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The Trace of an Automorphism on $H^0(J, \mathcal{O}(n\Theta))$

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1. Introduction

Let X be a projective smooth complex curve with group of automorphisms G. Let J be the Jacobian of X and let Θ be the theta divisor of J. Then G acts on J and Θ is invariant under the action of G. Given $h \in G$, our goal is to compute the trace of h on $H^0(J, \mathcal{O}(n\Theta))$ in order to decompose this space into a sum of irreducible representations of G. Dolgachev computed the decomposition of $H^0(J, \mathcal{O}(2\Theta))$ when X is the Klein quartic and used it to study some invariant vector bundles on this curve; see the proof of Corollary 6.3 in [4].

The strategy is as follows. Consider the exact sequence

$$0 \to \mathcal{O}(n\Theta) \to \mathcal{O}((n+1)\Theta) \to \mathcal{O}_\Theta((n+1)\Theta) \to 0. \tag{1}$$

By the Kodaira vanishing theorem we have

$$H^0(J, \mathcal{O}((n+1)\Theta)) = H^0(J, \mathcal{O}(n\Theta)) \oplus H^0(\Theta, \mathcal{O}((n+1)\Theta))$$

for $n \geq 1$. Then, all we need to do is to compute the decomposition for $H^0(\Theta, \mathcal{O}(n\Theta))$. The problem can be reduced to work with $H^0(S^{g-1}X, K_{S^{g-1}X}^{\otimes n})$, where $S^{g-1}X$ is the $g-1$ symmetric product of X, g is the genus of X, and $K_{S^{g-1}X}$ is the canonical line bundle of $S^{g-1}X$ (see Lemma 2.3). Now, to compute the trace of $h \in G$, we use the holomorphic Lefschetz theorem. There is no problem in applying this theorem if $\langle h\rangle \setminus \{1\}$ is contained in a conjugacy class of G (see Proposition 3.2), and the problem in general is how to compute the characteristic classes required in the theorem. If the fixed point set of h in $S^{g-1}X$ is finite, then it is still possible to compute the trace of h. If the components of the fixed point set of h in $S^{g-1}X$ have dimension at most 1, then by studying the function field of X one could proceed as in the example of [14] to compute the characteristic classes. We do not need to do the last in our examples; in fact, we have written a *Maple* program to compute the trace of h on $H^0(S^bX, K_{S^bX}^{\otimes n})$ when $\langle h\rangle \setminus \{1\}$ is contained in a conjugacy class of G. The program was used in our examples and can be obtained from me upon request. Our main results are Theorem 3.3 and the decomposition of $H^0(J, \mathcal{O}(n\Theta))$ for the Klein quartic, the Macbeath curve of genus 7, and the Bring curve of genus 4. This work is based on results from my thesis [13].

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2. The Symmetric Products

The aim of this section is to prove Lemma 2.3, which will allow us to apply the holomorphic Lefschetz theorem on the symmetric products of X rather than on the Jacobian. We start by mentioning some facts about the symmetric products of curves (see [12] for more details). In the cohomology of S^bX there are the classes $\eta, \vartheta, \sigma_i$ satisfying the following relations: $\vartheta = \sum_{i=1}^{g} \sigma_i$, $\sigma_i\sigma_j = \sigma_j\sigma_i$, and $\sigma_i^2 = 0$. For $b = a + d$ $(a, d \geq 0)$ and distinct $i_1, \ldots, i_a$, we have $\sigma_{i_1}\sigma_{i_2} \cdots \sigma_{i_a}\eta^d = \eta^b$. Then, if $a + d = b$,

$$\vartheta^a \eta^d = a! \binom{g}{a} \eta^b. \tag{2}$$

In general, the total Chern class of the tangent bundle of S^bX is given by

$$(1 + \eta t)^{(b-g+1)} e^{-\vartheta t/(1+\eta t)}, \tag{3}$$

where g is the genus of X (see [1, p. 339]).

Let $\alpha\colon S^{g-1}X \to \Theta$ be the Abel–Jacobi map and let $K = \alpha^* \mathcal{O}_\Theta(\Theta)$.

LEMMA 2.1. *We have* $K^n = K_{S^{g-1}X}^{\otimes n}$.

Proof. See [1, p. 258]. □

LEMMA 2.2. $\chi(K_{S^{g-1}X}^{\otimes n}) = n^g - (n-1)^g$.

Proof (see [15, Prop. 10.1(3)]). We have

$$\begin{aligned} td(S^dX) &= \left(\frac{\eta}{1 - e^{-\eta}}\right)^{d-g+1} \prod_{i=1}^{g} (1 + \sigma_i \tau) \\ &= \left(\frac{\eta}{1 - e^{-\eta}}\right)^{d-g+1} \sum_{i=0}^{g} \frac{\tau^i \vartheta^i}{i!}, \end{aligned} \tag{4}$$

where

$$\tau = \frac{\eta e^{-\eta} + e^{-\eta} - 1}{\eta(1 - e^{-\eta})}.$$

By formula (3), the Chern class of $K_{S^{g-1}X}^{\otimes n}$ is $1 + n\vartheta$; hence the Chern character of $K_{S^{g-1}X}^{\otimes n}$ is

$$\operatorname{ch}(K_{S^{g-1}X}^{\otimes n}) = e^{n\vartheta} = \prod_{i=1}^{g} (1 + n\sigma_i).$$

