $r = 1$ or $r = 2$. If $r = 1$ then necessarily $\alpha_1 = 0$ and $\gamma_1 = 1$, so $n = 3$. If $r = 2$ then we have $\alpha_1 = 0$, $\alpha_2 = 1$, and $\gamma_1 = \gamma_2 = 1$, so $n = 15$.

Assume now that n is even. If $n = 2^{\gamma}$, then $\tau(\lambda(n)) = 2$ is satisfied only for $n = 4$ or $n = 8$.

If n is not a power of 2, then Lemma 1 yields $\tau(\lambda^o(n')) \leq 2$. This can happen only if $\lambda(n') = 2^a$ or $\lambda(n') = 2^a p$ with p an odd prime. If $\lambda(n') = 2^a$ then

$$n = 2^{\gamma_0} \cdot (2^{2^{\alpha_1}} + 1)^{\gamma_1} \cdots (2^{2^{\alpha_r}} + 1)^{\gamma_r},$$

where $\gamma_j \geq 1$ ($j = 0, \dots, r$), $0 \leq \alpha_1 < \cdots < \alpha_r$, and again $2^{2^{\alpha_i}} + 1$ is prime for $i = 1, \dots, r$.

If we plug the preceding expression for n into the identity $\tau(\lambda(n)) = \omega(n) + 1$, we obtain

$$\max\{\tau(\lambda(2^{\gamma_0})), 2^{\alpha_r} + 1\} \cdot \gamma_1 \cdots \gamma_r = r + 2,$$

which can only be satisfied for $r \leq 3$ because $r + 2 \geq 2^{\alpha_r} + 1 \geq 2^{r-1} + 1$. A quick computation shows that $\gamma_j = 1$ for all $j \geq 1$, and we have only the following possibilities:

r	$(\alpha_1, \dots, \alpha_r)$	n
1	(0)	48
	(1)	10, 20, 40, 80
2	—	—
3	(0, 1, 2)	510, 1020, 2040, 4080, 8160, 16320

The next case to consider is when $\lambda(n') = 2^a p$, so that each odd prime dividing n is of the form $2^{2^{\alpha}} + 1$ or of the form $2^{\beta} p + 1$. Hence,

$$n = 2^{\gamma_0} \cdot (2^{2^{\alpha_1}} + 1)^{\gamma_1} \cdots (2^{2^{\alpha_r}} + 1)^{\gamma_r} \cdot (2^{\beta_1} p + 1)^{\gamma_{r+1}} \cdots (2^{\beta_s} p + 1)^{\gamma_{r+s}},$$

where $\gamma_j \geq 1$ ($j = 0, \dots, r + s$), $0 \leq \alpha_1 < \cdots < \alpha_r$, $2^{2^{\alpha_i}} + 1$ is prime for $i = 1, \dots, r$, $1 < \beta_1 < \cdots < \beta_s$, and $2^{\beta_k} p + 1$ is prime for $k = 1, \dots, s$.

We now distinguish two more subcases: $p^2 \mid n$ and $p^2 \nmid n$. If $p^2 \mid n$, then the equation $\tau(\lambda(n)) = \omega(n) + 1$ translates into

$$\max\{\tau(\lambda(2^{\gamma_0})), 2^{\alpha_r} + 1, \beta_s + 1\} \cdot \gamma_1 \cdots \gamma_{r+s} = r + s + 2. \tag{2}$$

In this case, there exists a $j \leq r$ such that $\gamma_j \geq 2$; since $\max\{a, b\} \geq (a + b)/2$, the LHS of (2) is greater than or equal to $2^{\alpha_r} + 1 + \beta_s + 1$. Using that $\alpha_r \geq r - 1$

and $\beta_s \geq s$, we once again obtain $2^{r-1} + 1 \leq r + 1$, which implies that $r = 1$ or $r = 2$.

If $r = 1$, then necessarily $\alpha_1 = 0$, $\gamma_1 = 2$, $s = 1$, and $\beta_1 = \gamma_2 = 1$. This implies that $n = 2^{\gamma_0} \cdot 3^2 \cdot 7$ and $\gamma_0 = 1, 2, 3$. If $r = 2$, then necessarily $\alpha_1 = 0, \alpha_2 = 1$, and $s \leq 2$ (since the LHS of (2) is greater than or equal to $2s + 2$). Checking all possibilities, we find that $n = 2^{\gamma_0} \cdot 3^2 \cdot 5 \cdot 7 \cdot 13$ and $\gamma_0 = 1, 2, 3, 4$.

For the other subcase, if $p^2 \nmid n$ then the equation $\tau(\lambda(n)) = \omega(n) + 1$ translates into

$$2 \cdot \max\{\tau(\lambda(2^{\gamma_0})), 2^{\alpha_r} + 1, \beta_s + 1\} \cdot \gamma_1 \cdots \gamma_{r+s} = r + s + 2. \tag{3}$$

For the same reasons as before, it follows that $r = 1$ or $r = 2$ and $s = 1$ or $s = 2$.

If $r = s = 1$ then we have the family of solutions $n = 2^{\gamma_0} \cdot 3 \cdot (2p + 1)$, where $\gamma_0 = 1, 2, 3$ and $2p + 1$ is prime with $p \geq 3$. If $r = s = 2$ then we have the solutions $n = 2^{\gamma_0} \cdot 3 \cdot 5 \cdot 7 \cdot 13$, where $\gamma_0 = 1, 2, 3, 4$. The remaining cases ($r = 1$, $s = 2$; $r = 2$, $s = 1$) produce for the RHS of (3) a value equal to 5 and so do not lead to any more solutions. $\square$

PROPOSITION 4. *We have that* $\mathcal{A}_2 = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3 \cup \mathcal{F}_4 \cup \mathcal{I}_1 \cup \mathcal{I}_2 \cup \mathcal{I}_3$, *where:*

$$\mathcal{F}_1 = \left\{ \begin{array}{l|l} 5, 2^4, 2^5 \cdot 3, 2^5 \cdot 5, 2^\beta \cdot 3^2, 2^6 \cdot 3 \cdot 5, & 1 \leq \alpha \leq 6, \\ 2^\alpha \cdot 3 \cdot 17,\ 2^\alpha \cdot 5 \cdot 17,\ 3 \cdot 5 \cdot 17, & 1 \leq \beta \leq 3 \\ 2^7 \cdot 3 \cdot 5 \cdot 17 & \end{array} \right\},$$

$$\mathcal{F}_2 = \left\{ \begin{array}{l|l} 2^\alpha \cdot 3^\beta \cdot 5 \cdot 7, 3^\beta \cdot 7, 3^\beta \cdot 5 \cdot 7 \cdot 13 & 1 \leq \alpha \leq 4, \\ 2^\alpha \cdot 3^\beta \cdot 5 \cdot 13, 2^\alpha \cdot 3^\beta \cdot 7 \cdot 13, & \beta = 1, 2 \\ 2^\alpha \cdot 5 \cdot 7 \cdot 13 & \end{array} \right\},$$

