Michigan Math. J. 52 (2004)

The Minimal Marked Length Spectrum of Riemannian Two-Step Nilmanifolds

Ruth Gornet & Maura B. Mast

Introduction

The purpose of this paper is to compare the minimal marked length spectrum and the Laplace spectrum on functions and on forms for Riemannian two-step nilmanifolds. A Riemannian nilmanifold is a closed manifold of the form $(\Gamma\backslash G, g)$, where G is a simply connected nilpotent Lie group, Γ is a cocompact (i.e., $\Gamma\backslash G$ compact) discrete subgroup of G, and g arises from a left invariant metric on G. Examples of nilmanifolds include flat tori and Heisenberg manifolds. The Laplace spectrum of a closed Riemannian manifold (M, g) is the set of eigenvalues of the Laplace–Beltrami operator Δ, counted with multiplicity. The Laplace–Beltrami operator may be extended to act on smooth p-forms by $\Delta = d\delta + \delta d$, where δ is the metric adjoint of d. Two manifolds have the same marked length spectrum if there exists an isomorphism between the fundamental groups such that corresponding free homotopy classes of loops can be represented by smoothly closed geodesics of the same length. Two manifolds have the same minimal marked length spectrum (resp., maximal marked length spectrum) if there exists an isomorphism between the fundamental groups such that the smallest (resp., longest) closed loops in corresponding free homotopy classes have the same length.

The main theorem of this paper is the following (see Theorem 2.5).

Theorem 1. *For a generic class of two-step nilmanifolds, if a pair of nilmanifolds in this class has the same minimal marked length spectrum, then the nilmanifolds necessarily share the same Laplace spectrum on functions and on forms and must also have the same marked length spectrum.*

We prove Theorem 2.5 by showing that the mapping that induces the marking between the fundamental groups must take the form of an almost inner automorphism composed with an isomorphism that is also an isometry. Work of Gordon and Wilson [GW1; G1] shows that almost inner automorphisms preserve the marked length spectrum and also preserve the Laplace spectrum on functions; DeTurck and Gordon [DG] showed that the Laplace spectrum on forms is preserved in this case.

We also prove the result without the generic hypothesis (see Theorem 4.1) in the class of nilmanifolds with a two-dimensional center. See Remark 3.4 for

Received July 26, 2003. Revision received May 20, 2004.

The first author was supported in part by the National Science Foundation under Grant no. DMS-0204648 and by the Texas Advanced Research Program under Grant no. 03644-0115.

additional information concerning the case where the dimension of the center of N is greater than 2. We do not prove the extension of Theorem 1 to all two-step nilmanifolds, but we do make the following conjecture.

CONJECTURE. *If any pair of two-step nilmanifolds has the same minimal marked length spectrum, then the nilmanifolds necessarily share the same Laplace spectrum on functions and on forms and must also have the same marked length spectrum.*

Finally, we give examples of nilmanifolds that satisfy the generic hypothesis (see Examples 2.3 and 2.4) and that do not (Example 3.1). For details of the generic condition in the statement of Theorem 2.5, see Definition 2.2.

Theorems 2.5 and 4.1 extend the work of Eberlein, who proved the following in [E1] (see also Theorem 1.22).

THEOREM 2. *If a pair of two-step nilmanifolds has the same maximal marked length spectrum, then the nilmanifolds necessarily share the same Laplace spectrum on functions and on forms and must also have the same marked length spectrum.*

Note that Eberlein's theorem holds for all pairs of two-step nilmanifolds, not just generic choices. Thus, for generic two-step nilmanifolds, the minimal marked length spectrum contains as much information about the spectrum on functions and on forms as the maximal marked length spectrum. The minimal length spectrum is geometrically more satisfying, since the smallest closed geodesic in any free homotopy class is just the smallest closed curve representing that class. Our proof of Theorem 2.5 is inspired by Eberlein's proof, but the technical difficulties that arise make the computation rather intricate, necessitating the generic hypothesis.

The generic hypothesis used in Theorem 1 may be relevant to another question: the density of closed geodesics. A compact manifold is said to have the *density of closed geodesic* property if the set of vectors tangent to smoothly closed geodesics is dense in the unit tangent bundle. While much is known about this property in the case of two-step nilmanifolds [E1; M; LP; De], the relationship between the density of closed geodesics property and the geometry of the nilmanifold is still not fully understood. In Lemma 2.15, we show that if a nilmanifold is generic then it satisfies a density condition that is closely related to the density of closed geodesics property. The authors suggest that generic nilmanifolds may provide an interesting context in which to study the density of closed geodesics property.

The relationship between the Laplace spectrum and lengths of closed geodesics arises from the study of the wave equation (see [DGu]) and—in the case of compact, hyperbolic manifolds—from the Selberg trace formula [S] (see also [C, Ch. XI]). The length spectrum is the set of lengths of closed geodesics, counted with multiplicity. The multiplicity of a length is the number of free homotopy classes of loops that can be represented by smoothly closed geodesics of that length. Clearly, if two manifolds have the same marked length spectrum, they have the same length spectrum.

Colin de Verdière [CdV] has shown that, generically, the Laplace spectrum determines the length spectrum. On Riemann surfaces, Huber showed that the length spectrum and the Laplace spectrum are equivalent notions (see [Bu] for references and an exposition). Note that nilmanifolds do not satisfy the genericity assumptions of Colin de Verdière and Duistermaat-Guillemin, since closed geodesics of any length come in large-dimensional families.

Two-step Riemannian nilmanifolds are of particular importance when considering these questions, as they have provided a rich source of examples of isospectral manifolds (see e.g. [DG; G1; G2; G3; GW1; GW2; GW3; Gt1; P1]). The Poisson summation formula gives the relationship between the Laplace spectrum and length spectrum of flat tori, with the result that pairs of flat tori are isospectral if and only if they share the same length spectrum (see [BGM] or [Bd]). Pesce [P2] has computed a Poisson-type formula relating the Laplace spectrum and length spectrum of Heisenberg manifolds, and he has also shown that pairs of Heisenberg manifolds that are isospectral must have the same lengths of closed geodesics. The authors have shown [GtM1; GtM2] that all known methods for constructing examples of isospectral two-step nilmanifolds necessarily produce examples with the same lengths of closed geodesics. Previously, Gordon [G1] and later Gornet [Gt1] exhibited examples of isospectral nilmanifolds that do not have the same length spectrum; that is, they exhibit different multiplicities. Recently, Miatello and Rossetti [MR] exhibited isospectral compact flat manifolds with this property. All known examples of isospectral manifolds for which the length spectrum has been studied have the same lengths of closed geodesics.

The marked length spectrum often contains significantly more geometric information than the length spectrum. Croke [Cr] and Otal [Ot1; Ot2] independently showed that, if a pair of compact surfaces with negative curvature has the same marked length spectrum, the surfaces are necessarily isometric. The same is true for flat tori (see [BGM] and [Bd]), and Miatello and Rossetti [MR] have generalized this to all compact flat manifolds with the same marked length spectrum. In contrast to Eberlein's result in [E1] for two-step nilmanifolds, Gornet [Gt3] has constructed continuous families of three-step nilmanifolds with the same marked length spectrum that are not isospectral on 1-forms. Also in contrast, the standard sphere and the Zoll sphere (see [Bs]) have the same marked length spectrum (trivially so, as they are both simply connected and by definition have the same lengths of closed geodesics), yet they are not even isospectral on functions. Indeed, any manifold isospectral to a standard sphere of dimension ≤ 6 must be isometric to it (see [BGM]).

In the cases studied by Croke and Otal, the marked length spectrum and the geodesic flow are, roughly speaking, equivalent notions. Gordon and Mao [GM] showed that generic pairs of two-step nilmanifolds with conjugate geodesic flows are isometric. Gordon, Mao, and Schueth [GMS] showed that pairs of two-step nilmanifolds with symplectically conjugate geodesic flows are isometric. Thus, the geodesic flow is significantly stronger than the marked length spectrum on two-step nilmanifolds.

The main tools we use to study the minimal marked length spectrum are Eberlein's proof of the maximal marked length spectrum, our previous work on the reformulation of lengths of closed geodesics at the Lie algebra level [GtM1], and our study of the length minimizing properties of geodesics [GtM2].

The authors wish to thank Patrick Eberlein and Carolyn S. Gordon for many useful suggestions.

1. Background

We study the Laplace spectrum on functions and forms and examine the marked length spectrum on Riemannian two-step nilmanifolds.

1.1. DEFINITIONS. The *Laplace spectrum* of a closed Riemannian manifold (M, g), denoted $\operatorname{spec}(M, g)$, is the collection of eigenvalues of the Laplace–Beltrami operator Δ, counted with multiplicity. The Laplace–Beltrami operator may be extended to act on smooth p-forms by $\Delta = d\delta + \delta d$, where δ is the metric adjoint of the differential d. We call Δ's eigenvalue spectrum the p-form spectrum. Since M is a closed manifold, the p-form spectrum is precisely the set of eigenvalues of Δ, each with finite multiplicity and a unique accumulation point at infinity, $p = 0, \dots, \dim M$. The *length spectrum* of M is the set of lengths of smoothly closed geodesics.

1.2. DEFINITIONS. (1) Two Riemannian manifolds (M, g) and (M', g') are said to have the same *marked length spectrum* if there exists an isomorphism $\Psi\colon \Pi_1(M) \to \Pi_1(M')$ between their fundamental groups such that the following property holds: For all $\sigma \in \Pi_1(M)$, there exists a closed geodesic of length α in the free homotopy class $[\sigma]$ of M if and only if there exists a closed geodesic of length α in the free homotopy class $[\Psi(\sigma)]$ of M'. In this case, we say that the isomorphism Ψ *marks* the length spectrum.

(2) Two Riemannian manifolds (M, g) and (M', g') are said to have the same *minimal marked* (resp., *maximal marked*) *length spectrum* if there exists an isomorphism $\Psi\colon \Pi_1(M) \to \Pi_1(M')$ such that the following property holds: For all $\sigma \in \Pi_1(M)$, the length of the shortest (resp., longest) closed geodesic in the free homotopy class $[\sigma]$ of M is equal to the length of the shortest (resp., longest) closed geodesic in the free homotopy class $[\Psi(\sigma)]$ of M'.

We may restate this as follows: Two Riemannian manifolds have the same *minimal marked* (resp., *maximal marked*) *length spectrum* if there exists an isomorphism between the fundamental groups such that the lengths of the shortest (resp., longest) closed geodesics representing corresponding free homotopy classes are equal. Note that a free homotopy class need not have a longest length (the canonical sphere, for example). Theorem 1.18 shows that, for two-step nilmanifolds, every free homotopy class has a longest length.

Our objects of study in this paper are two-step Riemannian nilmanifolds. To understand the marked length spectrum on nilmanifolds, we must understand geodesics on their simply connected covers: that is, two-step nilpotent Lie groups equipped with a left-invariant metric.

Let $\mathfrak{n}$ denote a finite-dimensional, real Lie algebra with Lie bracket $[\cdot, \cdot]$ and nontrivial center $\mathfrak{z}$. We say that $\mathfrak{n}$ is *two-step nilpotent* if $\mathfrak{n}$ is nonabelian and

$[X, Y] \in \mathfrak{z}$ for all $X, Y \in \mathfrak{n}$. A Lie group is said to be two-step nilpotent if its Lie algebra is two-step nilpotent. Let N denote the unique, simply connected Lie group with Lie algebra $\mathfrak{n}$.

The Lie group exponential map $\exp\colon \mathfrak{n} \to N$ is a diffeomorphism [R]. By the Campbell–Baker–Hausdorff formula [V], we may write the group operation of N in terms of the Lie algebra $\mathfrak{n}$ by

$$\exp(X)\exp(Y) = \exp\big(X + Y + \tfrac{1}{2}[X, Y]\big) \tag{1.3}$$

for all X, Y in $\mathfrak{n}$. Thus:

$$\begin{aligned} \exp(X)^{-1} &= \exp(-X); \\ \exp(X)\exp(Y)\exp(X)^{-1} &= \exp(Y + [X, Y]). \end{aligned} \tag{1.4}$$

Denote the inverse of exp by $\log\colon N \to \mathfrak{n}$. Then for $p, q \in N$, (1.3) becomes

$$\log(pq) = \log p + \log q + \tfrac{1}{2}[\log p, \log q]. \tag{1.5}$$

We assume that N has a Riemannian metric g that is *left invariant*—that is, left translations L_p are isometries for all p in N. Note that a left invariant metric on N determines an inner product on $\mathfrak{n} = T_e N$ while an inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{n}$ induces a left invariant metric on N. A Lie algebra together with an inner product $(\mathfrak{n}, \langle\cdot,\cdot\rangle)$ is called a *metric Lie algebra*. We use $\langle\cdot,\cdot\rangle$ to denote the inner product on $\mathfrak{n}$ and g to denote the corresponding left invariant metric on N. We denote the orthogonal complement of $\mathfrak{z}$ in $\mathfrak{n}$ by $\mathfrak{v}$, and we write $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$. Let $\pi_{\mathfrak{v}}\colon \mathfrak{n} \to \mathfrak{v}$ and $\pi_{\mathfrak{z}}\colon \mathfrak{n} \to \mathfrak{z}$ denote orthogonal projection onto $\mathfrak{v}$ and $\mathfrak{z}$, respectively.

To study the geometry of a two-step nilpotent Lie group equipped with a left invariant metric, it is helpful to consider a set of skew-symmetric linear transformations defined on the corresponding Lie algebra. These maps, first introduced by Kaplan [K], capture all of the geometry of a two-step nilpotent metric Lie group.