So, by Hirzebruch–Riemann–Roch:

$$\chi(K_{S^{g-1}X}^{\otimes n}) = \deg\left\{ \prod_{i=1}^{g} (1 + \sigma_i(\tau + n)) \right\}_{g-1}.$$

Notice that none of the terms in the expression

$$\prod_{i=1}^{g}(1+\sigma_i(\tau+n))$$

is divisible by a square of any σ_i, so we can assume $\sigma_1 = \cdots = \sigma_g = \eta$. Then what we want to compute is the coefficient of η^{g-1} in the following expression:

$$(1+\eta(\tau+n))^g = \left(\frac{\eta}{1-e^{-\eta}}\right)^g (n-(n-1)e^{-\eta})^g;$$

that is,

$$\chi(K_{S^{g-1}X}^{\otimes n}) = \operatorname*{Res}_{\eta=0}\left(\frac{n-(n-1)e^{-\eta}}{1-e^{-\eta}}\right)^g.$$

Then, setting $z = 1-e^{-\eta}$ yields

$$\operatorname*{Res}_{z=0}\frac{((n-1)z+1)^g}{z^g(1-z)} = \sum_{i=0}^{g-1}\binom{g}{i}(n-1)^i = n^g-(n-1)^g. \qquad \square$$

LEMMA 2.3. *For $n \geq 2$ we have $H^i(\Theta, \mathcal{O}(n\Theta)) \cong H^i(S^{g-1}X, K^n)$.*

We need Theorem 2.4 to prove this lemma.

A line bundle λ on a variety X is called *semi-ample* if, for some $\mu > 0$, the sheaf λ^μ is generated by global sections. Let X be a projective variety and let λ be an invertible sheaf on X. If $H^0(X, \lambda^\mu) \neq 0$, then the sections of λ^μ define a rational map

$$\phi_\mu = \phi_{\lambda^\mu}\colon X \to \mathbb{P}(H^0(X, \lambda^\mu)^*).$$

The *Iitaka dimension* $\kappa(\lambda)$ of λ is given by

$$\kappa(\lambda) = \begin{cases} -\infty & \text{if } H^0(X,\lambda^\mu) = 0\ \forall\mu, \\ \operatorname{Max}\{\dim\phi_\mu(X) \mid H^0(X,\lambda^\mu)\neq 0\} & \text{otherwise.}\end{cases}$$

THEOREM 2.4. *Let X be a projective manifold defined over a field K of characteristic* 0, *and let λ be an invertible sheaf on X. If λ is semi-ample and $\kappa(\lambda) = n = \dim X$, then*

$$H^b(X, \lambda^{-1}) = 0 \quad \textit{for } b < n.$$

Proof. See [8, Cor. 5.6(b)]. $\square$

Proof of Lemma 2.3. Notice that, since α is surjective, the natural map

$$\alpha^*\colon H^0(\Theta, \mathcal{O}(n\Theta)) \to H^0(S^{g-1}X, \alpha^*\mathcal{O}(n\Theta))$$

is injective. From the exact sequence (1) we see that $h^0(\Theta, \mathcal{O}(n\Theta)) = n^g-(n-1)^g$; thus, by Lemma 2.2, $\chi(K^n) = h^0(\Theta, \mathcal{O}(n\Theta))$. On the other hand, since $\mathcal{O}(n\Theta)$ is ample, $\alpha^*\mathcal{O}_\Theta(n\Theta)$ is semi-ample. Notice that the Iitaka dimension of K^n is $g-1 = \dim S^{g-1}X$, because α is a birational map between $S^{g-1}X$ and Θ. Then, by Theorem 2.4 and the Serre duality theorem,

$$H^i(S^{g-1}X, \alpha^*\mathcal{O}(n\Theta) \otimes K_{S^{g-1}X}) = 0 \quad \text{for } i > 0 \text{ and } n \geq 1.$$

Hence α^* is an isomorphism for $n \geq 2$. □

3. The Fixed Point Theorem

Let E be a vector bundle on a smooth variety X, and let G be a finite group acting on X. We say that G acts on E if, for each $g \in G$, there is an isomorphism of vector bundles $\phi_g : g^*E \to E$ such that, given $g, h \in G$, we have $\phi_{h \cdot g} = \phi_g \circ g^*(\phi_h)$ (see the definition of G-linearized vector bundle in [4]).

Suppose that X is a variety with a trivial action of a finite group G, that is, suppose every element of G acts as the identity. Let $V_1, \dots, V_m$ be the complex irreducible representations of G. Then any vector bundle E on X with action of G is isomorphic to a vector bundle of the form

$$\bigoplus_i V_i \otimes E_i,$$

where E_i is a unique vector bundle (which, of course, depends on E) with trivial action of G. For $h \in G$ and E as before, define

$$\mathrm{ch}_h(E) = \sum_i \chi_i(h) \cdot \mathrm{ch}(E_i), \tag{5}$$

where $\mathrm{ch}(E_i)$ is the Chern character of E_i and $\chi_i(h)$ represents the trace of $h|_{V_i}$ (see definition of $\mathrm{ch}\, u(g)$ in [2] just before 3.1).

If G acts on E and $h \in G$ acts trivially on X, then E has a decomposition $E = \bigoplus_{i=0}^{p} E(\nu^i)$, where $E(\nu^i)$ is the subvector bundle of E on which h acts as ν^j ($\nu = \mathbf{e}^{2i\pi/p}$) and p is the order of h. For each vector bundle $E(\nu^i)$, define the characteristic class

$$\mathcal{U}(E(\nu^i)) = \prod_j \left(\frac{1 - e^{-x_j}/\nu^i}{1 - 1/\nu^i} \right)^{-1}, \tag{6}$$

where $\{x_j\}_j$ are the Chern roots of $E(\nu^i)$; see [2, (4.5)].