$$\mathcal{F}_3 = \{2^\alpha \cdot 3 \cdot 5^2 \cdot 11 \mid 1 \leq \alpha \leq 4\},$$

$$\mathcal{F}_4 = \left\{ \begin{array}{l|l} 2^\delta \cdot 3^\beta \cdot 7 \cdot 19, & 1 \leq \alpha \leq 4, \\ 2^\alpha \cdot 3^\beta \cdot 5 \cdot 7 \cdot 13 \cdot 19 \cdot 37 & 1 \leq \beta, \delta \leq 3 \end{array} \right\};$$

$$\mathcal{I}_1 = \{2^\alpha \cdot (2p + 1) \mid 2p + 1,\ p \geq 3 \textit{ primes},\ 1 \leq \alpha \leq 3\},$$

$$\mathcal{I}_2 = \{3 \cdot (2p + 1) \mid 2p + 1,\ p \geq 3 \textit{ primes}\},$$

$$\mathcal{I}_3 = \left\{ 2^\alpha \cdot 3 \cdot 5 \cdot (2^\beta p + 1) \;\middle|\; \begin{array}{l} 2^\beta p + 1,\ p \geq 3 \textit{ primes}, \\ 1 \leq \alpha \leq 4,\ \beta = 1, 2 \end{array} \right\}.$$

Proof. We follow the same approach as in the previous results and obtain that, in order for n to satisfy $\tau(\lambda(n)) = \omega(n) + 2$, we must have $\lambda(n') = 2^\alpha p^\beta$ with $\alpha \geq 0$ and $\beta = 0, 1, 2$. This implies that n should be of the form

$$n = 2^{\gamma_0} \cdot A \cdot B \cdot C,$$

where A, B, and C are either 1 or of the respective forms

$$A = (2^{2^{\alpha_1}} + 1)^{\gamma_1} \cdots (2^{2^{\alpha_r}} + 1)^{\gamma_r},$$

$$B = (2^{\beta_1} p + 1)^{\gamma_{r+1}} \cdots (2^{\beta_s} p + 1)^{\gamma_{r+s}},$$

$$C = (2^{\delta_1} p^2 + 1)^{\gamma_{r+s+1}} \cdots (2^{\delta_t} p^2 + 1)^{\gamma_{r+s+t}}.$$

Here we assume the following conditions: $\gamma_j \geq 1$ for $j = 0, \dots, r+s+t$, $0 \leq \alpha_1 < \cdots < \alpha_r$, $2^{2^{\alpha_i}} + 1$ is prime for $i = 1, \dots, r$, $1 < \beta_1 < \cdots < \beta_s$, $2^{\beta_k} p + 1$ is prime for $k = 1, \dots, s$, $1 < \delta_1 < \cdots < \delta_t$, and $2^{\delta_l} p^2 + 1$ is prime for $l = 1, \dots, t$. We allow any one of r, s, t, γ_0 to be zero with the obvious meaning.

The equation $\tau(\lambda(n)) = \omega(n) + 2$ is equivalent to

$$\Theta \cdot \Lambda \cdot \gamma_1 \cdots \gamma_{r+s+t} = r + s + t + \min\{1, \gamma_0\} + 2, \tag{4}$$

where:

$$\Theta = \begin{cases} 1 & \text{if } (s+t > 0 \text{ and } p^3 \mid n) \text{ or } (s+t=0) \\ & \text{or } (t = 0, s > 0 \text{ and } p^2 \parallel n); \\ 3/2 & \text{if } t > 0 \text{ and } p^2 \parallel n; \\ 2 & \text{if } t = 0,\ s > 0, \text{ and } p^2 \nmid n; \\ 3 & \text{if } t > 0 \text{ and } p^2 \nmid n; \end{cases}$$

and $\Lambda = \max\{\tau(\lambda(2^{\gamma_0})), 2^{\alpha_r} + 1, \beta_s + 1, \delta_t + 1\}$. The terms $\beta_s + 1$ (resp. $\delta_t + 1$) should be omitted if $s = 0$ (resp. $t = 0$).

If $s = t = 0$, then these remarks imply that $r \leq 3$ and

$$n = 2^{\delta_0} \cdot 3^{\delta_1} \cdot 5^{\delta_2} \cdot 17^{\delta_3}.$$

In this case, all possible solutions of $\tau(\lambda(n)) = \omega(n) + 2$ are

r	$(\delta_0, \delta_1, \delta_2, \delta_3)$	n
0	$(4, 0, 0, 0)$	2^4
1	$(0, 0, 1, 0)$	5
	$(\delta, 2, 0, 0)$, $\delta = 1, 2, 3$	$2 \cdot 3^2, 2^2 \cdot 3^2, 2^3 \cdot 3^2$
	$(5, 1, 0, 0)$	$2^5 \cdot 3$
	$(5, 0, 1, 0)$	$2^5 \cdot 5$
2	$(6, 1, 1, 0)$	$2^6 \cdot 3 \cdot 5$
	$(\delta, 1, 0, 1)$, $1 \leq \delta \leq 6$	$2^\delta \cdot 3 \cdot 17$, $1 \leq \delta \leq 6$
	$(\delta, 0, 1, 1)$, $1 \leq \delta \leq 6$	$2^\delta \cdot 5 \cdot 17$, $1 \leq \delta \leq 6$
3	$(0, 1, 1, 1)$	$3 \cdot 5 \cdot 17$
	$(7, 1, 1, 1)$	$2^7 \cdot 3 \cdot 5 \cdot 17$

These solutions are exactly the 22 elements of $\mathcal{F}_1$.

When $t = 0$ and $s \neq 0$, equation (4) simplifies to

$$\Theta \cdot \Lambda \cdot \gamma_1 \cdots \gamma_{r+s} = r + s + \min\{1, \gamma_0\} + 2, \tag{5}$$

where:

$$\Theta = \begin{cases} 1 & \text{if } p^2 \mid n, \\ 2 & \text{if } p^2 \nmid n; \end{cases}$$

$\Lambda = \max\{\tau(\lambda(2^{\gamma_0})), 2^{\alpha_r} + 1, \beta_s + 1\}$, and the middle term is omitted if $r = 0$. In such a case, we have $p \nmid n$ and $s \leq \beta_s \leq (s + \min\{1, \gamma_0\})/2$. This is possible only for n even and $s = \beta_s = 1$. This implies that $n = 2^{\gamma_0}(2p+1)$ with $\gamma_0 = 1, 2, 3$, which are exactly the elements of $\mathcal{I}_1$.

If $r > 0$ then the LHS of (5) is greater than or equal to $2^{\alpha_r} + \beta_s + 2$, which implies that $2^{\alpha_r} \leq r + \min\{1, \gamma_0\}$. From this inequality it follows that $r \leq 2 + \min\{1, \gamma_0\}$.