1.6. Definition. Let $(\mathfrak{n}, \langle\cdot,\cdot\rangle)$ be a two-step nilpotent metric Lie algebra, $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$. Define a linear transformation $j\colon \mathfrak{z} \to \mathrm{so}(\mathfrak{v})$ by $j(Z)X = (\operatorname{ad} X)^* Z$ for $Z \in \mathfrak{z}$ and $X \in \mathfrak{v}$. Equivalently, for each $Z \in \mathfrak{z}$, $j(Z)\colon \mathfrak{v} \to \mathfrak{v}$ is the skew-symmetric linear transformation defined by

$$\langle j(Z)X, Y\rangle = \langle Z, [X, Y]\rangle$$

for all X, Y in $\mathfrak{v}$. Here $\operatorname{ad} X(Y) = [X, Y]$ for all $X, Y \in \mathfrak{n}$, and $(\operatorname{ad} X)^*$ denotes the (metric) adjoint of $\operatorname{ad} X$.

By skew-symmetry, $j(Z)$ has $\dim_{\mathbb{R}}(\mathfrak{v})$ purely complex eigenvalues counting (algebraic) multiplicities, and the nonzero eigenvalues occur in complex conjugate pairs; the eigenvalues of $j(Z)^2$ are then real and nonpositive.

Thus each two-step nilpotent metric Lie algebra carries with it the J-operator, j. On the other hand, given inner product spaces $\mathfrak{v}$ and $\mathfrak{z}$ and a linear transformation $j\colon \mathfrak{z} \to \mathrm{so}(\mathfrak{v})$, one can define a two-step nilpotent metric Lie algebra $(\mathfrak{v} \oplus \mathfrak{z}, \langle\cdot,\cdot\rangle)$ by requiring that $\mathfrak{z}$ be central and that $\oplus$ be orthogonal direct sum and then defining the Lie bracket $[\cdot,\cdot]$ via Definition 1.6. All two-step nilpotent metric Lie algebras are determined this way.

Understanding the geodesics of (N, g) requires understanding the invariant subspaces of $j(Z)$ for $Z \in \mathfrak{z}$. For this, we need the following.

1.7. DEFINITIONS AND NOTATION. Let $(\mathfrak{n}, \langle\cdot,\cdot\rangle)$ be a two-step nilpotent metric Lie algebra, and let $Z \in \mathfrak{z}$.

(1) Let $\mu(Z)$ denote the number of *distinct* eigenvalues of $j(Z)^2$. For ease of notation, we write μ rather than $\mu(Z)$ when Z is understood.
(2) Let $\vartheta_1(Z)^2, \dots, \vartheta_\mu(Z)^2$ denote the μ distinct eigenvalues of $-j(Z)^2$, with the assumption that $0 \le \vartheta_1(Z) < \vartheta_2(Z) < \cdots < \vartheta_\mu(Z)$. The distinct eigenvalues of $j(Z)$ are then $\{\pm\vartheta_1(Z)i, \dots, \pm\vartheta_\mu(Z)i\}$.
(3) Let $W_m(Z)$ denote the invariant subspace of $j(Z)$ corresponding to $\vartheta_m(Z)$, $m = 1, \dots, \mu$. Then $j(Z)^2|_{W_m(Z)} = -\vartheta_m(Z)^2 \operatorname{Id}|_{W_m(Z)}$; that is, $W_m(Z)$ is the eigenspace of $j(Z)^2$ with eigenvalue $-\vartheta_m(Z)^2$. In particular, if $\vartheta_1(Z) = 0$ then $W_1(Z) = \ker j(Z)$. By the skew-symmetry of $j(Z)$, $\mathfrak{v}$ is the orthogonal direct sum of the invariant subspaces $W_m(Z)$, and we write
$$\mathfrak{v} = \bigoplus_{m=1}^{\mu} W_m(Z).$$
(4) Let $X_0 + Z_0$ be a vector in $\mathfrak{n}$ with $X_0 \in \mathfrak{v}$ and $Z_0 \in \mathfrak{z}$. Define X_1 and X_2 by $X_0 = X_1 + X_2$ such that $X_1 \in \ker j(Z_0)$ and $X_2 \perp \ker j(Z_0)$. Let ξ_m denote the component of X_2 in $W_m(Z_0)$ for each m. We write $X_2 = \sum_m \xi_m$. Note that if $W_1(Z_0) = \ker j(Z_0)$ then $\xi_1 = 0$. When necessary, we assume $j(Z)^{-1}\xi_1 = 0$ if $\vartheta_1(Z) = 0$.
(5) Note that if $\vartheta_m(Z) \neq 0$ then
$$j(Z)^{-1}|_{W_m(Z)} = \frac{-1}{\vartheta_m(Z)^2} j(Z)|_{W_m(Z)}$$
and, for $m = 1, \dots, \mu$,
$$e^{sJ} = \cos(s\vartheta_m) \operatorname{Id} + \frac{\sin(s\vartheta_m)}{\vartheta_m} J \quad \text{on } W_m(Z),$$
where $J = j(Z)$ and $\vartheta_m = \vartheta_m(Z)$.

We will also need the following definition for our proof of the main theorem.

1.8. DEFINITION. Let $(\mathfrak{n}, \langle\cdot,\cdot\rangle)$ be a two-step nilpotent metric Lie algebra. Define $\mathcal{U} = \{Z \in \mathfrak{z} :$ there exists an open neighborhood $\mathcal{O}$ of Z such that μ is constant on $\mathcal{O}\}$. We call $\mathcal{U}$ the *simple subdomain* of $\mathfrak{z}$.

1.9. PROPOSITION [GtM1, Prop. 1.19]. *Let $(\mathfrak{n}, \langle\cdot,\cdot\rangle)$ be a two-step nilpotent metric Lie algebra. Then the following statements hold.*

(1) *The simple subdomain $\mathcal{U}$ is open and dense in $\mathfrak{z}$.*
(2) *The function $\mu(Z)$ is constant on $\mathcal{U}$.*
(3) *The function $\vartheta_m : \mathcal{U} \to \mathbb{R}$ is smooth on $\mathcal{U} - \{0\}$ for $m = 1, \dots, \mu(Z)$.*
(4) *If Z is a limit point of $\mathcal{U}$, then $\mu(Z) \le \mu(\mathcal{U})$.*

1.10. Lemma [LP, Lemma 3.2]. *Let* $(\mathfrak{n}, \langle\cdot,\cdot\rangle)$ *be a two-step nilpotent metric Lie algebra and let* $Z \in \mathcal{U}$. *Fix* $m \in \{1, \ldots, \mu(Z)\}$ *and let* $\xi_m \in W_m(Z)$. *Then* $[\xi_m, j(Z)\xi_m] = |\xi_m|^2 \vartheta_m(Z) \nabla\vartheta_m(Z)$, *where* ∇ *is the gradient.*

In the proof of Theorem 4.1, we shall need an analogue of Lemma 1.10 in the case $Z \notin \mathcal{U}$. To do this, we must introduce the notion of a *refined invariant subspace* of $W_\mu(Z)$.

Let $\zeta \notin \mathcal{U}$. Since $\mathcal{U}$ is dense in $\mathfrak{z}$, it follows that ζ is a limit point of $\mathcal{U}$. By Proposition 1.9 and continuity of the set of (unordered) eigenvalues [Kt, Sec. II.5], two of the eigenvalue curves must approach each other as Z approaches $\zeta \in \mathfrak{z} - \mathcal{U}$. Therefore, the counting function μ has a discontinuity at ζ. We proceed as though exactly two eigenvalue curves, $\vartheta_{m'}$ and $\vartheta_{m''}$, intersect at ζ; the statements generalize in the obvious manner to the case where more than two eigenvalue curves intersect at ζ. Hence, there exist $Z_s \to \zeta$ $(Z_s \in \mathcal{U})$ such that, as $s \to 0$,

$$\lim_{s\to 0} \vartheta_{m'}(Z_s) = \vartheta_m(\zeta) \quad \text{and} \quad \lim_{s\to 0} \vartheta_{m''}(Z_s) = \vartheta_m(\zeta).$$

We may assume in what follows that the curves $\vartheta_{m'}(Z_s)$ and $\vartheta_{m''}(Z_s)$ are analytic (in s).

For sufficiently small positive values of s, $j(Z_s)$ has invariant subspaces $W_{m'}(Z_s)$ and $W_{m''}(Z_s)$, respectively. By [A, Thm. 4.16], since $j(Z_s) \to j(\zeta)$ and skew-symmetry holds, we may define

$$W'(\zeta) = \lim_{s\to 0} W_{m'}(Z_s) \quad \text{and} \quad W''(\zeta) = \lim_{s\to 0} W_{m''}(Z_s).$$

Note that, since $W_{m'}(Z_s)$ and $W_{m''}(Z_s)$ are orthogonal and invariant subspaces of $j(Z_s)$ for all s, their limit spaces $W'(\zeta), W''(\zeta)$ are orthogonal and invariant subspaces of $j(\zeta)$, and

$$W_m(\zeta) = W'(\zeta) \oplus W''(\zeta).$$

We refer to $W'(\zeta)$ and $W''(\zeta)$ as *refined invariant subspaces* of $j(\zeta)$.

1.11. Proposition. *Let* $(\mathfrak{n}, \langle\cdot,\cdot\rangle)$ *be a two-step nilpotent metric Lie algebra and let* $\zeta \in \mathfrak{z} - \mathcal{U}$. *Let* Z_s *be an analytic* (*in* s) *curve in* $\mathcal{U}$ *such that* $\lim_{s\to 0} Z_s = \zeta$ *and, for* $m' \neq m''$, $\lim_{s\to 0} \vartheta_{m'}(Z_s) = \vartheta_m(\zeta)$ *and* $\lim_{s\to 0} \vartheta_{m''}(Z_s) = \vartheta_m(\zeta)$. *With notation as before, let* $\xi^s_{m'} \in W_{m'}(Z_s)$ *and* $\xi^s_{m''} \in W_{m''}(Z_s)$ *be such that* $\lim_{s\to 0} \xi^s_{m'} = \xi' \in W'(\zeta)$ *and* $\lim_{s\to 0} \xi^s_{m''} = \xi'' \in W''(\zeta)$. *Then*

$$[\xi' + \xi'', j(\zeta)(\xi' + \xi'')] = [\xi', j(\zeta)\xi'] + [\xi'', j(\zeta)\xi''].$$

In particular,

$$[\xi' + \xi'', j(\zeta)(\xi' + \xi'')] = |\xi'|^2 \vartheta_m(\zeta) \nabla\Theta_{m'}(\zeta) + |\xi''|^2 \vartheta_m(\zeta) \nabla\Theta_{m''}(\zeta),$$

where

$$\nabla\Theta_{m'}(\zeta) := \lim_{s\to 0} \nabla\vartheta_{m'}(Z_s) \quad \textit{and} \quad \nabla\Theta_{m''}(\zeta) := \lim_{s\to 0} \nabla\vartheta_{m''}(Z_s).$$

Proof. This result is contained in the proof of [GtM1, Lemma 4.6]. $\square$

We now partially describe geodesics in two-step nilpotent metric Lie groups. Since left translations are isometries in these Lie groups, it suffices to consider geodesics $\sigma(s)$ satisfying $\sigma(0) = e$, where e is the identity element in N. See [E1] for a complete description of the geodesics and more details.

1.12. PROPOSITION [E1, Prop. 3.1 & Prop. 3.5]. *Let (N, g) be a two-step nilpotent Lie group with a left invariant metric. Let $\sigma(s)$ be a curve through the identity with $\sigma(0) = e$ and $\sigma'(0) = X_0 + Z_0$, where $X_0 \in \mathfrak{v}$ and $Z_0 \in \mathfrak{z}$. Then $\sigma(s)$ may be written as $\exp(X(s) + Z(s))$, where $X(s) \in \mathfrak{v}$ and $Z(s) \in \mathfrak{z}$ for all s and where $X'(0) = X_0$ and $Z'(0) = Z_0$. Let $J = j(Z_0)$ and $\vartheta_m = \vartheta_m(Z_0)$ for $m = 1, \ldots, \mu$. The curve $\sigma(s)$ is a geodesic if and only if the following equations are satisfied for all $s \in \mathbb{R}$:*

$$X''(s) = j(Z_0)X'(s);$$

$$Z'(s) + \tfrac{1}{2}[X'(s), X(s)] \equiv Z_0.$$

If $\sigma(s)$ is a geodesic, then $X(s) = sX_1 + (e^{sJ} - \mathrm{Id})(J^{-1}X_2)$.

In [GtM2], we established several results about cut and conjugate points of geodesics. The following will be useful in our proof of the main theorem here.

1.13. THEOREM [GtM2, Cor. 2.10]. *Let (N, g) be a two-step nilpotent metric Lie group with Lie algebra $\mathfrak{n}$. Let σ be a unit speed geodesic in N with initial velocity $X_0 + Z_0$, where $Z_0 \in \mathfrak{z}$ and $X_0 \in \mathfrak{v}$. If $\vartheta_\mu(Z_0) = 0$ or $Z_0 = 0$, then σ has no cut point. Otherwise, the cut point for σ has a lower bound of $2\pi/\vartheta_\mu(Z_0)$.*

1.14. COROLLARY [cf. GtM2, Thm. 2.16]. *Let (N, g) be a two-step nilpotent metric Lie group. Let σ denote a unit speed geodesic in N with $\sigma(0) = e$ and $\sigma'(0) = \xi_\mu + Z_0$, where $Z_0 \in \mathfrak{z}$ and $\xi_\mu \in W_\mu(Z_0)$ and where $\vartheta_\mu(Z_0) \neq 0$ and $Z_0 \neq 0$. Then σ is length minimizing on the interval $[0, 2\pi/\vartheta_\mu(Z_0)]$.*

Let Γ denote a cocompact (i.e., $\Gamma\backslash N$ compact) discrete subgroup of N. The quotient manifold $\Gamma\backslash N$ obtained by letting Γ act by left translation on N is a *two-step nilmanifold,* and the left invariant metric g on N descends to a Riemannian metric on $\Gamma\backslash N$, also denoted by g. For details about cocompact discrete subgroups of nilpotent Lie groups, see [R, Ch. II] and [CG, Ch. 5].