THEOREM 3.1 (Holomorphic Lefschetz Theorem; see [2, Thm. 4.6]). *Let X be a compact complex manifold, V a holomorphic vector bundle over X, and h a finite-order automorphism of the pair (X, V). Let X^h denote the fixed point set of h and let $N^h = \bigoplus_{j=1}^{p-1} N(\nu^j)$ be the normal bundle of X^h decomposed according to the eigenvalues of h. Then*

$$\sum (-1)^i \operatorname{tr}(h|_{H^i(X,V)}) = \int_{X^h} \frac{\mathrm{ch}_h(V|_{X^h}) \cdot \prod_j \mathcal{U}(N(\nu^j)) \cdot td(X^h)}{\det(1 - (h|_{(N^h)^\vee})}.$$

Notice that Theorem 3.1 is a generalization of the Atiyah–Bott fixed point theorem.

Now let h be an automorphism of our curve X, and assume that h has order $p \neq 1$. Let b be a positive integer and choose integers m, l such that $b = mp + l$ with $m \geq 0$ and $0 \leq l < p$. From [14] we know that the components of the fixed point set of h in S^bX are isomorphic to S^kY, where $Y \cong X/\langle h\rangle$ and $0 \le k \le m$. The components of dimension k are parameterized by a set A_k of invariant divisors of degree $d_k = b - pk$; namely, for each $D \in A_k$ we have an embedding

$$i_D\colon S^kY \overset{i}{\hookrightarrow} S^{pk}X \overset{\mathcal{A}_D}{\hookrightarrow} S^{pk+d_k}X, \tag{7}$$

where i sends $Z \in S^kY$ to the divisor $f^*Z \in S^{pk}X$ ($f\colon X \to Y = X/\langle h\rangle$ is the quotient map) and $\mathcal{A}_D$ sends $Z \in S^{pk}X$ to $Z + D \in S^{pk+d_k}X$.

Let K denote the canonical line bundle of S^bX and let N_D be the normal bundle of the component of the fixed point set of h in S^bX corresponding to the divisor $D \in A_k$. Then

$$L(h, K^n) := \sum (-1)^i \operatorname{tr}(h|_{H^i(S^bX, K^n)}) = \sum_{k=0}^{m} \sum_{D \in A_k} \lambda(k, D),$$

where

$$\lambda(k, D) = \int_{S^kY} \frac{\operatorname{ch}_h(i_D^* K^n) \cdot \prod_j \mathcal{U}(N_D(\nu^j)) \cdot td(S^kY)}{\det(1 - h|_{N_D^\vee})}.$$

Let $N_{\mathcal{A}_D}$ be the normal bundle $N_{S^{pk}X/S^bX}$ with respect to the embedding $\mathcal{A}_D$ in (7). Given $D \in A_k$, define the *class* of D to be the vector $(r_1, \dots, r_{p-1})$, where r_j is the rank of $i^*N_{\mathcal{A}_D}(\nu^j)$.

Proposition 3.2. *Let $(r_1, \dots, r_{p-1})$ be the class of the divisor $D \in A_k$. Then*

(a) $\det(1 - h|_{N_D^\vee}) = p^k \prod_{j=1}^{p-1} (1 - \nu^{p-j})^{r_j}$ *and*

(b) $\operatorname{ch}_h(i_D^* K^n) = \nu^{n\alpha} e^{[(g-1-b)\eta + p\vartheta]nt}$,

where

$$\alpha = -k\frac{p(p-1)}{2} - \sum_{j=1}^{p-1} j r_j.$$

If $\langle h\rangle \setminus \{1\}$ is contained in a conjugacy class of $\operatorname{Aut}(X)$, *then*

$$\prod_{j=1}^{p-1} \mathcal{U}(N_D(\nu^j)) = p^A m(e^{-\eta t})^{-A} e^{t\vartheta q(e^{-\eta t})} \prod_{j=1}^{p-1} \left(\frac{1 - e^{-\eta t}/\nu^j}{1 - 1/\nu^j} \right)^{-r_j},$$

where $m(z) = \sum_{i=0}^{p-1} z^i$, $q(z) = -zm'(z)/m(z)$, $A = k + (\gamma - g)/(p-1)$, and γ is the genus of the quotient curve Y.

Proof. The action of h on i_D^*K is multiplication by $\det(h|_{i_D^* T_{S^bX}})^{-1}$. By Remark 3.4 in [14] we have part (a) and $\det(h|_{i_D^* T_{S^bX}})^{-1} = \nu^\alpha$. The Chern class of K_{S^bX} is $1 + [(g-1-b)\eta + \vartheta]t$ and so, using [14, Lemma 2.2], we see that i_D^*K has Chern class $1 + [(g-1-b)\eta + p\vartheta]t$. Thus the Chern character is given by $\operatorname{ch}(i_D^* K^{\otimes n}) = e^{[(g-1-b)\eta + p\vartheta]nt}$ and now we can use formula (5). The last formula is just [14, Thm. 3.8]. □

Theorem 3.3. *Let X be a hyperelliptic curve of genus g, and let h be the involution of X. Then, for $n \ge 1$, we have*

$$\operatorname{tr}(h|_{H^0(J,\mathcal{O}(n\Theta))}) = 1 + 2^{-g} \sum_{k=0}^{[(g-1)/2]} B(n,k)(1-(-1)^{k+1}) \binom{2g+2}{g-1-2k},$$

where

$$B(n,k) = n\frac{(-1)^{g-k-1}+1}{2} + \frac{(-1)^{n(g-k-1)}-1}{2} + (-1)^{g-k}.$$