We distinguish the two subcases $p = 3$ and $p > 3$. In the first subcase, $s \leq r + \min\{1, \gamma_0\}$ and $\beta_s \leq (r + s + \min\{1, \gamma_0\})/2$. This implies that

$$n = 2^{\delta_0} \cdot 3^{\delta_1} \cdot 5^{\delta_2} \cdot 7^{\delta_3} \cdot 13^{\delta_4}.$$

In this subcase, all possible solutions of $\tau(\lambda(n)) = \omega(n) + 2$ are

(r, s)	$(\delta_0, \delta_1, \delta_2, \delta_3, \delta_4)$	n
$(1, 1)$	$(0, \delta, 0, 1, 0)$, $\delta = 1, 2$	$3 \cdot 7, 3^2 \cdot 7$
$(1, 2)$	$(\delta, 1, 0, 1, 1)$, $1 \leq \delta \leq 4$	$2^\delta \cdot 3 \cdot 7 \cdot 13$
	$(\delta, 2, 0, 1, 1)$, $1 \leq \delta \leq 4$	$2^\delta \cdot 3^2 \cdot 7 \cdot 13$
	$(\delta, 0, 1, 1, 1)$, $1 \leq \delta \leq 4$	$2^\delta \cdot 5 \cdot 7 \cdot 13$
$(2, 2)$	$(0, 1, 1, 1, 1)$	$3 \cdot 5 \cdot 7 \cdot 13$
	$(0, 2, 1, 1, 1)$	$3^2 \cdot 5 \cdot 7 \cdot 13$
$(2, 1)$	$(\delta, 1, 1, 1, 0)$, $1 \leq \delta \leq 4$	$2^\delta \cdot 3 \cdot 5 \cdot 7$
	$(\delta, 2, 1, 1, 0)$, $1 \leq \delta \leq 4$	$2^\delta \cdot 3^2 \cdot 5 \cdot 7$
	$(\delta, 1, 1, 0, 1)$, $1 \leq \delta \leq 4$	$2^\delta \cdot 3 \cdot 5 \cdot 13$
	$(\delta, 2, 1, 0, 1)$, $1 \leq \delta \leq 4$	$2^\delta \cdot 3^2 \cdot 5 \cdot 13$

These solutions are exactly the 32 elements of $\mathcal{F}_2$.

In the subcase where $r > 0$, $s > 0$, $t = 0$, and $p > 3$, we have $\beta_s \geq 2s - 1$. Thus,

$$2^{\alpha_r} + 2s + 1 \leq 2^{\alpha_r} + \beta_s + 2 \leq r + s + \min\{\gamma_0, 1\} + 2$$

and $s \leq r + 1 + \min\{\gamma_0, 1\} - 2^{\alpha_r} \leq 1$, which implies that $s = 1$ and $\beta_1 \leq 2$. Note that $\alpha_r \leq 1$ and note also that r cannot be 3 since this would imply $s = 1$, $2^{\alpha_r} + 1 \geq 5$, $\tau(\lambda(n)) \geq 10$, and $\omega(n) \geq 8$, which is impossible because $\omega(n) \leq r + s + 3 \leq 7$.

Therefore,

$$n = 2^{\delta_0} \cdot 3^{\delta_1} \cdot 5^{\delta_2} \cdot (2^{\beta_1} p + 1)^{\delta_3}$$

with $p > 3$. If $5^2 \mid n$ then we have the solutions $n = 2^\alpha \cdot 3 \cdot 5^2 \cdot 11$ ($\alpha = 1, 2, 3, 4$), which are exactly the elements of $\mathcal{F}_3$. If $5^2 \nmid n$ then we have the solutions $n = 3 \cdot (2p + 1)$, which are elements of $\mathcal{I}_2$, and $n = 2^\alpha \cdot 3 \cdot 5 \cdot (2^\beta + 1)$ with $\alpha = 1, 2, 3, 4$ and $\beta = 1, 2$, which are elements of $\mathcal{I}_3$.

The last case to consider is when $t > 0$, so that there is a prime dividing n of the form $2^\beta \cdot p^2 + 1$. Now equation $\tau(\lambda(n)) = \omega(n) + 2$ is equivalent to

$$\Theta \cdot \Lambda \cdot \gamma_1 \cdots \gamma_{r+s+t} = r + s + t + \min\{1, \gamma_0\} + 2, \tag{6}$$

where

$$\Theta = \begin{cases} 1 & \text{if } p^3 \mid n, \\ 3/2 & \text{if } p^2 \parallel n, \\ 3 & \text{if } p^2 \nmid n, \end{cases}$$

and $\Lambda = \max\{\tau(\lambda(2^{\gamma_0})), 2^{\alpha_r}+1, \beta_s+1, \delta_t+1\}$. Here the terms $2^{\alpha_r}+1$ (resp. β_s+1) are to be omitted if $r=0$ (resp. $s=0$).

We claim that $r, s \neq 0$ (and will show this later). Hence, from (6) we may deduce that

$$2^{\alpha_r} + \beta_s + \delta_t + 3 \le r + s + t + \min\{1, \gamma_0\} + 2.$$

On one hand, this relation implies that $2^{\alpha_r} \le r - 1 + \min\{1, \gamma_0\}$, so that $\gamma_0 \ge 1$ and either $r=1$ and $\alpha_1 = 0$ or $r=2$, $\alpha_2 = 1$, and $\alpha_1 = 0$. On the other hand, the same relation implies that $s + t \le \beta_s + \delta_t \le 2r$.

If $r=1$ then $s = t = \beta_s = \delta_t = 1$, and since $2p^2+1$ is prime we necessarily have $p=3$. Therefore,

$$n = 2^{\gamma_0} \cdot 3^{\gamma_1} \cdot 7^{\gamma_2} \cdot 19^{\gamma_3},$$

and the only solutions of $\tau(\lambda(n)) = 6$ of this form are the first nine elements of $\mathcal{F}_4$.

If $r=2$, then $4 \le s+t \le \beta_s + \delta_t \le 4$. This implies that $s = t = 2$ and $(\beta_1, \beta_2, \delta_1, \delta_2) = (1,2,1,2)$; hence $p=3$,

$$n = 2^{\gamma_0} \cdot 3^{\gamma_1} \cdot 5^{\gamma_2} \cdot 7^{\gamma_3} \cdot 13^{\gamma_3} \cdot 19^{\gamma_4} \cdot 37^{\gamma_5},$$

and the only solutions of $\tau(\lambda(n)) = 9$ of this form are the last twelve elements of $\mathcal{F}_4$.