The following properties of cocompact discrete subgroups will be useful in our proof of the main theorem.

1.15. PROPOSITION. *Let (N, g) be a simply connected, two-step nilpotent metric Lie group, let Γ be a cocompact, discrete subgroup in N, and let $Z(N)$ denote the center of N. Then the following statements hold.*

(1) $\Gamma \cap Z(N)$ *is a lattice in* $Z(N)$ *and* $\log \Gamma \cap \mathfrak{z}$ *is a lattice in* $\mathfrak{z}$.
(2) $\pi_{\mathfrak{v}}(\log \Gamma)$ *is a lattice in* $\mathfrak{v}$.
(3) *Given* $\xi \in \log \Gamma$ *and* $\xi^*, \xi_1^*, \xi_2^* \in \log \Gamma \cap \mathfrak{z}$, *we obtain*:
  (a) $\xi_1^* + \xi_2^* \in \log \Gamma \cap \mathfrak{z}$;
  (b) $\xi + \xi^* \in \log \Gamma$; *and*
  (c) $k\xi \in \log \Gamma$ *for any integer* k.

Comments on Proof. Properties (1) and (2) follow from [CG, Ch. 5]. The properties listed in (3) follow from the multiplication rules (1.3), (1.4), and (1.5). See also [E1, Prop. 5.3]. □

We study closed geodesics on $(\Gamma\backslash N, g)$ by lifting them to the universal cover (N, g). Because N is simply connected, the free homotopy classes of $\Gamma\backslash N$ correspond to the conjugacy classes $[\sigma]_\Gamma$ in the fundamental group Γ. Note that $(N, g) \to (\Gamma\backslash N, g)$ is a Riemannian covering. Hence there exists a closed geodesic of length ω in the free homotopy class represented by $\gamma \in \Gamma$ if and only if there exists a unit speed geodesic $\sigma(s)$ on (N, g) such that γ translates σ with period ω. The geodesic σ then projects to a smoothly closed geodesic of length ω on $(\Gamma\backslash N, g)$ in the free homotopy class represented by γ.

1.16. Definition. Let σ be a unit speed geodesic in (N, g). A nonidentity element $\gamma \in N$ *translates* σ *by an amount* $\omega > 0$ if $\gamma\sigma(s) = \sigma(s + \omega)$ for all $s \in \mathbb{R}$. The number ω is called a *period* of γ. Note that if $\sigma(s)$ is a geodesic that is not unit speed and if $\gamma\sigma(s) = \sigma(s + \omega)$ for all $s \in \mathbb{R}$, then $\omega|\sigma'(0)|$ is a period of γ.

Eberlein [E1] proved many properties of periodic geodesics in two-step nilpotent metric Lie groups, including the fact that the maximal period may be calculated.

The shortest loop in a free homotopy class is just the smallest loop representing that class. The authors [GtM1] expressed arbitrary periods at the Lie algebra level. Stating these results requires the following definitions.

1.17. Definitions. (1) For $V \in \mathfrak{v}$, define $P_V : \mathfrak{z} \to [V, \mathfrak{n}]$ as orthogonal projection onto $[V, \mathfrak{n}]$. Define $P_V^\perp : \mathfrak{z} \to [V, \mathfrak{n}]^\perp$ as projection onto $[V, \mathfrak{n}]^\perp$, the orthogonal complement of $[V, \mathfrak{n}]$ in $\mathfrak{z}$.

(2) For $V \in \mathfrak{v}$ and $Z \in \mathfrak{z}$, define $Z_V = P_V(Z)$ and $Z_V^\perp = P_V^\perp(Z)$. We write $Z = Z_V + Z_V^\perp$.

(3) With notation as in Definition 1.7, define $K : \mathfrak{n} \to \mathfrak{z}$ by

$$K(X_0 + Z_0) = Z_0 + \frac{1}{2}\sum_{m=1}^{\mu}[j(Z_0)^{-1}\xi_m, \xi_m].$$

(4) Set $K_V = P_V \circ K$ and $K_V^\perp = P_V^\perp \circ K$.

1.18. Theorem [E1, pp. 632–634]. *Let γ be an arbitrary element of N and write $\gamma = \exp(V + Z)$ with $V \in \mathfrak{v}$ and $Z \in \mathfrak{z}$. Suppose σ is a unit speed geodesic in N that is translated by γ with period ω. As in Definition 1.7, write $\sigma'(0) = X_0 + Z_0$, where $X_0 = X_1 + X_2$ and $X_1 \in \ker(j(Z_0))$. Let $a = \sigma(0)$. Write $a^{-1}\sigma(s) = \exp(X(s) + Z(s))$, where $X(s) \in \mathfrak{v}$ and $Z(s) \in \mathfrak{z}$ for all s. Let $J = j(Z_0)$ and $\vartheta_m = \vartheta_m(Z_0)$ for $m = 1, \ldots, \mu$.*

(1) *Let $\omega^* = (|V|^2 + |Z_V^\perp|^2)^{1/2}$. Then $|V| \le \omega \le \omega^*$. Furthermore, ω^* is a period of γ, and is the largest possible period for γ.*
(2) $V = \omega X_1$.
(3) $Z(\omega) = \omega Z_0 + [V, J^{-1}X_2] + (\omega/2)\sum_{j=1}^{\mu}[J^{-1}\xi_j, \xi_j]$.

1.19. THEOREM [GtM1, Thm. 2.8]. *Let (N, g) be a simply connected two-step nilpotent metric Lie group with Lie algebra $\mathfrak{n}$. With notation as before, let $\gamma = \exp(V + Z_V + Z_V^\perp)$ be an element of N. Let β denote the angle between Z_0 and $K_V^\perp(X_2 + Z_0)$. The periods of γ are precisely*

$$\left\{ |V + Z_V^\perp|, \sqrt{|V|^2 + \frac{4\pi k_m(\vartheta_m(\bar{Z}_0)\cos\beta|Z_V^\perp| - \pi k_m)}{\vartheta_m(\bar{Z}_0)^2}} : X_2 + Z_0 \text{ satisfy conditions (i)–(iv)} \right\}, \quad (✠)$$

where $\bar{Z}_0 = Z_0/|Z_0|$. Given $X_2 \in \mathfrak{v}$ and $Z_0 \in \mathfrak{z}$, the conditions referred to in (✠) *are as follows*:

(i) $|X_2 + Z_0| = 1$;
(ii) $V \in \ker j(Z_0)$ *and* $X_2 \perp \ker j(Z_0)$;
(iii) $Z_V^\perp \in \mathrm{span}_{\mathbb{R}^+}\{K_V^\perp(X_2 + Z_0)\}$; *and*
(iv) *for all m such that $\xi_m \neq 0$, there exists a $k_m \in \mathbb{Z}^+$ such that*

$$k_m = \frac{|Z_V^\perp|\vartheta_m(Z_0)}{2\pi|K_V^\perp(X_2 + Z_0)|}.$$

Eberlein [E1] showed that an isomorphism marking the (maximal) marked length spectrum may be completely described in terms of a Γ-almost inner automorphism and an isometry. The set of Γ-almost inner automorphisms, whose importance in this context was first discovered by Gordon and Wilson [GW1], plays an important role in isospectral results (see [GW1; Gt1; Gt2; Gt3; DG]).

1.20. DEFINITION. Let Γ be a lattice in N and let ϕ be an automorphism of N. Then ϕ is said to be Γ-*almost inner* if for every element $\gamma \in \Gamma$ there exists an element $a \in N$, possibly depending on γ, such that $\phi(\gamma) = a^{-1}\gamma a$.

It is straightforward to show that ϕ is a Γ-almost inner automorphism of N if and only if, for any element $U \in \log\Gamma$, there exists $U^* \in \mathfrak{n}$ such that $\phi_*(U) = U + [U^*, U]$. This follows from the multiplication rule (1.3) and (1.5) for two-step nilpotent groups. Also, note that if ϕ is a Γ-almost inner automorphism and ψ is an automorphism of N, then $\psi \circ \phi \circ \psi^{-1}$ is a $\psi(\Gamma)$-almost inner automorphism.

1.21. THEOREM [G1; DG]. *Let N be a two-step nilpotent Lie group and let Γ be a cocompact discrete subgroup. Let ϕ be a Γ-almost inner automorphism of N. Then, for any choice of left invariant metric g on N*: $(\Gamma\backslash N, g)$ *and* $(\phi(\Gamma)\backslash N, g)$ *have the same marked length spectrum, are isospectral on functions, and are isospectral on p-forms for all p*, $p = 1, \dots, \dim(N)$.

1.22. THEOREM [E1, Thm. 5.20]. *Let $(\Gamma\backslash N, g)$ and $(\Gamma^*\backslash N^*, g^*)$ be Riemannian two-step nilmanifolds with the same maximal marked length spectrum. Let $\Psi\colon \Gamma \to \Gamma^*$ be an isomorphism inducing this marking. Then Ψ factors uniquely as $\Psi = (\Psi_1 \circ \Psi_2)|_\Gamma$, where Ψ_1 is a Γ_1-almost inner automorphism of N and Ψ_2*

is an isomorphism of N onto N^* that is also an isometry. Furthermore, $(\Gamma\backslash N, g)$ *and* $(\Gamma^*\backslash N^*, g^*)$ *have the same spectrum of the Laplacian on functions and on p-forms for all p, and they also have the same marked length spectrum.*

In the next section, we establish the corresponding version of Theorem 1.22 for the minimal marked length spectrum in the context of generic pairs of two-step nilmanifolds.

2. The Minimal Marked Length Spectrum in the Generic Case

Eberlein's result, stated in Theorem 1.22, holds for all pairs of two-step nilmanifolds with the same maximal marked length spectrum. In considering the minimal marked length spectrum, however, it is necessary to consider two separate cases. This section establishes the result for generic pairs of two-step nilmanifolds, where the condition determining genericity is defined in what follows. In the next section, we give an example illustrating the nongeneric (or exceptional) case. In Section 4, the authors demonstrate that the result from the main theorem also holds in the exceptional case with the additional assumption that the center is two-dimensional. Whether the result holds for all nilmanfolds—in particular, for exceptional nilmanifolds having higher-dimensional centers—remains an open question, although the authors conjecture that this is the case. See also Remark 3.4.

Recall from Definition 1.7 that for $Z \in \mathfrak{z}$ we denote the distinct eigenvalues of $j(Z)^2$ by $\{\pm\vartheta_1(Z)^2, \ldots, \pm\vartheta_\mu(Z)^2\}$, where $0 \le \vartheta_1(Z) < \cdots < \vartheta_\mu(Z)$. By Proposition 1.9, there exists a dense open subset $\mathcal{U}$ of $\mathfrak{z}$ such that $\vartheta_\mu(Z)$ is smooth on $\mathcal{U} - \{0\}$.

2.1. Lemma. *Let $Z \in \mathcal{U} - \{0\}$ such that $j(Z) \not\equiv 0$. Then $\nabla\vartheta_\mu(Z) \neq 0$.*

Proof. Let $Z \in \mathcal{U} - \{0\}$ such that $j(Z) \not\equiv 0$, and let $\bar{\xi}_\mu$ be a unit vector in $W_\mu(Z)$, the invariant subspace of $j(Z)$ corresponding to $\vartheta_\mu(Z)$. Note that $j(Z) \not\equiv 0$ implies $\vartheta_\mu(Z) \neq 0$. By Lemma 1.10,

$$[j(Z)^{-1}\bar{\xi}_\mu, \bar{\xi}_\mu] = \frac{\nabla\vartheta_\mu(Z)}{\vartheta_\mu(Z)}.$$

Since the left-hand side satisfies

$$\langle Z, [j(Z)^{-1}\bar{\xi}_\mu, \bar{\xi}_\mu]\rangle = |\bar{\xi}_\mu|^2 = 1$$

by the definition of $j(Z)$, the claim follows. □

The generic condition on a two-step nilpotent Lie group is determined by the behavior of the following map ∇ defined on the unit sphere of $\mathcal{U}$.

2.2. Definition. Let (N, g) be a two-step nilpotent metric Lie group. Let $S(\mathcal{O})$ denote the unit sphere in the set $\mathcal{O}$. If $\dim \mathfrak{z} = 1$, then N is *generic*. If $\dim \mathfrak{z} > 1$, define the map $\nabla\colon S(\mathcal{U}) \to S(\mathfrak{z})$ by

$$\nabla(Z) = \frac{\nabla\vartheta_\mu(Z)}{|\nabla\vartheta_\mu(Z)|}.$$

Denote by ∇_* the differential of ∇. We say that N is *exceptional* if there exists a nonempty open neighborhood $\hat{\mathcal{U}}$ of $S(\mathcal{U})$ such that ∇_* has less than full rank at every point $p \in \hat{\mathcal{U}}$. Otherwise, we say that N is *generic.*

We include all two-step nilmanifolds with $\dim \mathfrak{z} = 1$ in the definition of generic so that the family of generic two-step nilmanifolds is measure one in the family of all two-step nilmanifolds. Note also that, by Sard's theorem, if N is generic and if $\dim \mathfrak{z} > 1$ then ∇_* has full rank, except on a closed set of measure zero. Given a lattice Γ in N, we say that the nilmanifold $\Gamma\backslash N$ is generic (resp., exceptional) if the Lie group N is generic (resp., exceptional).