Proof. In this case, the quotient curve is $\mathbb{P}^1$ and h has $2g+2$ fixed points on X. We assume $b = g-1$. Each set A_k has $\binom{2g+2}{g-1-2k}$ elements, and all the divisors have the same class because there is only one eigenvalue; in fact, $r_1 = g-1-2k = \operatorname{rank} i^* N_{A_D}(-1)$. Because $\mathbb{P}^1$ has genus 0, the ϑ class is 0 in the cohomology ring of $S^k\mathbb{P}^1 \cong \mathbb{P}^k$ and η is the class of a hyperplane. We have

$$\sum_{D\in A_k} \lambda(k,D) = \int_{S^k\mathbb{P}^1} 2^{-g}(-1)^{-n(g-1-k)} \binom{2g+2}{g-1-2k} \left(\frac{1+e^{-\eta}}{1-e^{-\eta}}\right)^{k+1} \eta^{k+1}.$$

Now, $\int_{S^k\mathbb{P}^1} \left(\frac{1+e^{-\eta}}{1-e^{-\eta}}\right)^{k+1} \eta^{k+1}$ is the coefficient of η^k in $\left(\frac{1+e^{-\eta}}{1-e^{-\eta}}\right)^{k+1} \eta^{k+1}$. Hence

$$\begin{aligned}\int_{S^k\mathbb{P}^1} \left(\frac{1+e^{-\eta}}{1-e^{-\eta}}\right)^{k+1} \eta^{k+1} &= \operatorname*{Res}_{\eta=0} \left(\frac{1+e^{-\eta}}{1-e^{-\eta}}\right)^{k+1} \\ &= \operatorname*{Res}_{z=0} \left(\frac{2-z}{z}\right)^{k+1} \frac{dz}{1-z} = 1-(-1)^{k+1},\end{aligned}$$

and thus we have

$$L(h,K^n) = 2^{-g} \sum_{k=0}^{[(g-1)/2]} (-1)^{-n(g-1-k)}(1-(-1)^{k+1}) \binom{2g+2}{g-1-2k}.$$

Using the exact sequence (1) now yields $\operatorname{tr}(h|_{H^0(J,\mathcal{O}(2\Theta))}) = L(h,K^2)+1$, and the theorem follows by induction. □

With respect to this involution, we have $H^0(J,\mathcal{O}(n\Theta)) = \mathbb{C}^{\alpha(n)} \oplus V^{\beta(n)}$, where $\mathbb{C}$ and V are the one-dimensional representations on which h acts as 1 and -1 (respectively) and where $\alpha(n) = \frac{1}{2}[n^g + \operatorname{tr}(h|_{H^0(J,\mathcal{O}(n\Theta))})]$ and $\beta(n) = \frac{1}{2}[n^g - \operatorname{tr}(h|_{H^0(J,\mathcal{O}(n\Theta))})]$.

Next we compute the traces of automorphisms of specific curves—namely, the Klein quartic, the Macbeath curve of genus 7, and the Bring curve of genus 4. This will enable us to decompose $H^0(J,\mathcal{O}(n\Theta))$ into a sum of irreducible representations of the automorphism group of X.

Notice that, if $D \in A_k$ is supported on the fixed points of h in X, then the class of D can be computed using Remark 3.4 in [14]. So, if $\langle h\rangle \setminus \{1\}$ is contained in a conjugacy class of $\operatorname{Aut}(X)$, then $L(h,K^n)$ is completely determined by the following information:

- the dimension b of the symmetric product S^bX;
- the order p of the automorphism h;
- the genus g of the curve X;
- the number s of fixed points of h in the curve X; and

- a vector $\mathbf{u}_h = (a_1, \dots, a_s)$ in which a_i is a positive integer such that the automorphism h acts as ν^{a_i} on the tangent space T_{x_i} of the fixed point $x_i \in X$ ($\nu = e^{2i\pi/p}$).

This data is not enough in general; see, for instance, the example in [14].

NOTATION. If V is a representation of G and $h \in G$ belongs to the conjugacy class $**$, then we will write tr$**$ to denote the trace of h on V. Notice that, if $V = H^0(J, \mathcal{O}(n\Theta))$, then for $n \geq 1$ we have $\mathrm{tr}** = \sum(-1)^i \operatorname{tr}(h|_{H^i(J,\mathcal{O}(n\Theta))})$.

4. The Klein Quartic

Let X be the Klein quartic curve (see [7] for more details). This is a genus-3 curve with automorphism group $G = \mathbf{PSL}_2(\mathbb{F}_7)$. This group has six conjugacy classes: say 1A, 2A, 3A, 4A, 7A, 7B. If $h \in G$, then we derive the following tabulation.