Finally, we must prove the claim for $r, s \neq 0$. If $r=0$ and $s \neq 0$ then we deduce from (6) that

$$3(s+t+2)/2 \le 3(\beta_s + \delta_t + 2)/2 \le s + t + 3,$$

which implies $s + t \le 0$—a contradiction. A similar argument rules out the possibility $r=0$ and $s=0$. Lastly, if $r \neq 0$ and $s = 0$, then from (6) and from $\delta_t \ge t$ we deduce that

$$3(2^{\alpha_r} + t + 2)/2 \le r + t + 3,$$

which is also a contradiction and thus ends the proof of the proposition. □

3. Lower Bounds on the Counting Functions of $\mathcal{A}_k$

THEOREM 1. *$\mathcal{A}_k$ is nonempty for all nonnegative integers k.*

Proof. Let $p_1 = 3$, $p_2 = 5$, $p_3 = 13$, and $p_4 = 31$. Then, for each $m \ge 3$ and for each $t \in \{4,5,6,7\}$, the number $n = 2^{m+1} \cdot 7 \cdot 11 \cdot p_1 \cdots p_{t-3}$ satisfies $\omega(n) = t$ and $\tau(\lambda(n)) = \tau(2^{m-1} \cdot 3 \cdot 5) = 4m$. This means that $\tau(\lambda(n)) - \omega(n) = 4(m-1) - (t-4)$ can assume all possible values ≥ 8. Finally, $3 \in \mathcal{A}_0$, $4 \in \mathcal{A}_1$, $5 \in \mathcal{A}_2$, $7 \in \mathcal{A}_3$, $17 \in \mathcal{A}_4$, $13 \in \mathcal{A}_5$, $62 \in \mathcal{A}_6$, and $31 \in \mathcal{A}_7$. This completes the proof. □

In what follows, we show that if k is sufficiently large, then $\mathcal{A}_k$ contains "many" elements.

THEOREM 2. *For all $k \neq 0,1,2,3,4$, we have the lower bound*

$$\#\mathcal{A}_k(x) \gg_k \frac{x}{(\log x)^2} \quad \text{as } x \to \infty.$$

Proof. The proof uses the famous Theorem of Chen that we state in the following form (see also [4; 7, Lemma 1.2; 8, Chap. 11]).

LEMMA 1. *Let $a \in \mathbb{N}$ be an even number. There exists a constant $c = c(a)$ such that, if $x > x_0(a)$, then the number of primes $p \in [x/2, x]$ such that $p \equiv 1 \pmod{a}$ and $(p-1)/a$ has at most two prime factors, each of which exceeds $x^{1/10}$, is at least $c_a x/(\log x)^2$.*

We write $k = 4s + r$ for $s \geq 1$ and $r \in \{0, 1, 2, 3\}$, distinguishing two cases as follows:

Case 1. $r \neq 3$;
Case 2. $r = 3$.

In Case 1, we apply Chen's theorem with the choice $a = 2^s$ and obtain that there are either (a) at least $M_a \gg_a x/(\log x)^2$ primes $p \leq x/42$ with $p - 1 = 2^s q$ and q prime or (b) at least $N_a \gg_a x/(\log x)^2$ primes $p \leq x/42$ with $p - 1 = 2^s q_1 q_2$, where q_1 and q_2 are distinct primes exceeding $x^{1/10}$.

For (a), consider the M_a integers $n \leq x$ of the form $n = 7pT$, where

$$T = \begin{cases} 1 & \text{if } r = 2, \\ 2 & \text{if } r = 1, \\ 6 & \text{if } r = 0. \end{cases}$$

These choices yield $\omega(n) = 4 - r$, $\lambda(n) = 2^s \cdot 3 \cdot q$, and $\tau(\lambda(n)) = 4(s+1)$; therefore, $\tau(\lambda(n)) - \omega(n) = 4s + r = k$.

For (b), consider the N_a integers $n \leq x$ of the form $n = 2pT$, where

$$T = \begin{cases} 1 & \text{if } r = 2, \\ 3 & \text{if } r = 1, \\ 15 & \text{if } r = 0. \end{cases}$$

For $s \geq 2$ and for $s = 1$ with $r \neq 0$, we have $\omega(n) = 4 - r$, $\lambda(n) = 2^s \cdot q_1 \cdot q_2$, and $\tau(\lambda(n)) = 4(s+1)$, so that again $\tau(\lambda(n)) - \omega(n) = k$.

In Case 2, we apply Chen's theorem with the choice $a = 2^{s+1}$ and obtain that there are either (a) at least $M_a \gg_a x/(\log x)^2$ primes $p \leq x/510$ with $p - 1 = 2^{s+1}q$ and q prime or (b) at least $N_a \gg_a x/(\log x)^2$ primes $p \leq x/510$ with $p-1 = 2^{s+1}q_1q_2$, where q_1 and q_2 are distinct primes that exceed $x^{1/10}$.

Given (a), consider the M_a integers $n \leq x$ of the form $n = 210p$. For $s \geq 1$ we have $\omega(n) = 5$ and $\tau(\lambda(n)) = 4(s+2)$, so $\tau(\lambda(n)) - \omega(n) = 4s + 3 = k$. Given (b), consider the N_a integers $n \leq x$ of the form $n = 510p$. For $s \geq 3$ we have $\omega(n) = 5$ and $\tau(\lambda(n)) = 4(s+2)$, so again $\tau(\lambda(n)) - \omega(n) = 4s + 3 = k$.

Next assume that $k = 7$. Then we apply Chen's theorem with the choice $a = 2$ and obtain that there are either (a) at least $M_a \gg_a x/(\log x)^2$ primes $p \leq x/192$ with $p - 1 = 2q$ and q prime or (b) at least $N_a \gg_a x/(\log x)^2$ primes $p \leq x/192$ with $p - 1 = 2q_1q_2$, where q_1 and q_2 are distinct primes exceeding $x^{1/10}$.

For (a), consider the M_a integers $n \leq x$ of the form $n = 2^6 3p$. We have $\omega(n) = 3$ and $\tau(\lambda(n)) = \tau(2^4 p)$, so $\tau(\lambda(n)) - \omega(n) = 10 - 3 = 7$. For (b), consider the N_a integers $n \leq x$ of the form $n = p = 2q_1q_2 + 1$. We have $\omega(n) = 1$ and $\tau(\lambda(n)) = 8$, so again $\tau(\lambda(n)) - \omega(n) = 8 - 1 = 7$.

Finally, we treat the case $k = 11$. Here we apply Chen's theorem with the choice $a = 4$ and deduce that either (a) there exist $M \gg x/(\log x)^2$ primes $p \le x/4510$ such that $p - 1 = 4q$ with q prime or (b) there exist $N \gg x/(\log x)^2$ primes $p \le x/4510$ such that $p - 1 = 4q_1q_2$, where q_1 and q_2 are distinct primes that exceed $x^{1/10}$.

Given (a), we note that, for large x, all M positive integers $n = 2\cdot 5\cdot 11\cdot 41\cdot p = 4510p$ (where $p \le x$ is of the form $4q+1$) are $\le x$ and satisfy both $\omega(n) = 5$ and $\lambda(n) = 2^3\cdot 5\cdot q$; hence $\tau(\lambda(n)) = 16 = \omega(n)+11$. Given (b), it follows that, for large x, the N positive integers $n = p$ (where $p \le x$ is such that $p - 1 = 4q_1q_2$, with distinct primes q_1 and q_2 that exceed $x^{1/10}$) have the property that $\tau(\lambda(n)) = \tau(4q_1q_2) = 12 = \omega(n) + 11$. Thus, $\#\mathcal{A}_{11}(x) \ge \max\{M, N\} \gg x/(\log x)^2$, which completes the proof of this theorem. $\square$

The remaining cases are $k = 0, 1, 2, 3, 4$ and must be treated separately. Propositions 2, 3, and 4 address the first three cases, and certainly there is no hope of showing even that $\mathcal{A}_k$ is infinite for $k = 1, 2$. Although the next result is not as precise a characterization of $\mathcal{A}_k$ for $k = 3, 4$ as Propositions 2–4 for the smaller values of k, its aim is to demonstrate the impossibility of showing that either one of these two sets is infinite.