Observe that ∇ defines a mapping from an open neighborhood of the unit sphere in $\mathfrak{z}$ to the unit sphere in $\mathfrak{z}$. However, note that $\nabla\vartheta_\mu$ (where defined) is a mapping from $\mathcal{U} \subset \mathfrak{z}$ to $\mathfrak{z}$. See Examples 2.3, 2.4, and 3.1 for more details.

2.3. GENERIC EXAMPLE: $(2n+1)$-DIMENSIONAL HEISENBERG GROUP. For n a positive integer, let $\mathcal{H}_n$ denote the $(2n+1)$-dimensional real vector space with basis $\{X_1, \dots, X_n, Y_1, \dots, Y_n, Z\}$. Define a bracket on the basis elements by

$$[X_i, Y_i] = -[Y_i, X_i] = Z, \quad 1 \le i \le n,$$

with all other basis brackets equal to zero. Then $\mathcal{H}_n$ is a two-step nilpotent Lie algebra known as the $(2n+1)$-dimensional Heisenberg Lie algebra. The corresponding Lie group H_n is called the $(2n+1)$-dimensional Heisenberg Lie group. Give $\mathcal{H}_n$ the inner product, making the given basis an orthonormal basis. The center of $\mathcal{H}_n$ is $\mathfrak{z} = \operatorname{span}\{Z\}$, and $\mathfrak{v} = \operatorname{span}\{X_1, \dots, X_n, Y_1, \dots, Y_n\}$. Note that $j(Z^*) = -|Z^*| \operatorname{Id}|_{\mathfrak{v}}$ for all $Z^* \in \mathfrak{z}$ and that $\vartheta_1(Z^*) = |Z^*| = \vartheta_\mu(Z^*)$. Since the center of H_n is one-dimensional, H_n is a generic Lie group.

In Section 3, we construct an example of an exceptional Lie group using the direct sum of two three-dimensional Heisenberg Lie groups. A large class of Lie groups, generalized somewhat from the Heisenberg groups and the Heisenberg-type Lie groups, is the class of Heisenberg-like Lie groups. In the following example, we show that every Lie group that is Heisenberg-like is, in fact, generic.

2.4. GENERIC EXAMPLE: HEISENBERG-LIKE LIE GROUPS [GtM1]. A two-step nilpotent Lie group is said to be *Heisenberg-like* if, for every $m = 1, \dots, \mu$, there exists a $c_m \ge 0$ such that $\vartheta_m(Z) = c_m|Z|$ for every $Z \in \mathfrak{z}$. This class of Lie groups was introduced in [GtM1], and the reader is referred to that paper for examples and more information about these groups. A Lie group that is of Heisenberg type is one that is (up to scaling) Heisenberg-like with the added property that $\mu(Z) = 1$ for all $Z \in \mathfrak{z}$.

Let (N, g) be a two-step nilpotent metric Lie group that is Heisenberg-like. We claim that N is generic. As usual, we let $\mathfrak{n}$ be the Lie algebra corresponding to N and write $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$. Let $Z \in \mathfrak{z}$ and $|Z| = 1$. Then

$$\begin{aligned}\nabla(Z) &= \frac{\nabla\vartheta_\mu(Z)}{|\nabla\vartheta_\mu(Z)|}\\ &= \frac{c_\mu Z}{|c_\mu|}\\ &= Z.\end{aligned}$$

Since ∇ has maximal rank everywhere, we conclude that N is generic.

We are now ready to state and prove the main theorem of this paper.

2.5. MAIN THEOREM. *Suppose that* $(\Gamma\backslash N, g)$ *and* $(\Gamma^*\backslash N^*, g^*)$ *are two-step nilmanifolds that have the same minimal marked length spectrum. Assume that the metric Lie group* (N, g) *is generic as defined in Definition 2.2. Let* $\Psi\colon \Gamma \to \Gamma^*$ *be an isomorphism that induces the marking. Then* $\Psi = (\Psi_1 \circ \Psi_2)|_\Gamma$, *where* Ψ_2 *is a* Γ*-almost inner automorphism and* Ψ_1 *is an isomorphism from* N *onto* N^* *that is also an isometry. In particular,* $(\Gamma\backslash N, g)$ *and* $(\Gamma^*\backslash N^*, g^*)$ *must be isospectral on functions and on* p*-forms for all* $p = 1, \dots, \dim(N)$ *and must have the same marked length spectrum.*

2.6. REMARK. The basic outline of the proof in the generic case was inspired by Eberlein's proof of Theorem 1.22 for the maximal marked length spectrum [E1, Thm. 5.20]. There is a major distinction, however: A step that appears early in Eberlein's proof is left until the end in our proof, when we must restrict to the generic case. This is due to the fact that, for the maximal marked length spectrum, the maximal period of an element of the form $\exp(Z) \in \Gamma$ for $Z \in \mathfrak{z}$ is the length of the vector Z. Consequently, the metric on $\mathfrak{z}$ can be read directly from the maximal marked length spectrum. The fundamental distinction between the proof of our result and that of Eberlein's lies in the difficulty in extracting knowledge of lengths of central vectors from the minimal marked length spectrum.

Proof of Theorem 2.5. The uniqueness of the decomposition is straightforward; see [E1, p. 656] for details.

Let $\Psi\colon \Gamma \to \Gamma^*$ mark the minimal marked length spectrum. Then $\Psi\colon \Gamma \to \Gamma^*$ is an isomorphism and, since N and N^* are nilpotent, Ψ extends uniquely to an isomorphism from N onto N^*, which we also denote by Ψ.

Let $\mathfrak{n}$ (resp., $\mathfrak{n}^*$) be the Lie algebra of N (resp., N^*). Let $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ and $\mathfrak{n}^* = \mathfrak{v}^* \oplus \mathfrak{z}^*$ as in Definition 1.6.

An isomorphism between Lie groups induces an isomorphism between Lie algebras. Thus, $\Psi_*\colon \mathfrak{n} \to \mathfrak{n}^*$ satisfies the condition that $\Psi_*(\mathfrak{z}) = \mathfrak{z}^*$. We decompose Ψ_* as

$$\Psi_*(V + Z) = A(V) + B(V) + C(Z)$$

for all $V \in \mathfrak{v}$ and $Z \in \mathfrak{z}$. Here $A\colon \mathfrak{v} \to \mathfrak{v}^*$ and $B\colon \mathfrak{v} \to \mathfrak{z}^*$ are the linear transformations obtained by projecting $\Psi_*(\mathfrak{v})$ onto $\mathfrak{v}^*$ and $\mathfrak{z}^*$, respectively, and $C = \Psi_*|_{\mathfrak{z}}$. Note that $C\colon \mathfrak{z} \to \mathfrak{z}^*$.

We briefly outline the rest of the proof. We first show that A is an isometry and that there exists an isomorphism $\Psi_1 : N \to N^*$ such that $\Psi_{1*}(V+Z) = A(V) + C(Z)$ for all $V \in \mathfrak{v}$ and $Z \in \mathfrak{z}$. We then define a linear isomorphism $T : \mathfrak{n} \to \mathfrak{n}$ by $T(V+Z) = V + Z + (C^{-1} \circ B)(V)$ for all $V \in \mathfrak{v}$ and $Z \in \mathfrak{z}$ and demonstrate that $T = \Psi_{2*}$ for some Γ-almost inner automorphism Ψ_2 of N. We then show that $\Psi_* = \Psi_{1*} \circ \Psi_{2*}$. The last step is to restrict to the generic case and show that, in this case, the map C is an isometry. Hence Ψ_1 is an isometry in the generic case, and the result follows.

2.7. LEMMA. *For all $V, V' \in \mathfrak{v}$,*

$$[A(V), A(V')]^* = C([V, V']),$$

where $[\cdot,\cdot]^*$ denotes the Lie bracket in $\mathfrak{n}^*$.

Proof. This follows immediately from the fact that Ψ_* is a Lie algebra isomorphism. □

The proof that A is an isometry requires several steps, which we write as lemmas. The next result is needed in the proof of Lemma 2.9. Recall from Proposition 1.15 that $\pi_{\mathfrak{v}}(\log \Gamma)$ is a lattice in $\mathfrak{v}$.

2.8. LEMMA [E2]. *Let $V \in \pi_{\mathfrak{v}}(\log \Gamma)$ and let $\varepsilon > 0$. Then there exists a $U \in \log \Gamma$ such that $U = kV + Z_0$ for some $k \in \mathbb{Z}^+$ and some $Z_0 \in \mathfrak{z}$, $|Z_0| < \varepsilon$.*

Proof. Given $V \in \pi_{\mathfrak{v}}(\log \Gamma)$ and $\varepsilon > 0$, let $U^* \in \log \Gamma$ be an element such that $U^* = V + Z^*$ for some $Z^* \in \mathfrak{z}$. Let R be a positive number larger than the diameter of a fundamental domain centered at the origin in $\mathfrak{z}$ for the additive lattice $\log \Gamma \cap \mathfrak{z}$. For each $k \in \mathbb{Z}^+$, there exists an element $U_k \in \log \Gamma \cap \mathfrak{z}$ such that $|kZ^* - U_k| \le R$. Let $\alpha_k = kZ^* - U_k$. By the compactness of the closed ball of radius R centered at the origin in $\mathfrak{z}$, there exist integers $m > n \ge 1$ such that $|\alpha_m - \alpha_n| < \varepsilon$. If we set $U = (m-n)V - (U_m - U_n) + (m-n)Z^*$, then $U = (m-n)U^* - (U_m - U_n) \in \log \Gamma$ by Proposition 1.15. Moreover, $U = (m-n)V + (\alpha_m - \alpha_n)$ and hence U satisfies the desired condition. □

2.9. LEMMA. *Let $V \in \pi_{\mathfrak{v}}(\log \Gamma)$. Then*

$$|A(V)| \le |V|,$$

with equality if and only if $B(V) \in [A(V), \mathfrak{n}^*]^*$.

Proof. Let $V \in \pi_{\mathfrak{v}}(\log \Gamma)$ and suppose that $V = 0$. Since $\Psi_* : \mathfrak{n} \to \mathfrak{n}^*$ is an isomorphism, $A : \mathfrak{v} \to \mathfrak{v}^*$ is one-to-one. Then $A(V) = 0$ and the result follows.

Now suppose $V \ne 0$. Then $A(V) \ne 0$. By Lemma 2.8, for all $n \in \mathbb{Z}^+$ there exists a $U_n \in \log \Gamma$ such that $U_n = k_n V + Z_n$ with $k_n \in \mathbb{Z}^+$, $Z_n \in \mathfrak{z}$, and $|Z_n| < 1/n$. Let $p_n = \exp(U_n) \in \Gamma$. Set $\Psi(p_n) = p_n^* = \exp(\Psi_*(U_n)) \in \Gamma^*$. Now $\Psi_*(U_n) = k_n A(V) + k_n B(V) + C(Z_n)$.

Using Theorem 1.18 and the fact that $|(Z_n)^{\perp}_{V}| \leq |Z_n|$, the minimal length ω_n in $[p_n]_\Gamma$ satisfies

$$|k_n V|^2 \leq \omega_n^2 \leq |k_n V|^2 + |Z_n|^2.$$

But ω_n is also the minimal length in $[p_n^*]_{\Gamma^*}$, and by Theorem 1.18 this likewise satisfies

$$|k_n A(V)|^2 \leq \omega_n^2.$$

Therefore,

$$|A(V)|^2 \leq \frac{\omega_n^2}{k_n^2} \leq |V|^2 + \frac{1}{k_n^2}|Z_n|^2 \leq |V|^2 + \frac{1}{n^2}.$$

Letting $n \to \infty$, we obtain

$$|A(V)|^2 \leq |V|^2.$$

We next prove the equality condition.

Let $B(V)^\perp$ be the component of $B(V)$ orthogonal to $[A(V), \mathfrak{n}^*]^*$. Suppose $B(V)^\perp = 0$, that is, $B(V) \in [A(V), \mathfrak{n}^*]^*$. By Theorem 1.18,

$$|k_n V|^2 \leq \omega_n^2 \leq |k_n A(V)|^2 + |k_n B(V)^\perp + C(Z_n)^\perp|^2.$$

Since $B(V)^\perp = 0$, after dividing by k_n^2 we obtain

$$|V|^2 \leq \frac{\omega_n^2}{k_n^2} \leq |A(V)|^2 + \frac{1}{k_n^2}|C(Z_n)^\perp|^2.$$

Letting $n \to \infty$ and then using $|Z_n| < 1/n$ and the continuity of the linear operator $C\colon \mathfrak{z} \to \mathfrak{z}^*$, we conclude

$$|V|^2 \leq |A(V)|^2,$$

as desired.

To prove the converse direction, we assume $|V| = |A(V)|$. We must show $B(V)^\perp = 0$.

On N^*, let $\sigma_n(s)$ be a unit-speed geodesic of shortest length ω_n representing the free homotopy class $[p_n^*]_{\Gamma^*}$. Set $\sigma_n(0) = a_n \in N^*$, so $\sigma_n(\omega_n) = p_n^* a_n$. Also, $\sigma_n(s + \omega_n) = p_n^* \sigma_n(s)$ for all real s. Let $\dot\sigma_n(0) = L_{a_n *}(X_{0n} + Z_{0n})$, where $X_{0n} \in \mathfrak{v}^*$ and $Z_{0n} \in \mathfrak{z}^*$, and set $a_n^{-1}\sigma_n(s) = \exp(X_n(s) + Z_n(s))$, where $X_n(s) \in \mathfrak{v}^*$ and $Z_n(s) \in \mathfrak{z}^*$.