Conjugacy class of h	Number of fixed points	u_h
2A	4	$(1, 1, 1, 1)$
3A	2	$(1, 2)$
4A	0	—
7A	3	$(1, 2, 4)$
7B	3	$(3, 5, 6)$

For h in 4A, let $p_1, \dots, p_4$ be the four fixed points of h^2 in X. We can assume that $p_3 = hp_1$ and $p_4 = hp_2$. Then the fixed points of h in S^2X are $p_1 + hp_1$ and $p_2 + hp_2$. We have $(T_{S^2X})_{p_1+hp_1} = (T_X)_{p_1} \oplus (T_X)_{hp_1}$, and h induces the two linear maps $\alpha\colon (T_X)_{p_1} \to (T_X)_{hp_1}$ and $\beta\colon (T_X)_{hp_1} \to (T_X)_{p_1}$. Then the automorphism induced on $(T_{S^2X})_{p_1+hp_1}$ has a matrix conjugate to $A = \left(\begin{smallmatrix} 0 & a \\ b & 0 \end{smallmatrix}\right)$.

Since $p_1 + hp_1$ is a fixed point of $h^2 \in 2\mathrm{A}$, we see that $A^2 = -\mathrm{Id}_{(T_{S^2X})_x}$. Then we see that A is conjugate to $\left(\begin{smallmatrix} i & * \\ 0 & -i \end{smallmatrix}\right)$. That is, the divisors $p_1 + hp_1$ and $p_2 + hp_2$ have class $(1, 0, 1)$.

It is not hard to compute $L(h, K^n)$ for this curve. Let $\zeta = e^{2\pi i/7}$; then, by induction, we obtain the values in Table 1. The trace for $h \in 7\mathrm{B}$ is the complex conjugate of the trace of an automorphism in 7A.

Now let $\chi_1, \chi_3, \bar{\chi}_3, \chi_6, \chi_7, \chi_8$ be the irreducible representations of G. If $V = \chi_1^a \oplus \chi_3^b \oplus \bar{\chi}_3^c \oplus \chi_6^d \oplus \chi_7^e \oplus \chi_8^f$ then, from the character table of G (see [3]), we have

$$
\begin{array}{cccccccl}
a & +3b & +3c & +6d & +7e & +8f & = & \mathrm{tr}\,1\mathrm{A} \\
a & -b & -c & +2d & -e & & = & \mathrm{tr}\,2\mathrm{A} \\
a & & & & +e & -f & = & \mathrm{tr}\,3\mathrm{A} \\
a & +b & +c & & -e & & = & \mathrm{tr}\,4\mathrm{A} \\
a & +\alpha b & +\bar{\alpha} c & -d & & +f & = & \mathrm{tr}\,7\mathrm{A} \\
a & +\bar{\alpha} b & +\alpha c & -d & & +f & = & \mathrm{tr}\,7\mathrm{B}
\end{array}
$$

Table 1

Conjugacy class of h	$\sum(-1)^i \operatorname{tr}\big(h\vert_{H^i(J,\mathcal{O}(n\Theta))}\big)$
1A	n^3
2A	$n(3+(-1)^n)/2$
3A, 4A	n
7A	$\big(-\frac{2}{7}\zeta^3-\frac{4}{7}\zeta^4-\frac{4}{7}\zeta^5-\frac{4}{7}\big)(\zeta^{n+1})^4 + \big(\frac{2}{7}\zeta^5+\frac{4}{7}\zeta^4+\frac{4}{7}\zeta+\frac{4}{7}\zeta^3\big)(\zeta^{n+1})^2 + \big(-\frac{2}{7}\zeta+\frac{2}{7}\zeta^5-\frac{2}{7}\zeta^3+\frac{2}{7}\zeta^4+\frac{2}{7}\zeta^2-\frac{2}{7}\big)\zeta^{n+1} - \frac{2}{7}\zeta-\frac{1}{7}-\frac{2}{7}\zeta^4-\frac{2}{7}\zeta^2$

where $\alpha = \big(-1+i\sqrt{7}\big)/2$. The general solution of this system of equations is given by

$$\begin{pmatrix} a\\ b\\ c\\ d\\ e\\ f\end{pmatrix} = \begin{pmatrix} \frac{\operatorname{tr}1\mathrm{A}}{168}+\frac{\operatorname{tr}2\mathrm{A}}{8}+\frac{\operatorname{tr}7\mathrm{B}}{7}+\frac{\operatorname{tr}7\mathrm{A}}{7}+\frac{\operatorname{tr}4\mathrm{A}}{4}+\frac{\operatorname{tr}2\mathrm{A}}{3}\\ -\frac{\operatorname{tr}2\mathrm{A}}{8}+\frac{\operatorname{tr}1\mathrm{A}}{56}-\frac{i\sqrt{7}\operatorname{tr}7\mathrm{A}}{14}+\frac{i\sqrt{7}\operatorname{tr}7\mathrm{B}}{14}-\frac{\operatorname{tr}7\mathrm{B}}{14}-\frac{\operatorname{tr}7\mathrm{A}}{14}+\frac{\operatorname{tr}4\mathrm{A}}{4}\\ \frac{\operatorname{tr}4\mathrm{A}}{4}-\frac{\operatorname{tr}7\mathrm{A}}{14}+\frac{i\sqrt{7}\operatorname{tr}7\mathrm{A}}{14}-\frac{\operatorname{tr}7\mathrm{B}}{14}-\frac{i\sqrt{7}\operatorname{tr}7\mathrm{B}}{14}-\frac{\operatorname{tr}2\mathrm{A}}{8}+\frac{\operatorname{tr}1\mathrm{A}}{56}\\ \frac{\operatorname{tr}1\mathrm{A}}{28}+\frac{\operatorname{tr}2\mathrm{A}}{4}-\frac{\operatorname{tr}7\mathrm{B}}{7}-\frac{\operatorname{tr}7\mathrm{A}}{7}\\ \frac{\operatorname{tr}2\mathrm{A}}{3}+\frac{\operatorname{tr}1\mathrm{A}}{24}-\frac{\operatorname{tr}2\mathrm{A}}{8}-\frac{\operatorname{tr}4\mathrm{A}}{4}\\ \frac{\operatorname{tr}7\mathrm{B}}{7}+\frac{\operatorname{tr}1\mathrm{A}}{21}+\frac{\operatorname{tr}7\mathrm{A}}{7}-\frac{\operatorname{tr}2\mathrm{A}}{3} \end{pmatrix}.$$