PROPOSITION 5. *Assume that $\mathcal{A}_3 \cup \mathcal{A}_4$ is infinite. Then there exists an even positive integer c such that the set of primes of the form $p = cq^\beta + 1$, with q prime and $\beta \le 4$, is infinite.*

Proof. Assume that $n \in \mathcal{A}_3 \cup \mathcal{A}_4$. Then $\tau(\lambda(n)) \le \omega(n) + 4$. Write $m = \lambda(n)$ and note that $\omega(n)$ is at most the number of divisors of m of the form $p - 1$ for some prime p. Hence, m can have at most four divisors d such that $d + 1$ is composite. Write $m = 2^\alpha\ell$, where ℓ is odd. If $\alpha \ge 9$, then $2^3, 2^5, 2^6, 2^7, 2^9$ are five divisors of m and none of the form $p - 1$ for some prime p. Therefore, $\alpha \le 8$. If $\tau(\ell) \ge 6$ then ℓ (and hence m) has at least five odd divisors > 1, and certainly none of them is of the form $p-1$ for some prime p. Thus $\tau(\ell) \le 5$, which shows that either $\ell = q^\beta$ for some prime q and some $\beta \le 4$ or $\ell = q_1q_2$, where q_1 and q_2 are distinct primes.

Assume that $\ell = q^\beta$ holds for infinitely many n. Then there exist infinitely many primes p of the form $p - 1 = 2^{\alpha_0}q^\beta$ for some $\alpha_0 \in \{1, \dots, 9\}$ and $\beta \in \{1, \dots, 4\}$, which implies the conclusion of the proposition.

Assume now that $\ell = q_1q_2$ holds for infinitely many n. Suppose further that $q_1 < q_2$. We then distinguish two cases. The first case is when q_1 remains bounded for infinitely many such n. Then $2^\alpha q_1$ can take only finitely many values. Since we have infinitely many values for n, there must exist some fixed even positive integer c (an even divisor of a number of the form 2^9q_1 over all the finitely many possibilities for q_1) such that $p - 1 = cq_2$ holds for infinitely many primes p, which implies the conclusion of the proposition. The second case is when q_1 tends to infinity as n tends to infinity in $\mathcal{A}_3 \cup \mathcal{A}_4$. If for infinitely many such n we have that either $2q_1 + 1$ or $2q_2 + 1$ is prime, then the conclusion of the proposition follows with $c = 2$. Assuming this is not the case, we derive a contradiction.

Observe first that $\alpha \le 3$, for otherwise $2^3, q_1, q_2, 2q_1, 2q_2$ are five divisors of n, none of which is of the form $p-1$ for some odd prime p. Assume now that $\alpha = 1$. Then $\tau(\lambda(n)) = \tau(2q_1q_2) = 8$ and so $\omega(n) \ge 4$. Since the only prime factors of n are in $\{2, 3, 2q_1+1, 2q_2+1, 2q_1q_2+1\}$, we deduce that one of $2q_1+1$ and $2q_2+1$ must be prime—a contradiction. Finally, if $\alpha = 2$ then $\tau(\lambda(n)) = \tau(4q_1q_2) = 12$, so $\omega(n) \ge 8$. Because all the prime factors of n belong to $\{2, 3, 5, 2q_1+1, 2q_2+1, 4q_1+1, 4q_2+1, 2q_1q_2+1, 4q_1q_2+1\}$, again it follows that one of $2q_1+1$ or $2q_2+1$ must be a prime, which is the final contradiction. □

4. Upper Bounds on the Counting Functions of $\mathcal{A}_k$

Our first result in this section shows that, for numbers $n \in \mathcal{A}_k$, $\omega(n)$ is bounded in terms of k.

PROPOSITION 6. *If $n \in \mathcal{A}_k$, then $\omega(n) \le 2(k+1)^2 + 1$.*

Proof. We use the same idea and notation as in the proof of Proposition 5. Let $n \in \mathcal{A}_k$ and put $m = \lambda(n) = 2^\alpha \ell$, where α is a nonnegative integer and ℓ is odd. If $\alpha \ge 2k+3$, then $2^3, 2^5, \dots, 2^{2k+3}$ are $k+1$ divisors of m and none of the form $p-1$ for some prime p, which is a contradiction. If $\tau(\ell) \ge k+2$ then ℓ (and hence m) has $k+1$ odd divisors > 1, and obviously none of them is of the form $p-1$ for some prime p, which is again a contradiction. Hence $\alpha \le 2k+2$ and $\tau(\ell) \le k+1$, so

$$\begin{aligned}\omega(n) &= \tau(\lambda(n)) - k = \tau(2^\alpha \ell) - k = (\alpha+1)\tau(\ell) - k\\ &\le (2k+3)(k+1) - k = 2(k+1)^2 + 1.\end{aligned}$$ □

An upper bound for the counting function $\#\mathcal{A}_k(x)$ of $\mathcal{A}_k$ follows from Proposition 6 with a little extra work. Let us set

$$b_k = 2(k+1)^2 + 3 + \lfloor \log_2(2(k+1)^2 + k + 1) \rfloor.$$

We then have the following result.

THEOREM 3. *For all nonnegative integers k we have the upper bound*

$$\#\mathcal{A}_k(x) \ll_k \frac{x(\log\log x)^{b_k}}{(\log x)^2} \quad \textit{as } x \to \infty.$$

Proof. Let $K \ge 2$ be any fixed positive integer. Let $\pi_K(x)$ be the number of primes $p \le x$ such that $\omega(p-1) \le K$. We begin with the following lemma.

LEMMA 2. *There exists an absolute constant c_0 such that the following estimate holds*:

$$\pi_K(x) \ll \frac{x(\log\log x + c_0)^{K+1}}{(K-1)!\,(\log x)^2} \quad \textit{as } x \to \infty.$$

Proof. Let $\mathcal{P}(x) = \{p \le x : \omega(p-1) \le K\}$. Put $y = x^{1/\log\log x}$ and $u = \log x/\log y = \log\log x$. For a positive integer n we write $P(n)$ for the largest prime factor of n. Let

$$\Psi(x, y) = \{n \le x : P(n) \le y\}.$$

By a result of de Bruijn ([2]; see also [3; 9, Cor. 3; 13, Chap. III.5]), the bound

$$\#\Psi(x, y) \le x \exp(-(1 + o(1))u \log u) < \frac{x}{(\log x)^2} \tag{7}$$

holds as $u \to \infty$, where $u = \log x / \log y$ and provided that $u \le y^{1/2}$, which is satisfied for our choice of y.