Let $J_n = j(Z_{0n})\colon \mathfrak{v}^* \to \mathfrak{v}^*$. We write $\mathfrak{v}^* = \mathfrak{v}^*_{1n} \oplus \mathfrak{v}^*_{2n}$, where $\mathfrak{v}^*_{1n} = \ker J_n$ and $\mathfrak{v}^*_{2n}$ is the orthogonal complement of $\mathfrak{v}^*_{1n}$ in $\mathfrak{v}^*$. Let $X_{0n} = X_{1n} + X_{2n}$, where $X_{1n} \in \mathfrak{v}^*_{1n}$ and $X_{2n} \in \mathfrak{v}^*_{2n}$. By Theorem 1.18,

$$\pi_{\mathfrak{v}^*}(\log(p_n^*)) = k_n A(V) = \omega_n X_{1n}.$$

Now $a_n^{-1}\sigma_n(s+\omega_n) = (a_n^{-1} p_n^* a_n) a_n^{-1}\sigma_n(s)$, where $a_n^{-1}\sigma_n(s)$ is a geodesic through the identity e^* in N^*. Set $a_n^{-1} p_n^* a_n = \exp(V_n^* + Z_n^*)$. By (1.5),

$$\begin{aligned}\log(a_n^{-1} p_n^* a_n) &= \log(p_n^*) + [\log(p_n^*), \log(a_n)]^* \\ &= k_n A(V) + k_n B(V) + C(Z_n) + [k_n A(V), \log(a_n)]^*,\end{aligned}$$

where

$$V_n^* = k_n A(V),$$
$$Z_n^* = k_n B(V) + C(Z_n) + [k_n A(V), \log(a_n)]^*,$$

and

$$Z_n^{*\perp} = k_n B(V)^{\perp} + C(Z_n)^{\perp}. \tag{$\star$}$$

Here $Z_n^{*\perp}$ is the component of Z_n^* that is orthogonal to $[A(V), \mathfrak{n}^*]^*$.

By Theorem 1.18,

$$k_n^2|V|^2 \leq \omega_n^2 \leq k_n^2|V|^2 + |Z_n^{\perp}|^2,$$

and by assumption, $|V| = |A(V)|$. Hence

$$0 \leq \omega_n^2 - k_n^2|A(V)|^2 \leq \frac{1}{n^2}$$

and therefore

$$\lim_{n\to\infty} \frac{\omega_n^2}{k_n^2} = |A(V)|^2.$$

Since $k_n^2|A(V)|^2 = \omega_n^2|X_{1n}|^2$, we have

$$\lim_{n\to\infty} |X_{1n}| = \lim_{n\to\infty} \frac{k_n^2}{\omega_n^2}|A(V)|^2 = 1$$

and

$$0 \leq \omega_n^2(1 - |X_{1n}|^2) \leq \frac{1}{n^2}.$$

Since $1 = |X_{1n}|^2 + |X_{2n}|^2 + |Z_{0n}|^2$, it follows that

$$\lim_{n\to\infty} |Z_{0n}| = 0,$$
$$\lim_{n\to\infty} |X_{2n}| = 0, \tag{$\dagger$}$$

and

$$0 \leq \omega_n^2(|X_{2n}|^2 + |Z_{0n}|^2) \leq \frac{1}{n^2}.$$

In particular,

$$\lim_{n\to\infty} \omega_n|X_{2n}| = 0. \tag{$\ddagger$}$$

By Theorem 1.18,

$$Z_n^* = \omega_n Z_{0n} + [V_n^*, J_n^{-1}X_{2n}]^* + \frac{\omega_n}{2}\sum_{r=1}^{\mu_n}[J_n^{-1}\xi_{rn}, \xi_{rn}]^*,$$

where ξ_{rn} is the component of X_{2n} contained in the rth invariant subspace of J_n and μ_n is the number of distinct eigenvalues of J_n^2, as defined in Definition 1.7. Now

$$\begin{aligned} Z_n^{*\perp} &= \omega_n Z_{0n}^{\perp} + \frac{\omega_n}{2}\sum_{r=1}^{\mu_n}[J_n^{-1}\xi_{rn}, \xi_{rn}]^{*\perp} \\ &= k_n B(V)^{\perp} + C(Z_n)^{\perp} \end{aligned}$$

by ($\star$). Thus

$$B(V)^{\perp} = \frac{\omega_n}{k_n} Z_{0n}^{\perp} - \frac{C(Z_n)^{\perp}}{k_n} + \frac{\omega_n}{2k_n} \sum_{r=1}^{\mu_n} [J_n^{-1}\xi_{rn}, \xi_{rn}]^{*\perp}.$$

Therefore,

$$B(V)^{\perp} = \lim_{n\to\infty}\left(\frac{\omega_n}{k_n} Z_{0n}^{\perp}\right) - \lim_{n\to\infty} \frac{C(Z_n)^{\perp}}{k_n} + \lim_{n\to\infty} \frac{\omega_n}{2k_n} \sum_{r=1}^{\mu_n} [J_n^{-1}\xi_{rn}, \xi_{rn}]^{*\perp}.$$

Note that the first two terms on the right-hand side are zero because $|Z_{0n}|$ approaches zero, ω_n/k_n approaches $|A(V)| \neq 0$, and Z_n approaches zero. Thus

$$|B(V)^{\perp}| \leq \frac{1}{2}|A(V)| \lim_{n\to\infty} \sum_{r=1}^{\mu_n} |[J_n^{-1}\xi_{rn}, \xi_{rn}]^{*\perp}|.$$

By the continuity of the bracket $[\cdot,\cdot]^*$, it remains to show that $\lim_{n\to\infty}|\xi_{rn}| = 0$ and $\lim_{n\to\infty} J_n^{-1}\xi_{rn} = 0$, for then $B(V)^{\perp} = 0$ and $B(V) \in [A(V), \mathfrak{n}^*]^*$ as desired.

Since $|\xi_{rn}| \leq |X_{2n}|$, which goes to zero by (†), we have

$$\lim_{n\to\infty} |\xi_{rn}| = 0.$$

Now $X_n(\omega_n) = V_n^*$ since $\sigma_n(\omega_n) = a_n^{-1}p_n^* a_n$, but by Definition 1.7(5) and the geodesic equations of Proposition 1.12,

$$X_n(\omega_n) = V_n^* = \omega_n X_{1n} + \sum_{r=1}^{\mu_n} (\cos(\omega_n\vartheta_{rn}) - 1) J_n^{-1}\xi_{rn} + \frac{\sin(\omega_n\vartheta_{rn})}{\vartheta_{rn}}\xi_{rn},$$

where ϑ_{rn} is the rth eigenvalue of J_n. Since $\omega_n X_{1n} = V_n^*$ and since the invariant subspaces of J_n are orthogonal to each other,

$$0 = (\cos(\omega_n\vartheta_{rn}) - 1) J_n^{-1}\xi_{rn} + \frac{\sin(\omega_n\vartheta_{rn})}{\vartheta_{rn}}\xi_{rn}$$

for all r and n. In particular, if $\xi_{rn} \neq 0$, then $\omega_n\vartheta_{rn} = 2h_{rn}\pi$ for some $h_{rn} \in \mathbb{Z}^+$.

Finally, if $\xi_{rn} \neq 0$ then $|J_n^{-1}\xi_{rn}| = (1/\vartheta_{rn})|\xi_{rn}|$, by Definitions 1.6 and 1.7, so that

$$|J_n^{-1}\xi_{rn}| = \frac{1}{\vartheta_{rn}}|\xi_{rn}| = \frac{\omega_n}{2h_{rn}\pi}|\xi_{rn}| < \omega_n|\xi_{rn}| \leq \omega_n|X_{2n}|,$$

which goes to zero by (‡). □

2.10. Lemma. *For all $V \in \mathfrak{v}$,*

$$|A(V)| \leq |V|.$$

Proof. This follows from the linearity and continuity of both sides of the inequality and by the density of $\{V/|V| : V \in \pi_{\mathfrak{v}} \log \Gamma\}$ in $\mathfrak{v}$, since Γ is a lattice in N. □

2.11. Lemma. *The mapping $A\colon \mathfrak{v} \to \mathfrak{v}^*$ is an isometry. Moreover, $B(V) \in [A(V), \mathfrak{n}^*]^*$ for all $V \in \pi_{\mathfrak{v}}(\log \Gamma)$.*

Proof. Clearly Ψ^{-1} marks the minimal marked length spectrum from $\Gamma^*\backslash N^*$ to $\Gamma\backslash N$. Define $\Psi_*^{-1} = A^* + B^* + C^*$ analogously to Ψ_*. By Lemma 2.10, $|A^*(V^*)| \leq |V^*|$ for all $V^* \in \mathfrak{v}^*$, and clearly $A^* = A^{-1}$. We conclude that $|A(V)| = |V|$ for all $V \in \mathfrak{v}$ and, by Lemma 2.9, it follows that $B(V) \in [A(V), \mathfrak{n}^*]$ for all $V \in \pi_{\mathfrak{v}}(\log \Gamma)$. $\square$

Note that, by Lemma 2.7, the mapping $A + C$ is a Lie algebra isomorphism. Let $\Psi_1 : N \to N^*$ be the Lie group isomorphism such that $\Psi_{1*} = A + C$.

Let $T : \mathfrak{n} \to \mathfrak{n}$ be the linear isomorphism defined by

$$T(V + Z) = V + Z + (C^{-1} \circ B)(V)$$

for all $V \in \mathfrak{v}$ and all $Z \in \mathfrak{z}$. Then $\Psi_* = \Psi_{1*} \circ T$, and T is a Lie algebra isomorphism. Let $\Psi_2 : N \to N$ be the map defined by $\Psi_{2*} = T$.

2.12. LEMMA. *The map Ψ_2 just defined is a Γ-almost inner automorphism.*

Proof. By the remark preceding Theorem 1.21, it suffices to show that

$$\Psi_{2*}(V + Z) - V - Z \in [V, \mathfrak{n}]$$

for all $V \in \pi_{\mathfrak{v}}(\log \Gamma)$. By definition, $\Psi_{2*} = T$. Note that $T(V + Z) - V - Z = (C^{-1} \circ B)(V)$. Furthermore,

$$B(V) \in [A(V), \mathfrak{n}^*]^* = [A(V), A(\mathfrak{v})]^* = C([V, \mathfrak{v}])$$

by Lemma 2.7. Thus $(C^{-1} \circ B)(V) \in [V, \mathfrak{n}]$ for all $V \in \pi_{\mathfrak{v}}(\log \Gamma)$. The result follows. $\square$

Note that $\Psi = \Psi_1 \circ \Psi_2$. It remains to show that Ψ_1 is an isometry. Recall that $\Psi_{1*} = A + C$. We have already shown that $A : \mathfrak{v} \to \mathfrak{v}^*$ is an isometry. The last step is to show that $|C(Z)| = |Z|$ for all $Z \in \mathfrak{z}$. To establish this, we will ultimately need to restrict to the generic case.

By Definition 1.20 and Theorem 1.21, $\Psi_1 \circ \Psi_2 \circ \Psi_1^{-1}$ marks the minimal marked length spectrum from

$$(\Psi_1(\Gamma)\backslash N^*, g^*) \to (\Psi_1 \circ \Psi_2(\Gamma)\backslash \Psi_1 \circ \Psi_2(N), g^*) = (\Gamma^*\backslash N^*, g^*).$$

Hence we may drop the Γ-almost inner automorphism Ψ_2, and we have reduced to the case where Ψ marks the minimal length spectrum from $(\Gamma\backslash N, g)$ to $(\Gamma^*\backslash N^*, g^*)$, $\Psi_*(\mathfrak{v}) = \mathfrak{v}^*$, and $A = \Psi_* : \mathfrak{v} \to \mathfrak{v}^*$ is an isometry.

Note also that the entire proof (so far) may be repeated by replacing $\mathfrak{z}$ with $[\mathfrak{n}, \mathfrak{n}] = \mathfrak{n}^{(1)} \subset \mathfrak{z}$. Euclidean factors are then included in the subalgebra $\mathfrak{v}$. We conclude that C is an isometry on Euclidean factors. We may thus assume that $\vartheta_\mu(Z) \neq 0$ for all $Z \in \mathfrak{z} - \{0\}$.

On (N, g), let σ be a unit-speed geodesic representing the minimal period $\omega > 0$ of $\gamma \in \Gamma$; that is, $\gamma\sigma(s) = \sigma(s + \omega)$, and ω is the smallest period of γ. Write $\sigma(s) = p \exp(X(s) + Z(s))$ where $p \in N$, $X(s) \in \mathfrak{v}$, and $Z(s) \in \mathfrak{z}$. Then $\dot\sigma(0) = L_{p*}(X_0 + Z_0)$.

2.13. Lemma. *Let σ be as before. Then $|Z_0| \leq |C(Z_0)|$.*

Proof. Reparameterize σ so that $\sigma(0) = p$ and $\sigma(1) = \gamma p$. Then

$$\begin{aligned}
\omega &= \int_0^1 |\dot\sigma(s)|\, ds \\
&= \int_0^1 \left| L_{\sigma(s)*}\left(X'(s) + Z'(s) + \frac{1}{2}[X'(s), X(s)] \right) \right| ds \\
&= \int_0^1 |X'(s) + Z_0|\, ds \\
&= \int_0^1 \sqrt{|X'(s)|^2 + |Z_0|^2}\, ds.
\end{aligned}$$

The second equality follows from [E1, Prop. 3.2]. The third equality follows from Proposition 1.12 and the fact that g is left invariant.