Then, if $H^0(J,\mathcal{O}(n\Theta)) = \chi_1^{a(n)}\oplus\chi_3^{b(n)}\oplus\bar{\chi}_3^{c(n)}\oplus\chi_6^{d(n)}\oplus\chi_7^{e(n)}\oplus\chi_8^{f(n)}$, it follows that for $n=1,\dots,10$ we have

$$\begin{pmatrix} a(n)\\ b(n)\\ c(n)\\ d(n)\\ e(n)\\ f(n)\end{pmatrix} = \begin{pmatrix} 1&2&2&4&4&6&7&10&11&14\\ 0&0&1&1&3&4&6&9&14&18\\ 0&0&1&1&3&4&8&9&14&18\\ 0&1&2&4&6&11&14&22&28&41\\ 0&0&1&2&5&8&14&20&30&40\\ 0&0&0&2&4&8&14&22&32&44 \end{pmatrix}.$$

5. The Macbeath Curve of Genus 7

There exists a Hurwitz curve of genus 7 with group of automorphisms $G = \mathbf{PSL}_2(\mathbb{F}_8)$. Equations for this curve were first computed in [10] by Macbeath, and we refer to his paper for more details. The group G is simple and has 504 elements. There are 9 conjugacy classes: 1A, 2A, 3A, 7A, 7B*2, 7C*4, 9A, 9B*2,

Table 2

Conjugacy class of h	$\sum(-1)^i \operatorname{tr}\big(h\vert_{H^i(J,\mathcal{O}(n\Theta))}\big)$
1A	n^7
2A	$n^3(3+(-1)^n)/2$
3A	$\frac{8}{3}w^n n + \frac{8}{3}(w^n)^2 n + \frac{11}{3}n$
7A, 7B, 7C, 9A, 9B, 9C	n

and 9C*4. An element in each class has order $1, 2, 3, 7, 7, 7, 9, 9, 9$ respectively. Letting $h \in G$ then yields the following tabulation.

Conjugacy class of h	Number of fixed points	u_h
2A	4	$(1, 1, 1, 1)$
3A	6	$(1, 1, 1, 2, 2, 2)$
7A	2	$(\alpha, 7-\alpha)$
7B	2	$(2\alpha, 14-2\alpha)$
7C	2	$(4\alpha, 28-4\alpha)$
9A, 9B, 9C	0	—

Suppose that $h \in G$ has order 7, and let $H = \langle h \rangle$. Since h has two fixed points in X, the normalizer $N(H)$ of H has order 14. Let $t \in N(H)$ be of order 2. We have $N(H) = \langle t, h \rangle$, so if p_1 is a fixed point of h then the other fixed point is tp_1. Now $tht = h^k$. Observe that $k \neq 1$, for otherwise $N(H)$ would be cyclic and there would be an element of order 14 in G. Hence $k \equiv -1 \bmod 7$. From this we see that, if h acts as ζ^α ($\zeta = e^{2i\pi/7}$) on T_{p_1}, then h acts as $\zeta^{-\alpha}$ on T_{tp_1}. The value of $\alpha \in \{1, 2, 3, 4, 5, 6\}$ depends on the conjugancy class of h, although in this case $L(h, K^n)$ is independent of the value of α.

Now suppose that $z \in G$ has order 9. The six fixed points of z^3 in X have the form $p_1, zp_1, z^2p_1, p_2, zp_2, z^2p_2$, where z^3 acts on T_{p_1} as ω and as ω^2 on T_{p_2}. The fixed point set of z in S^6X consists of the three points $2p_1 + 2zp_1 + 2z^2p_1$, $2p_2 + 2zp_2 + 2z^2p_2$, and $p + zp + z^2p + p_2 + zp_2 + z^2p_2$. Similarly to the case 4A in the Klein quartic example (and to the proof of Lemma 3.1 in [14]), one can see that the matrix corresponding to the action of h on the tangent space of these divisors has characteristic polynomial $q(\lambda) = (\lambda^3 - \omega)(\lambda^3 - \omega^2)$. That is, the three divisors have class $(1, 1, 0, 1, 1, 0, 1, 1)$.

Now we can compute $L(h, K^n)$ and use induction to obtain the values in Table 2.