Therefore, if $\mathcal{P}_1(x) = \mathcal{P}(x) \cap \Psi(x, y)$ then

$$\#\mathcal{P}_1(x) \ll \frac{x}{(\log x)^2}.$$

Now let $\mathcal{P}_2(x) = \{p \le x : q^2 \mid p - 1 \text{ for some } q \ge y\}$. For a fixed $q \ge y$, the number of $1 < n \le x$ such that $q^2 \mid n - 1$ and is $\le x/q^2$. Thus,

$$\#\mathcal{P}_2(x) \le \sum_{q \ge y} \frac{x}{q^2} \ll x \int_y^\infty \frac{dt}{t^2} \ll \frac{x}{y} = o\left(\frac{x}{(\log x)^2}\right).$$

Put $\mathcal{P}_3(x) = \mathcal{P}(x) \backslash (\mathcal{P}_1(x) \cup \mathcal{P}_2(x))$. Write $p - 1 = Pm$, where $P = P(p-1)$. Since $P > y$ and $p \notin \mathcal{P}_2(x)$, we deduce that $P(m) < P$. Thus, $\omega(m) \le K - 1$. Fix m. By Brun's sieve (see e.g. [8, Thm. 2.3]), the number of primes $p \le x$ such that $p - 1 = mP$ for some prime P is

$$\ll \frac{x}{\varphi(m)} \frac{1}{(\log x/m)^2} \ll \frac{x}{\varphi(m)(\log y)^2} \ll \frac{x(\log\log x)^2}{\varphi(m)(\log x)^2}.$$

Summing now over all the acceptable values of m, we obtain

$$\begin{aligned}
\#\mathcal{P}_3(x) &\ll \frac{x(\log\log x)^2}{(\log x)^2} \sum_{\substack{m \le x \\ \omega(m) \le K-1}} \frac{1}{\varphi(m)} \\
&\le \frac{x(\log\log x)^2}{(\log x)^2} \sum_{k=1}^{K-1} \sum_{\substack{m \le x \\ \omega(m) = k}} \frac{1}{\varphi(m)} \\
&\le \frac{x(\log\log x)^2}{(\log x)^2} \sum_{k=1}^{K-1} \frac{1}{k!} \left(\sum_{p^\alpha \le x} \frac{1}{p^{\alpha-1}(p-1)} \right)^k \\
&\ll \frac{x(\log\log x)^2}{(\log x)^2} \sum_{k=1}^{K-1} \frac{1}{k!} \left(\sum_{p \le x} \frac{1}{p-1} + O(1) \right)^k \\
&\ll \frac{x(\log\log x)^2}{(\log x)^2} \sum_{k=1}^{K-1} \frac{1}{k!} (\log\log x + c_0)^{k-1}.
\end{aligned}$$

It remains only to observe that the last term dominates as x tends to infinity, which finishes the proof of Lemma 2. □

We are now ready to prove Theorem 3. Assume that $k \ge 3$, since otherwise the result follows immediately (from Propositions 2–4 and Brun's sieve) even with a smaller b_k (i.e., $b_0 = 0$, $b_1 = 1$, and $b_2 = 1$).

Now note that if $p \mid n$ and $n \in \mathcal{A}_k$, then

$$2^{\omega(p-1)} \leq \tau(p-1) \leq \tau(\lambda(n)) = \omega(n) + k \leq 2(k+1)^2 + k + 1$$

(by Proposition 6) and so $\omega(p-1) \leq K = \lfloor \log_2(2(k+1)^2 + k + 1) \rfloor$. Lemma 2 shows that

$$\#\{p \leq x : \omega(p-1) \leq K\} \ll_K \frac{x(\log\log x)^{K+1}}{(\log x)^2}. \tag{8}$$

We put $\mathcal{A}_{k,1}(x)$ for the set of $n \in \mathcal{A}_k(x)$ such that either $P \leq y = x^{1/\log\log x}$ or P^2 divides n. As in the proof of Lemma 2,

$$\#\mathcal{A}_{k,1} \ll \frac{x}{(\log x)^2}. \tag{9}$$

Let $\mathcal{A}_{k,2}(x)$ stand for the complement of $\mathcal{A}_{k,1}(x)$ in $\mathcal{A}_k(x)$. Now write $n \in \mathcal{A}_{k,2}(x)$ as $n = Pm$, where $P = P(n)$. Hence $P > y = x^{1/\log\log x}$, P^2 does not divide n, and $\omega(m) = \omega(n) - 1 \leq 2(k+1)^2$. Fixing m, the number of values for $P \leq x/m$ such that $\omega(P-1) \leq K$ is, by (8),

$$\begin{aligned}\pi_K(x/m) &\ll_k \frac{x(\log\log(x/m))^{K+1}}{m(\log(x/m))^2} \ll_k \frac{x(\log\log x)^{K+1}}{m(\log y)^2}\\ &\ll_k \frac{x(\log\log x)^{K+3}}{m(\log x)^2}.\end{aligned}$$

If we sum the preceding inequality over all the values of $m \leq x$ with $\omega(m) \leq 2(k+1)^2$, then it follows that the number of possibilities is

$$\begin{aligned}\#\mathcal{A}_{k,2}(x) &\ll_k \frac{x(\log\log x)^{K+3}}{(\log x)^2} \sum_{\substack{m \leq x\\ \omega(m) \leq 2(k+1)^2}} \frac{1}{m}\\ &\ll_k \frac{x(\log\log x)^{K+3}}{(\log x)^2} \sum_{\ell=0}^{2(k+1)^2} \frac{1}{\ell!}\Bigg(\sum_{p^\alpha \leq x} \frac{1}{p^\alpha}\Bigg)^{\ell}\\ &\ll_k \frac{x(\log\log x)^{K+3+2(k+1)^2}}{(\log x)^2};\end{aligned}$$

this, together with (9), completes the proof of Theorem 3. □

A more careful analysis (along the lines of the proof of [1, Thm. 4.1]) shows that Theorem 3 holds with a somewhat smaller b_k. Furthermore, it is clear that one could write down a formula for the implied constant in terms of k. We do not enter into such details.

5. A More General Statement

Let $f(x) \geq 1$ be any function that tends to infinity with n and that is monotonically decreasing for $x > x_0$. Let

$$\mathcal{B}_f = \{n : \tau(\lambda(n)) - \omega(n) < \exp((\log\log n)/f(n))\}. \tag{10}$$

We can then show the following result.

THEOREM 4. *If $\mathcal{B}_f$ is the set appearing in* (10), *then*

$$\#\mathcal{B}_f(x) \le \frac{x}{(\log x)^{2+o(1)}} \quad \text{as } x \to \infty.$$

We start by proving the following lemma.