Now the length of $\Psi(\sigma(s))$ from $s = 0$ to $s = 1$ is greater than or equal to the length of $\sigma(s)$, since the length of the smallest curve representing $[\Psi(\gamma)]_{\Gamma^*}$ is the same as the length of the smallest curve representing $[\gamma]_\Gamma$. Therefore,

$$\begin{aligned}
\omega &\leq \int_0^1 |\Psi_*(\dot\sigma(s))|\, ds \\
&= \int_0^1 \left| L_{\Psi(\sigma(s))*} \circ \Psi_* \left(X'(s) + Z'(s) + \frac{1}{2}[X'(s), X(s)] \right) \right| ds \\
&= \int_0^1 \sqrt{|A(X'(s))|^2 + |C(Z_0)|^2}\, ds \\
&= \int_0^1 \sqrt{|X'(s)|^2 + |C(Z_0)|^2}\, ds.
\end{aligned}$$

Hence $|Z_0| \leq |C(Z_0)|$, and we have proved Lemma 2.13. □

We now apply Lemma 2.13 to the family of geodesics that represent minimal periods of *central* elements of Γ, that is, elements in $\Gamma \cap Z(N)$. Since these periods can be represented by geodesics starting at the identity e in N, studying them is much easier than studying all Γ-periodic geodesics.

To proceed, we need the following definition.

2.14. Definition. Define $\mathcal{D} = \{Z_0/|Z_0| \in S(\mathfrak{z})$: there exists $X_0 \in \mathfrak{v}$ such that $X_0 + Z_0$ is the initial velocity of a minimal γ-periodic unit speed geodesic in (N, g) for some $\gamma \in Z(N) \cap \Gamma\}$. Define $\mathcal{D}^*$ in $\mathfrak{n}^*$ analogously.

It follows from Lemma 2.13 that if $\mathcal{D}$ and $\mathcal{D}^*$ are dense in $S(\mathfrak{z})$ and $S(\mathfrak{z}^*)$, respectively, then C is an isometry. We now use the generic hypothesis to conclude that, for most two-step nilmanifolds, $\mathcal{D}$ is dense in $S(\mathfrak{z})$.

2.15. Lemma. *Assume that $(\Gamma\backslash N, g)$ is generic. Then $\mathcal{D}$ is dense in the unit sphere in $\mathfrak{z}$.*

Proof. Suppose $\dim \mathfrak{z} = 1$. Note that $\Gamma \cap Z(N)$ is a lattice in $Z(N)$. A geodesic translated by $\gamma \in \Gamma \cap Z(N)$ must have a central component. By projecting the initial velocity of these geodesics onto $S(\mathfrak{z})$, it follows that $\mathcal{D}$ will contain both unit vectors in $\mathfrak{z}$. Clearly, $\mathcal{D}$ is dense in $S(\mathfrak{z})$.

We may assume $\dim \mathfrak{z} > 1$. Let $\bar{Z}_0$ be a unit vector in $\mathcal{U} \subset \mathfrak{z}$. Without loss of generality, assume that ∇ is nonsingular at $\bar{Z}_0$. Let $\bar{\xi}_\mu$ be a unit vector in $W_\mu(\bar{Z}_0)$. Consider the unit speed geodesic $\sigma_r(s)$ with initial velocity $\sin(r)\bar{\xi}_\mu + \cos(r)\bar{Z}_0$. We show that $\sigma_r(s)$ comes arbitrarily close to a lattice element at $s_0 = 2\pi/(\cos(r)\vartheta_\mu(\bar{Z}_0))$. This must be a minimal geodesic up to s_0 by Corollary 1.14.

Denote $\xi_\mu^r = \sin(r)\bar{\xi}_\mu$ and $Z_0^r = \cos(r)\bar{Z}_0$. By Theorem 1.18,

$$\begin{aligned}\log \sigma_r\left(\frac{2\pi}{\vartheta_\mu(Z_0^r)}\right) &= \frac{2\pi}{\vartheta_\mu(Z_0^r)} Z_0^r + \frac{\pi}{\vartheta_\mu(Z_0^r)}[j(Z_0^r)^{-1}\xi_\mu^r, \xi_\mu^r] \\ &= \frac{2\pi}{\vartheta_\mu(\bar{Z}_0)} \bar{Z}_0 + \tan^2(r)\frac{\pi}{\vartheta_\mu(\bar{Z}_0)}[j(\bar{Z}_0)^{-1}\bar{\xi}_\mu, \bar{\xi}_\mu].\end{aligned}$$

Now, by Lemma 1.10, on the dense open subset $\mathcal{U}$ we have

$$\frac{\pi}{\vartheta_\mu(\bar{Z}_0)}[j(\bar{Z}_0)^{-1}\bar{\xi}_\mu, \bar{\xi}_\mu] = \frac{\pi}{\vartheta_\mu(\bar{Z}_0)^2}\nabla\vartheta_\mu(\bar{Z}_0).$$

By hypothesis, ∇ is analytic and nonsingular on small neighborhoods of $\bar{Z}_0$. Thus, by the inverse function theorem, ∇ is a local diffeomorphism near $\bar{Z}_0$. Hence, for vectors close to $\bar{Z}_0$, $\nabla(Z)$ covers an open neighborhood of $S(\mathfrak{z})$.

Let $N_\varepsilon(\bar{Z}_0)$ denote a neighborhood of $\bar{Z}_0$ in $S(\mathcal{U})$ of unit vectors Z near $\bar{Z}_0$. The set

$$\left\{2Z + \tan^2 r \frac{\nabla\vartheta_\mu(Z)}{\vartheta_\mu(Z)} : Z \in N_\varepsilon(\bar{Z}_0),\ r \in (0, \pi/2)\right\}$$

covers an unbounded neighborhood of $\mathfrak{z}$. In particular, it is unbounded in all directions (since ∇ covers an open neighborhood of the sphere) and so must contain a lattice element.

We have thus proved Lemma 2.15. □

Observe that if $(\Gamma\backslash N, g)$ is generic then, under the hypotheses of Theorem 2.5, so is $(\Gamma^*\backslash N^*, g^*)$. To see this, note that $N^* = \Psi(N)$ and $\Gamma^* = \Psi(\Gamma)$. Hence $(\Gamma^*\backslash N^*, g^*)$ is isometric to $(\Gamma\backslash N, \Psi^{-1*}g^*)$. Using that Ψ is an isometry from $\mathfrak{v}$ to $\mathfrak{v}^*$, the claim is just a basic exercise in the properties of j and ϑ_μ under changes in left invariant metrics that affect only the center. (See the proof of Theorem 4.1 for more details.)

To conclude the proof of Theorem 2.5, note that the sets $\mathcal{D}$ and $\mathcal{D}^*$ are dense in the unit spheres in $\mathfrak{z}$ and $\mathfrak{z}^*$, respectively. Applying Lemma 2.13 to $\Psi^{-1}\colon \mathfrak{z}^* \to \mathfrak{z}$, we conclude that C is an isometry.

Therefore, Ψ is the composition of an isometry and a Γ-almost inner automorphism. It follows that $\Gamma\backslash N$ and $\Gamma^*\backslash N^*$ are isospectral on functions and on p-forms, for all p, and must have the same marked length spectrum. □

3. An Illuminating Nongeneric Example

Here we construct a metric Lie group (N, g) and show directly that it is exceptional (i.e., not generic). We then construct a family of lattices such that, for each choice of lattice Γ, the set $\mathcal{D}$ defined in Section 2 is not dense in the unit sphere in $\mathfrak{z}$. Thus, the density argument used at the end of the proof in the generic case fails for this example. This example generalizes in an interesting manner; see Remark 3.4 for further information.

We will show in Theorem 4.1 that, even though this example is not generic, there cannot exist another two-step nilmanifold with the same minimal marked length spectrum other than those explained by Γ-almost inner automorphisms and isometries.

3.1. Exceptional Example. Let $(\mathfrak{n}, \langle\cdot,\cdot\rangle)$ be the orthogonal sum of two three-dimensional Heisenberg Lie algebras with the standard inner product. That is, $\mathfrak{n}$ has orthonormal basis $\{X_1, Y_1, Z_1, X_2, Y_2, Z_2\}$ with Lie bracket given by

$$[X_i, Y_i] = -[Y_i, X_i] = Z_i \quad \text{for } i = 1, 2,$$

with all other basis brackets equal to zero. Equivalently, $(\mathfrak{n}, \langle\cdot,\cdot\rangle)$ is defined by $\mathfrak{z} = \operatorname{span}_{\mathbb{R}}\{Z_1, Z_2\}$ and $\mathfrak{v} = \operatorname{span}_{\mathbb{R}}\{X_1, Y_1, X_2, Y_2\}$, and the J-operator is given by

$$j(Z_1) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad j(Z_2) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

where we extend the definition of j to $\mathfrak{z}$ by linearity.

Let N be the simply connected Lie group with Lie algebra $\mathfrak{n}$. The inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{n}$ determines a left invariant metric on N, which we denote by g.

The eigenvalues of $j(z_1 Z_1 + z_2 Z_2)$ are $\{\pm z_1 i, \pm z_2 i\}$, so that the ϑ_m curves (defined in Definition 1.7) are given by $\{|z_1|, |z_2|\}$. Note that $\vartheta_\mu(z_1 Z_1 + z_2 Z_2) = \max\{|z_1|, |z_2|\}$. If $|z_1| = |z_2|$, then $\vartheta_\mu(z_1 Z_1 + z_2 Z_2) = |z_1| = |z_2|$. Let

$$\mathcal{U} = \{z_1 Z_1 + z_2 Z_2 : z_1 \neq \pm z_2\},$$

and note that $\mathcal{U}$ is open and dense in $\mathfrak{z}$. Then $\vartheta_m(Z)$ is smooth on $\mathcal{U}$ for $m = 1, 2$ (see Proposition 1.9).

Observe that if $|z_i| > |z_j|$, $i \neq j$, then $\nabla\vartheta_\mu(z_1 Z_1 + z_2 Z_2) = \pm Z_i$ (depending on the sign of z_i). If we restrict to the neighborhood of $\mathfrak{z}$ where $z_1 > z_2 > 0$, then the calculation of $\nabla\colon S(\mathcal{U}) \to S(\mathfrak{z})$ is as follows. Parameterize the unit sphere in the neighborhood by $\theta \mapsto \cos\theta Z_1 + \sin\theta Z_2$. Then

$$\nabla(\cos\theta Z_1 + \sin\theta Z_2) = (\nabla\vartheta_\mu)(\cos\theta Z_1 + \sin\theta Z_2) = Z_1 = \cos 0 Z_1 + \sin 0 Z_2.$$

In local coordinates,

$$\nabla(\theta) = 0$$

for all $-\pi/4 < \theta < \pi/4$. In particular, ∇ is a constant map on this interval and clearly never has full rank. A similar calculation holds on the intervals $(\pi/4, 3\pi/4)$, $(-3\pi/4, 3\pi/4)$, and $(-\pi/4, -3\pi/4)$. Thus, by Definition 2.2, (N, g) is not generic.

Let $c_1 > 2\pi$ and $c_2 > 2\pi$. Let Γ be any lattice in N that satisfies

$$\Gamma \cap Z(N) = \{\exp(k_1c_1Z_1 + k_2c_2Z_2) : k_i \in \mathbb{Z},\ i = 1,2\}. \tag{3.2}$$

We claim that if (3.2) holds then the set $\mathcal{D}$ (as defined in Definition 2.14) is

$$\mathcal{D} = \left\{ \pm Z_1, \pm Z_2, \frac{\pm 1}{\sqrt{2}}(Z_1 + Z_2), \frac{\pm 1}{\sqrt{2}}(Z_1 - Z_2)\right\}.$$

Certainly this is *not* a dense subset in the unit circle in $\mathfrak{z}$.

We now review known results about geodesics on (a) the sum of two Riemannian manifolds and (b) the Heisenberg group with a left invariant metric. Note that a geodesic $\sigma(s)$ on (N, g) may be decomposed as $\sigma(s) = (\sigma_1(s), \sigma_2(s))$, where $\sigma_i(s)$ is a geodesic on the (Heisenberg) Lie subgroup H_i of N with Lie algebra $\mathfrak{h}_i = \mathrm{span}_{\mathbb{R}}\{X_i, Y_i, Z_i\}$ for $i = 1,2$.

It is known from [E1, Prop. 4.9] that, for central elements $\gamma \in Z(N)$, the unit speed geodesic $\sigma(s)$ starting at e is γ-periodic with period τ if and only if $\sigma(\tau) = \gamma$. Furthermore, every period of the central element γ can be achieved by a geodesic starting at e. Thus, we are interested in unit speed geodesics $\sigma(s)$ starting at the identity and eventually hitting central elements. Clearly, $\sigma(0) = e \in N$ if and only if $\sigma_i(0) = e_i \in H_i$, where e_i is the identity of H_i, $i = 1,2$. Also $\sigma(\tau) \in Z(N)$ if and only if $\sigma_i(\tau) \in Z(H_i)$, where $Z(H_i)$ is the center of H_i, $i = 1,2$.

Let H be the simply connected Lie group with metric Lie algebra determined by the orthonormal basis $\{X, Y, Z\}$ and with Lie bracket determined on the basis by $[X, Y] = -[Y, X] = Z$ and $[X, Z] = [Y, Z] = 0$. Clearly, H is isometric to H_i, $i = 1,2$.

There are two categories of unit speed geodesics $\sigma(s)$ in (H, g) satisfying $\sigma(0) = e$ and $\sigma(\tau) \in Z(H)$ for some $\tau > 0$.

Type I: $\sigma(s) = \exp(\pm sZ)$. In this case, $\dot\sigma(0) = \pm Z$.