Let $H^0(J, \mathcal{O}(n\Theta)) = \mathbb{C}^{a_1(n)} \oplus V_2^{a_2(n)} \oplus V_3^{a_3(n)} \oplus \cdots \oplus V_9^{a_9(n)}$. From the character table of G (see [3]), we obtain a system of linear equations whose solutions are

Table 3 Character table of S_5

	1a	2a	2b	3a	6a	4a	5a
χ_1	1	1	1	1	1	1	1
χ_2	1	−1	1	1	−1	−1	1
χ_3	4	−2	0	1	1	0	−1
χ_4	4	2	0	1	−1	0	−1
χ_5	5	1	1	−1	1	−1	0
χ_6	5	−1	1	−1	−1	1	0
χ_7	6	0	−2	0	0	0	1

$$
\begin{aligned}
a_1(n) &= \tfrac{16}{21}\operatorname{tr} 7\mathrm{A} + \tfrac{1}{9}\operatorname{tr} 3\mathrm{A} + \tfrac{1}{504}\operatorname{tr} 1\mathrm{A} + \tfrac{1}{8}\operatorname{tr} 2\mathrm{A},\\
a_2(n) &= \tfrac{1}{72}\operatorname{tr} 1\mathrm{A} - \tfrac{1}{8}\operatorname{tr} 2\mathrm{A} + \tfrac{1}{3}\operatorname{tr} 7\mathrm{A} - \tfrac{2}{9}\operatorname{tr} 3\mathrm{A},\\
a_3(n) &= a_4(n) = a_5(n) = \tfrac{1}{9}\operatorname{tr} 3\mathrm{A} + \tfrac{1}{72}\operatorname{tr} 1\mathrm{A} - \tfrac{1}{8}\operatorname{tr} 2\mathrm{A},\\
a_6(n) &= \tfrac{2}{21}\operatorname{tr} 7\mathrm{A} + \tfrac{1}{63}\operatorname{tr} 1\mathrm{A} - \tfrac{1}{9}\operatorname{tr} 3\mathrm{A},\\
a_7(n) &= a_8(n) = a_9(n) = -\tfrac{1}{7}\operatorname{tr} 7\mathrm{A} + \tfrac{1}{56}\operatorname{tr} 1\mathrm{A} + \tfrac{1}{8}\operatorname{tr} 2\mathrm{A}.
\end{aligned}
$$

For the first ten values of n we have

$$
\begin{pmatrix} a_1(n)\\ a_2(n)\\ a_3(n)\\ a_4(n)\\ a_5(n)\\ a_6(n)\\ a_7(n)\\ a_8(n)\\ a_9(n) \end{pmatrix}
=
\begin{pmatrix}
1 & 4 & 13 & 52 & 175 & 620 & 1683 & 4296 & 9597 & 20100\\
0 & 0 & 22 & 212 & 1070 & 3824 & 11396 & 29000 & 66324 & 138640\\
0 & 0 & 30 & 212 & 1070 & 3840 & 11396 & 29000 & 66348 & 138640\\
0 & 0 & 30 & 212 & 1070 & 3840 & 11396 & 29000 & 66348 & 138640\\
0 & 0 & 30 & 212 & 1070 & 3840 & 11396 & 29000 & 66348 & 138640\\
0 & 2 & 32 & 260 & 1240 & 4438 & 13072 & 33288 & 75912 & 158730\\
0 & 4 & 42 & 308 & 1410 & 5052 & 14748 & 37576 & 85500 & 178820\\
0 & 4 & 42 & 308 & 1410 & 5052 & 14748 & 37576 & 85500 & 178820\\
0 & 4 & 42 & 308 & 1410 & 5052 & 14748 & 37576 & 85500 & 178820
\end{pmatrix}.
$$

6. The Bring Curve of Genus 4

The Bring curve is the only genus-4 curve admitting the symmetric group $G = S_5$ as its group of automorphisms. Some information about this curve can be found in [5; 6; 16]. This curve can be defined in $\mathbb{P}^4$ using the equations

$$\sum_{i=1}^{5} x_i = 0, \quad \sum_{i=1}^{5} x_i^2 = 0, \quad \sum_{i=1}^{5} x_i^3 = 0.$$

The group acts by permuting coordinates. One can use [9] to produce Table 3, the character table of S_5, and some information about its subgroups. There are seven conjugacy classes for G: $\overline{1}$, $\overline{(1,2)}$, $\overline{(1,2)(3,4)}$, $\overline{(1,2,3)}$, $\overline{(1,2,3)(4,5)}$, $\overline{(1,2,3,4)}$, $\overline{(1,2,3,4,5)}$ of orders $1,2,2,3,6,4,5$ and sizes $1,10,15,20,20,30,24$ respectively.

Denote by 1a, 2a, 2b, 3a, 6a, 4a, and 5a the conjugacy classes of G. The Fuchsian group that yields X and G has period partition $(2,4,5)$. Again letting $h \in G$, we have the following tabulation.

Conjugacy class of h	Number of fixed points	u_h
2a	6	$(1,1,1,1,1,1)$
2b	2	$(1,1)$
3a	0	—
6a	0	—
4a	2	$(i,-i)$
5a	4	$(1,2,3,4)$

The normalizer of $\langle(1,2)(3,4)\rangle$ is $H = \langle(3,4),(1,2),(1,3)(2,4)\rangle$ and has eight elements. An element in 2b is the square of an element in 4a, so it is the image of an element in a maximal cyclic subgroup of order 4 of the Fuchsian group that yields S_5 as the group of automorphisms of X. Then, by [11, Thm. 1], we see that there are two fixed points in X for an automorphism in 2b.

The normalizer of $\langle(1,2,3,4)\rangle$ is $\langle(1,2,3,4),(2,4)\rangle$ and also has eight elements. So if $h \in$ 4a then the two fixed points p_1, p_2 of h^2 are the fixed points of h in X. Since h and h^3 are conjugate to each other, we see that h acts as i and $-i$ on the tangent spaces of the two fixed points. The fixed points of h in S^3X are $3p_1$, $2p_1+p_2$, p_1+2p_2, and $3p_2$. From Remark 3.4 in [14] we see that these divisors have class $(1,1,1)$.