LEMMA 3. *Let $\mathcal{P}_f = \{p : \omega(p-1) < 2(\log\log p)/\sqrt{f(p)}\}$. Then*

$$\#\mathcal{P}_f(x) \le \frac{x}{(\log x)^{2+o(1)}} \quad \text{as } x \to \infty. \tag{11}$$

Proof. Let x be large, put $y = x^{1/\log\log x}$, and let

$$\mathcal{P}_2(x) = \{p \in \mathcal{P}_f(x) : p-1 \notin \Psi(x,y)\}.$$

If $p \in \mathcal{P}_2(x)$ then $p-1 = Qm$, where $Q = P(p-1) > y$ and $m \le x/y$. Fix m. By Brun's method, the number of primes $Q \le x/m$ such that $p = Qm+1$ is also prime is

$$\ll \frac{x}{\varphi(m)(\log(x/m))^2} \le \frac{x}{\varphi(m)(\log y)^2} \le \frac{x(\log\log x)^2}{\varphi(m)(\log x)^2}.$$

Using the minimal order $\varphi(m)/m \gg 1/\log\log x$ of the Euler function in the interval $[1,x]$, we get that if m is fixed then the number of acceptable primes $p \in \mathcal{P}_2(x)$ with $(p-1)/P(p-1) = m$ is

$$\ll \frac{x(\log\log x)^3}{m(\log x)^2}.$$

Let $\mathcal{M}(x)$ be the set of acceptable values for m. Since $\omega(p-1) \le 2(\log\log p)/\sqrt{f(p)}$, f is increasing for large x, and $p > y$ for all $p \in \mathcal{P}_2(x)$, it follows that

$$z = \max\{2(\log\log p)/\sqrt{f(p)} : p \in \mathcal{P}_2(x)\} \le \frac{2\log\log x}{\sqrt{f(y)}} = o(\log\log x) \tag{12}$$

as $x \to \infty$. Furthermore, $\mathcal{M}(x) \subseteq \{m \le x : \omega(m) \le z\}$. As a result,

$$\#\mathcal{P}_2(x) \ll \frac{x(\log\log x)^3}{(\log x)^2} \sum_{m \in \mathcal{M}(x)} \frac{1}{m} \ll \frac{x(\log\log x)^3}{(\log x)^2} \sum_{k \le z} \sum_{\substack{m \le x \\ \omega(m)=k}} \frac{1}{m}. \tag{13}$$

Put

$$\mathcal{S}_k(x) = \sum_{\substack{m \le x \\ \omega(m)=k}} \frac{1}{m}.$$

Then unique factorization, the multinomial formula, and Stirling's formula imply that

$$\mathcal{S}_k(x) \le \frac{1}{k!}\left(\sum_{p \le x} \sum_{\alpha \ge 1} \frac{1}{p^\alpha}\right)^k \le \left(\frac{e\log\log x + O(1)}{k}\right)^k,$$

where we have used the obvious fact that

$$\sum_{p \ge 2} \sum_{\alpha \ge 2} \frac{1}{p^\alpha} = O(1)$$

together with Mertens's formula

$$\sum_{p \le x} \frac{1}{p} = \log\log x + O(1).$$

For every fixed value of $A > 1$, the function $(eA/t)^t$ is increasing for $t < A$ and so

$$\begin{aligned} \mathcal{S}_k(x) &\le \left(\frac{e \log\log x + O(1)}{z}\right)^z = \exp(z \log(e(\log\log x + O(1))/z)) \\ &\le \exp\left(\frac{2\log\log x}{\sqrt{f(y)}} \log(O(\sqrt{f(y)}))\right) = \exp(o(\log\log x)) \\ &= (\log x)^{o(1)} \quad \text{for } k \le z. \end{aligned} \tag{14}$$

Hence, the inequalities (12) and (13) together with the estimate (14) yield

$$\begin{aligned} \#\mathcal{P}_2(x) &\ll \frac{x(\log\log x)^3}{(\log x)^2} \sum_{k \le z} \mathcal{S}_k(x) \\ &\ll \frac{x(\log\log x)^4}{(\log x)^2} \max\{\mathcal{S}_k(x) : k \le z\} = \frac{x}{(\log x)^{2+o(1)}}; \end{aligned}$$

combining this with estimate (7) implies inequality (11) and completes the proof of Lemma 3. □

Now partial summation immediately yields our next result.

COROLLARY 2. *If $\mathcal{P}_f$ is the set of primes appearing in Lemma 3, then*

$$\sum_{p \in \mathcal{P}_f} \frac{1}{p} = O(1).$$

Proof of Theorem 4. Let again $y = x^{1/\log\log x}$, $w = x/(\log x)^2$, and

$$\mathcal{B}_1(x) = \{n \le w\} \cup \Psi(x, y).$$

By (7) we have

$$\#\mathcal{B}_1(x) \le \frac{2x}{(\log x)^2} \tag{15}$$

once x is large. Let $\mathcal{B}_2(x) = \{n \le x : \omega(n) > 10\log\log x\}$. It follows from results of Norton [11; 12] that

$$\#\mathcal{B}_2(x) \ll \frac{x}{(\log x)^{\lambda}},$$

where $\lambda = 1 + 10\log(10/e) > 2$; therefore,

$$\#\mathcal{B}_2(x) < \frac{x}{(\log x)^2}. \tag{16}$$

Now put

$$\mathcal{B}_3(x) = \mathcal{B}_f(x) \backslash (\mathcal{B}_1(x) \cup \mathcal{B}_2(x)),$$

and assume that $n \in \mathcal{B}_3(x)$. Replacing $f(x)$ with $\min\{f(x), \log\log\log x\}$, we may assume that $f(x) \le \log\log\log x$. Then $p - 1 \mid \lambda(n)$ for all prime factors p of n and so

$$\begin{aligned}2^{\omega(p-1)} &\le \tau(\lambda(n)) \le \omega(n) + \exp((\log\log n)/f(n))\\ &< 10\log\log x + \exp((\log\log x)/f(w))\\ &< \exp\left(\frac{1.1(\log\log x)}{f(w)}\right),\end{aligned}$$

so

$$\omega(p-1) < \frac{1.6(\log\log x)}{\sqrt{f(w)}}, \tag{17}$$

where we used the fact that $1.1/\log 2 < 1.6$. Let $\mathcal{B}_4(x) = \{n \in \mathcal{B}_3(x) : P(n) > w\}$. Since $w \ge p/(\log p)^2$ holds for all $p \in [w, x]$ once x is large, it follows that if $p = P(n)$ for $n \in \mathcal{B}_4(x)$ then

$$\omega(p-1) < \frac{1.6(\log\log x)}{f(p/(\log p)^2)} < \frac{2(\log\log p)}{\sqrt{g(p)}}$$

holds for large x. Here g is the function $g(t) = (f(t/(\log t)^2))^2$, which is increasing for large t. Thus, $p \in \mathcal{P}_g$. Let us now write $n = Pm$, where $m < x/p < (\log x)^2$, and let us fix m. Then $p \in \mathcal{P}_g(x/m)$ and, by Lemma 3, the number of such choices for p is

$$\#\mathcal{P}_g(x/m) \le \frac{x}{m(\log x/m)^{2+o(1)}} = \frac{x}{m(\log x)^{2+o(1)}}.$$