Type II: $\sigma(s) = \exp(V(s) + Z(s))$. In this case, $\dot\sigma(0) = xX + yY + zZ$, where $x^2 + y^2 + z^2 = 1$, $x^2 + y^2 \neq 0$, and $z \neq 0$. Let J_z denote the matrix

$$J_z = \begin{pmatrix} 0 & -z \\ z & 0 \end{pmatrix};$$

then $J_zX = zY$ and $J_zY = -zX$. The geodesic $\sigma(s)$ first hits the center at $\tau = 2\pi/|z|$ and is not length minimizing past this point [W]. By Theorem 1.18(3),

$$\log\sigma(\tau) = \pm\pi\left(\frac{1 + z^2}{z^2}\right)Z, \tag{3.3}$$

where the sign of $\log\sigma(\tau)$ is determined by the sign of z.

The periods of $\gamma = \exp(cZ) \in Z(H)$ are given as follows (see [E1, Prop. 5.16]):

$$\left\{|c|, \sqrt{4\pi k(|c| - \pi k)} : 1 \le k < \frac{|c|}{2\pi}\right\}.$$

Note that, if $|c| \le 2\pi$, then γ has unique period $|c|$ and corresponds to a geodesic of Type I. If $|c| > 2\pi$, then the minimal period of γ is $\sqrt{4\pi(|c| - \pi)}$ and corresponds to a geodesic of Type II.

We return now to $N = H_1 \oplus H_2$ and let $\gamma \in Z(N)$. Write $\gamma = (\gamma_1, \gamma_2)$ under this decomposition, where $\gamma_i \in Z(H_i)$ for $i = 1, 2$. Clearly, a geodesic $\sigma(s) = (\sigma_1(s), \sigma_2(s))$ is γ-periodic if and only if $\sigma_i(s)$ is γ_i-periodic, $i = 1, 2$.

Let $\sigma(s)$ be a unit speed γ-periodic geodesic of period τ, where $\gamma_1 = \exp(k_1 Z_1)$ and $\gamma_2 = \exp(k_2 Z_2)$. We wish to determine conditions on γ and τ such that σ is γ-periodic of period τ, and we may easily conclude as follows.

If $\gamma_1 = 0$ but $\gamma_2 \neq 0$, then the minimal period of γ is equal to the minimal period of γ_2. In particular, any such unit speed γ-periodic geodesic $\sigma(s)$ in N through the identity $e \in N$ must have $\pi_{\mathfrak{z}}(\dot{\sigma}(0)) \in \operatorname{span}_{\mathbb{R}}\{Z_2\}$. Recall from Section 1 that $\pi_{\mathfrak{z}}$ denotes orthogonal projection onto the center $\mathfrak{z} = \operatorname{span}_{\mathbb{R}}\{Z_1, Z_2\}$ of $\mathfrak{n}$. In this case, the normalized central component of the initial velocity is contained in $\mathcal{D}$, as desired. The case $\gamma_1 \neq 0$ but $\gamma_2 = 0$ is identical.

We thus restrict our attention to the case $\log \gamma = k_1 Z_1 + k_2 Z_2$, where $k_1 \neq 0$ and $k_2 \neq 0$. We assume $k_1 > 0$ and $k_2 > 0$ to simplify calculations; the cases $k_1 < 0$ or $k_2 < 0$ are nearly identical.

Observe that if $V = V_1 + V_2$ with $V_i \in \mathfrak{h}_i$, then $|V|^2 = |V_1|^2 + |V_2|^2$. Now, for $i = 1, 2$, the curve $\sigma_i(s)$ is a γ_i-periodic geodesic that is not, in general, unit speed. Let

$$v_i = |\dot{\sigma}_i(0)|.$$

Then $\alpha_i(s) = \sigma_i(s/v_i)$ is a unit speed γ_i-periodic geodesic with period $v_i\tau$. Note that this implies $\tau^2 = (v_1\tau)^2 + (v_2\tau)^2$, since $v_1^2 + v_2^2 = 1$. One easily checks that τ is a period of γ if and only if $\tau = \sqrt{\tau_1^2 + \tau_2^2}$, where τ_i is a period of γ_i, $i = 1, 2$.

Suppose that $\sigma(s)$ is a centrally periodic and unit speed geodesic in N with period τ. As before, we write $\sigma(s) = (\sigma_1(s), \sigma_2(s))$ and let $v_i = |\dot{\sigma}_i(0)|$. Then there are four possible choices for such a geodesic, corresponding to the possible choices for Type I or Type II centrally periodic geodesics in H_i, $i = 1, 2$:

1. $\sigma(s) = (\exp(sv_1 Z_1), \exp(sv_2 Z_2))$;
2. $\sigma(s) = (\exp(sv_1 Z_1), \exp(V_2(sv_2) + Z_2(sv_2)))$;
3. $\sigma(s) = (\exp(V_1(sv_1) + Z_1(sv_1)), \exp(sv_2 Z_2))$;
4. $\sigma(s) = (\exp(V_1(sv_1) + Z_1(sv_1)), \exp(V_2(sv_2) + Z_2(sv_2)))$.

We shall examine each case separately.

Case 1. Here $\alpha_i(s) = \exp(sv_i Z_i)$ is a unit speed γ_i-periodic geodesic with period $v_i\tau$, $i = 1, 2$. Both geodesics are of Type I. In order for these to be minimal geodesics, we must have $v_i\tau \leq 2\pi$. The geodesics expressed in case 1 are minimally γ-periodic if and only if $|\log \gamma_i| \leq 2\pi$ for $i = 1, 2$. These account for all length minimizing geodesics from $e \in N$ to $\exp(k_1 Z_1 + k_2 Z_2) \in Z(N)$, where $k_i \leq 2\pi$ for $i = 1, 2$.

Case 2. Here $\exp(sZ_1)$ is γ_1-periodic of period $v_1\tau$ and is also of Type I. In order for this to be a minimal geodesic, we must have $v_1\tau \leq 2\pi$, which (as before) implies $k_1 \leq 2\pi$. Furthermore, $\alpha_2(s) = \exp(V_2(s) + Z_2(s))$ is γ_2-periodic of period $v_2\tau$ and is of Type II. For this to be a minimal geodesic it must satisfy $v_2\tau > 2\pi$, which implies $k_2 > 2\pi$. Hence such a geodesic is minimally γ-periodic if and only if $|\log \gamma_1| \le 2\pi$ and $|\log \gamma_2| > 2\pi$. These account for all length minimizing geodesics from $e \in N$ to $\exp(k_1 Z_1 + k_2 Z_2)$ where $0 < k_1 \le 2\pi$ and $k_2 > 2\pi$.

Case 3. This is identical to Case 2 with the subscripts 1 and 2 interchanged.

Case 4. One easily concludes that this case accounts for all length minimizing geodesics from $e \in N$ to $\exp(k_1 Z_1 + k_2 Z_2)$ where $k_i > 2\pi$, $i = 1, 2$. In fact, for each such geodesic σ in this case, $\pi_{\mathfrak{z}}(\dot\sigma(0)) \in \operatorname{span}_{\mathbb{R}}(Z_1 + Z_2)$. Let $\sigma(s)$ satisfy choice 4. Let $\alpha_i(s) = \sigma_i(s v_i) = \exp(V_i(s v_i) + Z_i(s v_i))$ for $i = 1, 2$. Then $\alpha_i(s)$ is a unit speed γ_i-periodic geodesic of period $\tau_i = v_i \tau$. Since $\tau_i = v_i \tau$ is the minimal period of γ_i and since $|\log \gamma_i| > 2\pi$, we must have

$$\tau_i = \sqrt{4\pi(k_i - \pi)}, \quad i = 1, 2.$$

On the other hand, if $\dot\alpha_i(0) = x_i X_i + y_i Y_i + z_i Z_i$, then by (3.3) we have

$$k_i = \frac{\pi(1 + z_i^2)}{z_i^2},$$

which for $i = 1, 2$ implies that

$$z_i = \sqrt{\frac{\pi}{k_i - \pi}}.$$

Now $\dot\sigma_i(0) = v_i x_i X_i + v_i y_i Y_i + v_i z_i Z_i$. It remains to show that $v_1 z_1 = v_2 z_2$. But

$$v_i \tau z_i = \sqrt{4\pi(k_i - \pi)}\sqrt{\frac{\pi}{k_i - \pi}} = \sqrt{4\pi^2},$$

$i = 1, 2$, which is clearly independent of i.

Given the lattice Γ determined in (3.2), it follows that no elements of $\log \Gamma \cap \mathfrak{z}$ have minimal geodesics that satisfy Case 1, Case 2, or Case 3. Therefore, either $k_i = 0$ for at least one value of $i \in \{1, 2\}$, so that the length minimizing geodesic to γ must lie in $\operatorname{span}_{\mathbb{R}} Z_i$, or else Case 4 holds. This implies that the normalized central components of the length minimizing initial velocities to central lattice elements lie in

$$\mathcal{D} = \left\{ \pm Z_1, \pm Z_2, \frac{\pm 1}{\sqrt{2}}(Z_1 \pm Z_2) \right\},$$

as claimed.

3.4. REMARK. We claim that the example in this section may be generalized as follows. Let N be isomorphic to the (finite) direct sum of at least three $(2n+1)$-dimensional Heisenberg groups, with the additional assumption that the metric on N is compatible with the Heisenberg structure so that all eigenvalues of $j(Z)$ are linear in Z. Then there cannot exist another two-step nilmanifold with the same minimal marked length spectrum except for those explained by Γ-almost inner automorphisms and isometries. Note that, in this case, $\dim \mathfrak{z} > 2$; see [GtM3] for further details.

4. The Two-Dimensional Center Case

The purpose of this section is to prove Theorem 2.5 without the generic hypothesis in the special case where $\dim \mathfrak{z} = 2$. Thus, even though Example 3.1 is not generic, Theorem 4.1 shows that there cannot exist another two-step nilmanifold with the same minimal marked length spectrum besides those explained by Γ-almost inner automorphisms and isometries. The methods employed in our proof depend heavily on the fact that $\dim \mathfrak{z} = 2$. The authors conjecture that the generic hypothesis can be dropped for $\dim \mathfrak{z} > 2$. See Remark 3.4 for a discussion of a special case.

4.1. THEOREM. *Suppose that* $(\Gamma_1 \backslash N_1, g_1)$ *and* $(\Gamma_2 \backslash N_2, g_2)$ *are two-step nilmanifolds that have the same minimal marked length spectrum. Assume that the dimension of the center of* N_1 *(and hence* N_2*) is* 2. *Let* $\Psi\colon \Gamma_1 \to \Gamma_2$ *be an isomorphism that induces the marking. Then* $\Psi = (\Psi_1 \circ \Psi_2)|_{\Gamma_1}$, *where* Ψ_2 *is a* Γ_1*-almost inner automorphism and* Ψ_1 *is an isomorphism from* N_1 *onto* N_2 *that is also an isometry. In particular,* $(\Gamma_1 \backslash N_1, g_1)$ *and* $(\Gamma_2 \backslash N_2, g_2)$ *must be isospectral on functions and on p-forms, for all* $p = 1, \ldots, \dim(N_1)$, *and must have the same marked length spectrum.*

Proof. Recall that Lemmas 2.7–2.13 in the proof of Theorem 2.5 did not depend on the genericity condition. Hence, we may assume these lemmas hold under the hypotheses of Theorem 4.1 and so reduce to the case where the isomorphism $\Psi\colon \Gamma_1 \to \Gamma_2$ marks the minimal length spectrum from $(\Gamma_1 \backslash N_1, g_1)$ to $(\Gamma_2 \backslash N_2, g_2)$, $\Psi_*\colon \mathfrak{v}_1 \to \mathfrak{v}_2$ is an isometry, and $\Psi_*(\mathfrak{v}_1) = \mathfrak{v}_2$. It remains to show that $\Psi_*\colon \mathfrak{z}_1 \to \mathfrak{z}_2$ is also an isometry, where $\dim_{\mathbb{R}} \mathfrak{z}_1 = \dim_{\mathbb{R}} \mathfrak{z}_2 = 2$. As in the proof of Theorem 2.5, we may eliminate Euclidean factors (see the remarks after Lemma 2.12). In particular, we may assume that $\vartheta_\mu(Z) \neq 0$ for all $Z \in \mathfrak{z}_1 - \{0\}$.

Recall that Ψ extends to an isomorphism $\Psi\colon N_1 \to N_2$. Thus, $(\Gamma_2 \backslash N_2, g_2)$ is (trivially) isometric to $(\Gamma_1 \backslash N_1, \Psi^{-1*} g_2)$. For ease of notation let $\Gamma \backslash N = \Gamma_1 \backslash N_1$, and write $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ as usual. Note then that ${g_1}_{\mathfrak{v}\otimes\mathfrak{v}}$ and $\Psi^{-1*} {g_2}_{\mathfrak{v}\otimes\mathfrak{v}}$ are isometric. One easily shows that there exists a linear mapping $F\colon \mathfrak{z} \to \mathfrak{z}$ that is symmetric and positive with respect to the g_1-inner product on $\mathfrak{z}$ such that, for all $Z, Z' \in \mathfrak{z}$,

$$g_2(Z, Z') = g_1(Z, FZ'). \tag{4.2}$$

We have thus reduced to the case where $(\Gamma \backslash N, g_1)$ and $(\Gamma \backslash N, g_2)$ have the same minimal marked length spectrum, and the identity mapping $\Gamma \to \Gamma$ induces this marking. Moreover, $\mathfrak{n}$ has the same orthogonal decomposition $\mathfrak{v} \oplus \mathfrak{z}$ with respect to both g_1 and g_2, the metrics g_1 and g_2 are isometric on $\mathfrak{v}$, and g_1 and g_2 are related via F on $\mathfrak{z}$ (see (4.2)).