An automorphism in 6a has no fixed points in X. If $h \in$ 6a then $h^3 \in$ 2a. Hence the fixed points of h^3 in X are of the form p_1, hp_1, h^2p_1, p_2, hp_2, and h^2p_2; the fixed points of h in S^3X are $p_1+hp_1+h^2p_1$ and $p_2+hp_2+h^2p_2$. The matrix corresponding to the action of h on the tangent spaces at these divisors has characteristic polynomial $q(\lambda) = \lambda^3+1$, that is, the three divisors have class $(1,0,1,0,1)$.

The normalizer of $\langle(1,2,3,4,5)\rangle$ is $\langle(1,2,3,4,5),(2,5)(3,4),(2,4,5,3)\rangle$ and has twenty elements. So there are four fixed points in X for an automorphism h in 5a, and since the four powers of h belong to this same class we see that h acts as $\nu^1,\dots,\nu^4$ ($\nu = e^{2i\pi/5}$) on the tangent spaces of these points. Computing $L(h, K^n)$ and using induction, we obtain the values in Table 4.

Let $H^0(J, \mathcal{O}(n\Theta)) = \mathbb{C}^{a_1(n)} \oplus V_2^{a_2(n)} \oplus V_3^{a_3(n)} \oplus \cdots \oplus V_7^{a_7(n)}$. Then, from the character table of G, we have

$$\begin{pmatrix} a_1(n) \\ a_2(n) \\ a_3(n) \\ a_4(n) \\ a_5(n) \\ a_6(n) \\ a_7(n) \end{pmatrix} = \begin{pmatrix} \frac{\mathrm{tr\,1a}}{120} + \frac{\mathrm{tr\,6a}}{6} + \frac{\mathrm{tr\,5a}}{5} + \frac{\mathrm{tr\,4a}}{4} + \frac{\mathrm{tr\,2b}}{8} + \frac{\mathrm{tr\,2a}}{12} + \frac{\mathrm{tr\,3a}}{6} \\ -\frac{\mathrm{tr\,2a}}{12} + \frac{\mathrm{tr\,1a}}{120} - \frac{\mathrm{tr\,6a}}{6} + \frac{\mathrm{tr\,5a}}{5} - \frac{\mathrm{tr\,4a}}{4} + \frac{\mathrm{tr\,2b}}{8} + \frac{\mathrm{tr\,3a}}{6} \\ \frac{\mathrm{tr\,1a}}{30} + \frac{\mathrm{tr\,6a}}{6} - \frac{\mathrm{tr\,2a}}{6} + \frac{\mathrm{tr\,3a}}{6} - \frac{\mathrm{tr\,5a}}{5} \\ -\frac{\mathrm{tr\,6a}}{6} + \frac{\mathrm{tr\,1a}}{30} + \frac{\mathrm{tr\,2a}}{6} + \frac{\mathrm{tr\,3a}}{6} - \frac{\mathrm{tr\,5a}}{5} \\ -\frac{\mathrm{tr\,3a}}{6} + \frac{\mathrm{tr\,1a}}{24} + \frac{\mathrm{tr\,2b}}{8} - \frac{\mathrm{tr\,4a}}{4} + \frac{\mathrm{tr\,2a}}{12} + \frac{\mathrm{tr\,6a}}{6} \\ \frac{\mathrm{tr\,4a}}{4} + \frac{\mathrm{tr\,1a}}{24} - \frac{\mathrm{tr\,2a}}{12} + \frac{\mathrm{tr\,2b}}{8} - \frac{\mathrm{tr\,6a}}{6} - \frac{\mathrm{tr\,3a}}{6} \\ \frac{\mathrm{tr\,5a}}{5} + \frac{\mathrm{tr\,1a}}{20} - \frac{\mathrm{tr\,2b}}{4} \end{pmatrix}.$$

Table 4

Conjugacy class of h	$\sum(-1)^i \operatorname{tr}\big(h\vert_{H^i(J,\mathcal{O}(n\Theta))}\big)$
1a	n^4
2a	$\frac{3}{4}+\frac{5(-1)^n}{4}+\frac{3n^2}{2}$
2b, 3a	n^2
4a, 6a	$\frac{3}{2}+\frac{(-1)^n}{2}$
5a	$\frac{9}{5}+\frac{4(\nu^n)^4}{5}+\frac{4(\nu^n)^3}{5}+\frac{4(\nu^n)^2}{5} + \big(-\frac{4}{5}\nu^4-\frac{4}{5}\nu-\frac{4}{5}\nu^2-\frac{4}{5}\nu^3\big)\nu^n$

For $n=1,\dots,10$ we have

$$\begin{pmatrix} a_1(n)\\ a_2(n)\\ a_3(n)\\ a_4(n)\\ a_5(n)\\ a_6(n)\\ a_7(n)\end{pmatrix} = \begin{pmatrix} 1 & 3 & 5 & 10 & 17 & 27 & 41 & 62 & 89 & 127\\ 0 & 0 & 2 & 4 & 10 & 16 & 28 & 44 & 68 & 100\\ 0 & 0 & 2 & 7 & 18 & 40 & 76 & 131 & 212 & 324\\ 0 & 2 & 6 & 15 & 30 & 58 & 100 & 163 & 252 & 374\\ 0 & 1 & 4 & 12 & 28 & 57 & 104 & 176 & 280 & 425\\ 0 & 0 & 2 & 8 & 22 & 48 & 92 & 160 & 260 & 400\\ 0 & 0 & 2 & 9 & 26 & 56 & 108 & 189 & 308 & 476 \end{pmatrix}.$$

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