Summing this inequality for $m \le (\log x)^2$, we have

$$\begin{aligned}\#\mathcal{B}_4(x) &\le \sum_{m \le (\log x)^2} \#\mathcal{P}_g\left(\frac{x}{m}\right)\\ &\le \frac{x}{(\log x)^{2+o(1)}} \sum_{m \le (\log x)^2} \frac{1}{m}\\ &= \frac{x}{(\log x)^{2+o(1)}},\end{aligned} \tag{18}$$

because

$$\sum_{m \le (\log x)^2} \frac{1}{m} \ll \log\log x = (\log x)^{o(1)}.$$

From now on we assume that $n \in \mathcal{B}_5(x) = \mathcal{B}_3(x) \backslash \mathcal{B}_4(x)$. Let $n = Pm$, where $P = P(n) \in [y, w]$. Since $1.6\log\log x < 2\log\log y \le 2\log\log P$ for large x and since $f(w) \ge f(P)$, it follows that

$$\omega(P-1) < \frac{1.6(\log\log x)}{f(w)} < \frac{2(\log\log P)}{f(P)}.$$

In particular, $P \in \mathcal{P}_{f^2}$. By Lemma 3, if $m \le x/y$ is fixed then the number of choices for P is at most

$$\#\mathcal{P}_{f^2}(x/m) \le \frac{x}{m(\log(x/m))^{2+o(1)}} \le \frac{x}{m(\log y)^{2+o(1)}} \le \frac{x}{m(\log x)^{2+o(1)}},$$

where we have used that $x/m \ge y$ and $\log y = \log x/\log\log x = (\log x)^{1+o(1)}$. Let $\mathcal{M}(x)$ be the set of acceptable values of m. Then

$$\#\mathcal{B}_5(x) \le \sum_{m\in\mathcal{M}(x)} \frac{x}{m(\log x)^{2+o(1)}} \le \frac{x}{(\log x)^{2+o(1)}} \sum_{m\in\mathcal{M}(x)} \frac{1}{m}. \tag{19}$$

Let $\mathcal{Q}(x)$ be the set of primes dividing some $m \in \mathcal{M}(x)$, and note that $\mathcal{Q}(x)$ consists of the primes $q \le x$ satisfying inequality (17). We put

$$v = \exp\left(\exp\left(\frac{\log\log x}{\sqrt{f(w)}}\right)\right)$$

and split the primes in $\mathcal{Q}$ into two subsets as follows:

$$\mathcal{Q}_1 = \{q \le v\} \cap \mathcal{Q};$$
$$\mathcal{Q}_2 = \mathcal{Q} \cap [v, w].$$

Observe that if $q \in \mathcal{Q}_2$ then

$$\frac{2\log\log q}{\sqrt{f(q)}} \ge \frac{2\log\log x}{\sqrt{f(q)f(w)}} \ge \frac{2\log\log x}{f(w)} > \omega(q-1);$$

therefore, $\mathcal{Q}_2 \subset \mathcal{P}_f$. This argument shows that

$$\begin{aligned}\sum_{m\in\mathcal{M}(x)} \frac{1}{m} &\le \prod_{q\in\mathcal{Q}_1\cup\mathcal{Q}_2}\left(\sum_{\alpha\ge 0}\frac{1}{q^\alpha}\right)\\ &\le \exp\left(\sum_{q\in\mathcal{Q}_1}\frac{1}{q} + \sum_{q\in\mathcal{Q}_2}\frac{1}{q} + O\left(\sum_{q\ge 2}\sum_{\alpha\ge 2}\frac{1}{q^\alpha}\right)\right).\end{aligned} \tag{20}$$

Since

$$\sum_{q\in\mathcal{Q}_1}\frac{1}{q} \le \sum_{q\le v}\frac{1}{v} = \log\log v + O(1) = o(\log\log x)$$

(by Mertens's formula),

$$\sum_{q\in\mathcal{Q}_2}\frac{1}{q} \le \sum_{q\in\mathcal{P}_f}\frac{1}{q} = O(1)$$

(by Corollary 2), and

$$\sum_{q\ge 2}\sum_{\alpha\ge 2}\frac{1}{q^\alpha} = O(1),$$

it follows from (20) that

$$\sum_{m\in\mathcal{M}(x)}\frac{1}{m} \le \exp(o(\log\log x)) = (\log x)^{o(1)},$$

which together with (19) gives

$$\#\mathcal{B}_5(x) \le \frac{x}{(\log x)^{2+o(1)}}. \tag{21}$$

Since $\mathcal{B}_3(x) \subseteq \mathcal{B}_4(x) \cup \mathcal{B}_5(x)$, by estimates (18) and (21) we have that

$$\#\mathcal{B}_3(x) < \frac{x}{(\log x)^{2+o(1)}}, \tag{22}$$

which together with estimates (15) and (16) completes the proof of Theorem 4. $\square$

6. Average and Normal Orders of $\tau(\lambda(n)) - \omega(n)$

Our last result addresses average and normal orders of the function

$$h(n) = \tau(\lambda(n)) - \omega(n).$$

THEOREM 5. (i) *There exist positive constants c_0, c_1 such that the inequalities*

$$\exp\left(c_0\sqrt{\frac{\log x}{\log\log x}}\right) \le \frac{1}{x}\sum_{n\le x} h(n) \le \exp\left(c_1\sqrt{\frac{\log x}{\log\log x}}\right) \tag{23}$$

hold for all $x \ge 1$.

(ii) *The inequality*

$$h(n) = 2^{0.5(1+o(1))(\log\log n)^2}$$

holds for almost all positive integers n.

Proof. (i) In [10] it is shown that inequalities (23) hold with some constants c_0 and c_1 for the function $\tau(\lambda(n)) = h(n) + \omega(n)$. Since the average value of $\omega(n)$ is $\log\log x = \exp(o(\sqrt{\log x/\log\log x}))$, the required inequality follows.

(ii) In [5] it is shown that the normal order of both $\omega(\varphi(n))$ and $\Omega(\varphi(n))$ is $0.5(\log\log n)^2$. Since $\omega(\lambda(n)) = \omega(\varphi(n))$ and $\Omega(\lambda(n)) \le \Omega(\varphi(n))$, it follows that the normal order of both $\omega(\lambda(n))$ and $\Omega(\lambda(n))$ is also $0.5(\log\log n)^2$. Finally, since

$$2^{\omega(\lambda(n))} \le \tau(\lambda(n)) \le 2^{\Omega(\lambda(n))}$$

and since the normal order of $\omega(n)$ is $\log\log n = 2^{o((\log\log n)^2)}$, the desired inequalities follow. $\square$

7. Remarks

We suspect that for every $k \ge 1$ there exist constants $a_k > 0$ and $c_k \ge 0$ such that

$$\#\mathcal{A}_k(x) = a_k(1+o(1))\frac{x(\log\log x)^{c_k}}{(\log x)^2} \quad \text{as } x \to \infty. \tag{24}$$

Widely believed conjectures concerning the distribution of Sophie Germain primes p together with Proposition 3 seem to support conjecture (24) at $k = 1$ (with $c_1 = 0$ and some $a_1 > 0$). Note that an upper bound for this shape is given in Theorem 3.

We would like to leave this conjecture as an open problem.

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