Recall from Definition 2.2 that $\nabla\colon S(\mathcal{U}) \to S(\mathfrak{z})$, where $\mathcal{U}$ is a dense open subset of the center $\mathfrak{z}$ on which $\nabla\vartheta_\mu$ is defined. Recall also from Definition 2.14 that $\mathcal{D} = \{Z_0/|Z_0| \in S(\mathfrak{z})$: there exists $X_0 \in \mathfrak{v}$ such that $X_0 + Z_0$ is the initial velocity of a minimal γ-periodic unit speed geodesic in (N, g) for some $\gamma \in Z(N) \cap \Gamma\}$. That is, $\mathcal{D}$ is the set of normalized central component directions of all initial velocities of all closed geodesics that represent the smallest length in a central free homotopy class. Let $\mathcal{D}^*$ denote the analogous set for $(\Gamma\backslash N, g_2)$.

Assume that $\mathcal{D}$ is not dense in $S(\mathfrak{z})$. Then there exists an open neighborhood $\mathcal{O}$ of $S(\mathfrak{z})$ such that $\mathcal{D}\cap\mathcal{O}=\emptyset$ and such that $\nabla\vartheta_\mu$ is well-defined on $\mathcal{O}$. This implies that ∇_* does not have full rank on an open subset of $S(\mathcal{U})$, which in turn (because $S(\mathcal{U})$ is one-dimensional) implies that ∇_* has rank 0. In particular, ∇ is constant and ϑ_μ is linear on $\mathcal{O}$ (or possibly on a smaller open subset of $\mathcal{O}$, which we also denote $\mathcal{O}$). Let A denote the constant $\nabla\vartheta_\mu(Z)$, where $Z\in\mathcal{O}$. By the definition of the gradient, for $Z\in\mathcal{O}$ we have

$$\vartheta_\mu(Z)=g_1(Z,A).$$

We claim that $A=\nabla\vartheta_\mu(Z)$ is a rational direction, that is, $\operatorname{span}_{\mathbb{R}}\{\nabla\vartheta_\mu(Z)\}=\operatorname{span}_{\mathbb{R}}\{\log\gamma\}$ for some $\gamma\in\Gamma\cap Z(N)$. To see this, we use the proof of Lemma 2.15 to show that either $\operatorname{span}_{\mathbb{R}}\{A\}$ is dense when projected to the central torus $Z(N)/\Gamma\cap Z(N)$ or A is a rational direction. Let $Z\in\mathcal{O}$ and let $\xi\in W_\mu(Z)$ be a unit vector. Then the geodesic $\sigma_r(s)$ starting at the identity with initial velocity $\sigma_r'(0)=\sin r\xi+\cos rZ$ first hits the center at $\tau_r=2\pi/\vartheta_\mu(\cos rZ)$ and, by Corollary 1.14, is length minimizing on $[0,\tau_r]$. Using the geodesic equations and simplifying as in Lemma 2.15, it follows that

$$\begin{aligned}\log\sigma_r(\tau_r)&=\frac{2\pi}{\vartheta_\mu(\cos rZ)}\left(\cos rZ+\frac{1}{2}\sin^2 r\frac{\nabla\vartheta_\mu(Z)}{\vartheta_\mu(\cos rZ)}\right)\\&=\frac{2\pi}{\vartheta_\mu(Z)}Z+\frac{\pi\tan^2 r}{\vartheta_\mu(Z)^2}A.\end{aligned}$$

Note that the set

$$Ray:=\{\log\sigma_r(\tau_r):r\in(0,\pi/2)\}$$

spans a ray starting at $2\pi Z/\vartheta_\mu(Z)$ in the direction A. If $\operatorname{span}_{\mathbb{R}}\{A\}$ is dense when projected to the torus $Z(N)/\Gamma\cap Z(N)$, then the set *Ray* comes arbitrarily close to lattice elements. Since $\vartheta_\mu(Z)$ is continuous, by varying Z slightly we can construct a ray that actually hits a lattice element, and thus there are elements in $\mathcal{O}$ that are also in $\mathcal{D}$. This contradicts our hypothesis on $\mathcal{O}$, and we conclude that A is a rational direction.

Let $\alpha(Z)=g_1(Z,A)$ for $Z\in S(\mathfrak{z})$. Then α is an analytic eigenvalue curve. Note that $|\alpha(Z)|=\vartheta_\mu(Z)$ for $Z\in\mathcal{O}$; that is, $|\alpha(Z)|$ is maximal on $\mathcal{O}$. However, for $Z=A^\perp$, where $A^\perp$ is orthogonal to A, we have $\alpha(A^\perp)=0$. Since Euclidean factors have been eliminated, this implies that $|\alpha(Z)|$ is not maximal for all $Z\in S(\mathfrak{z})$. As Z varies in the unit circle $S(\mathfrak{z})$, $|\alpha|$ must eventually cross another analytic eigenvalue curve, which becomes the new maximal eigenvalue curve. Let $\beta(Z)$ be an analytic eigenvalue curve and suppose that $|\beta|$ crosses $|\alpha(Z)|$ at $Z=\zeta\in S(\mathfrak{z})$ so that $|\beta(Z)|$ is maximal as Z approaches ζ from one side in $S(\mathfrak{z})$. In particular, $\vartheta_\mu(\zeta)=|\alpha(\zeta)|=|\beta(\zeta)|$. Observe that there will be two possible choices for β (and hence ζ), since $\dim\mathfrak{z}=2$ and $\dim S(\mathfrak{z})=1$. Without loss of generality, we choose one curve β and the corresponding value for ζ.

Each eigenvalue curve $\vartheta_m(Z)$ has the property that $\vartheta_m(Z_1+sZ_2)$ is a convex curve [Kt, Sec. II.7.2]; in particular this holds for α, which is linear on $\mathcal{O}$. Given a sufficiently small open neighborhood of ζ, it follows that α and β only cross at ζ in that neighborhood and so $\nabla\alpha(\zeta)$ and $\nabla\beta(\zeta)$ are linearly independent.

We claim that $\zeta \in \mathcal{D}$. To show this, we use the notion of refined invariant subspaces, defined in Section 1. Let ξ_α be a unit vector in the refined invariant subspace $W_\alpha(\zeta) \subset W_\mu(\zeta)$ associated with $\alpha(\zeta) = \vartheta_\mu(\zeta)$, and let ξ_β be a unit vector in the refined invariant subspace $W_\beta(\zeta)$ associated with $\beta(\zeta) = \vartheta_\mu(\zeta)$. Then the geodesic $\sigma_{\delta,\eta}(s)$ starting at the identity with initial velocity

$$\sin\delta(\cos\eta\xi_\alpha + \sin\eta\xi_\beta) + \cos\delta\zeta$$

first hits the center at

$$\tau_\delta = \frac{2\pi}{\vartheta_\mu(\cos\delta\zeta)},$$

and it is length minimizing on $[0, \tau_\delta]$ by Corollary 1.14. Also,

$$\begin{aligned} Z_{\delta,\eta} &= \log\sigma_{\delta,\eta}(\tau_\delta) \\ &= \frac{2\pi}{\vartheta_\mu(\cos\delta\zeta)}\left(\cos\delta\zeta + \frac{1}{2}\sin^2\delta\left(\cos^2\eta\frac{\nabla\alpha(\zeta)}{\alpha(\cos\delta\zeta)} + \sin^2\eta\frac{\nabla\beta(\zeta)}{\beta(\cos\delta\zeta)}\right)\right). \end{aligned}$$

The fact that $Z_{\delta,\eta}$ reduces to this form follows from Proposition 1.11 and Theorem 1.18.

Note that the set

$$\mathit{Wedge} := \{Z_{\delta,\eta} : \delta \in (0, \pi/2),\ \eta \in [0, \pi]\}$$

covers an unbounded wedge starting at the vector

$$\frac{2\pi}{\vartheta_\mu(\zeta)}\zeta$$

and determined by positive linear combinations of $\nabla\alpha(\zeta)$ and $\nabla\beta(\zeta)$. Infinitely many lattice elements are contained in *Wedge*. Hence there exists a $\gamma \in \Gamma \cap Z(N)$ such that $\gamma = Z_{\delta,\eta}$ for some $\delta \in (0, \pi/2)$ and $\eta \in [0, \pi]$. It follows that $\zeta \in \mathcal{D}$.

Let ϑ_μ^* denote the maximal eigenvalue curve using the metric g_2, and let $\nabla^*\vartheta_\mu^*$ denote its gradient for g_2. Then the following hold:

$$\begin{aligned} \vartheta_\mu^*(Z) &= \vartheta_\mu(FZ); \\ \nabla^*\vartheta_\mu^*(Z) &= \nabla\vartheta_\mu(FZ). \end{aligned} \tag{4.3}$$

Thus the g_2-eigenvalue curve $\alpha^*(Z) = \alpha(FZ)$ crosses the g_2-eigenvalue curve $\beta^*(Z) = \beta(FZ)$ at $F^{-1}\zeta$. In order to normalize this to a g_2-unit vector, we let $f^2 = g_2(F^{-1}\zeta, F^{-1}\zeta) = g_1(\zeta, F^{-1}\zeta)$. If we define

$$\zeta' = \frac{1}{f}F^{-1}\zeta, \tag{4.4}$$

then ζ' is a g_2-unit vector in the same direction as ζ.

Observe that $\zeta' \in \mathcal{D}^*$ by an argument similar to the foregoing. Also, the unbounded wedge corresponding to *Wedge* determined by ζ' starts at the vector

$$\frac{2\pi}{\vartheta_\mu^*(\zeta')}\zeta'$$

and is determined by positive linear combinations of the same two vectors $\nabla\alpha(\zeta)$ and $\nabla\beta(\zeta)$. To see this, note that $\nabla^*\alpha^*(\zeta') = \nabla\alpha(\zeta)$ and $\nabla^*\beta^*(\zeta') = \nabla\beta(\zeta)$. This follows from (4.3) and the fact that, for all $Z \in \mathfrak{z}$, $\vartheta_m(cZ) = |c|\vartheta_m(Z)$ holds for all eigenvalue curves and so $\nabla\vartheta_m(cZ) = \nabla\vartheta_m(Z)$ for $c \neq 0$ and $m = 1, \dots, \mu$.

Because $\dim \mathfrak{z} = 2$, these two-dimensional wedges, both determined by positive linear multiples of $\nabla\alpha(\zeta)$ and $\nabla\beta(\zeta)$, must overlap in an infinite wedge. Thus, infinitely many lattice elements in the center are hit by geodesics that are of this type for both g_1 and g_2.

Let γ be an arbitrary central lattice element that is contained in the overlapping wedges, so that γ is hit by $\sigma_{\delta,\eta}(s)$ at $2\pi/\vartheta_\mu(\cos\delta\zeta)$ in g_1 and by $\sigma^*_{\delta',\eta'}(s)$ at $2\pi/\vartheta^*_\mu(\cos\delta'\zeta')$ in g_2.

Using the fact that both of these elements have the same distance from e in either metric, we must have

$$\tau_\delta = \frac{2\pi}{\vartheta_\mu(\cos\delta\zeta)} = \frac{2\pi}{\vartheta^*_\mu(\cos\delta'\zeta')}.$$

Thus

$$\cos\delta = \frac{1}{f}\cos\delta',$$

which implies that

$$\tan^2\delta' = \frac{1}{f^2}(\tan^2\delta + 1 - f^2). \tag{4.5}$$

Then

$$\begin{aligned}\frac{2\pi}{\vartheta_\mu(\zeta)}\zeta &+ \frac{\pi}{\vartheta_\mu(\zeta)^2}\tan^2\delta(\cos^2\eta\nabla\alpha(\zeta) + \sin^2\eta\nabla\beta(\zeta)) \\ &= \frac{2\pi}{\vartheta^*_\mu(\zeta')}\zeta' + \frac{\pi}{\vartheta^*_\mu(\zeta')^2}\tan^2\delta'(\cos^2\eta'\nabla^*\alpha^*(\zeta') + \sin^2\eta'\nabla^*\beta^*(\zeta'))\end{aligned}$$

for infinitely many values of δ, δ', η, and η'. Converting ζ' to ζ and δ' to δ via (4.4) and (4.5), we obtain

$$\begin{aligned}\frac{2\pi}{\vartheta_\mu(\zeta)}\zeta &+ \frac{\pi}{\vartheta_\mu(\zeta)^2}\tan^2\delta(\cos^2\eta\nabla\alpha(\zeta) + \sin^2\eta\nabla\beta(\zeta)) \\ &= \frac{2\pi}{\vartheta_\mu(\zeta)}F^{-1}\zeta + \frac{\pi}{\vartheta_\mu(\zeta)^2}(\tan^2\delta + 1 - f^2)(\cos^2\eta'\nabla\alpha(\zeta) + \sin^2\eta'\nabla\beta(\zeta)).\end{aligned}$$

This implies that

$$\begin{aligned}2\vartheta_\mu(\zeta)(I - F^{-1})\zeta &= \nabla\alpha(\zeta)(\cos^2\eta'(\tan^2\delta + 1 - f^2) - \cos^2\eta\tan^2\delta) \\ &\quad + \nabla\beta(\zeta)(\sin^2\eta'(\tan^2\delta + 1 - f^2) - \sin^2\eta\tan^2\delta)\end{aligned}$$

for infinitely many values of δ, η, and η'. Take the inner product (in the g_1-metric) of both sides with ζ. Using that $\alpha(\zeta) = g_1(\zeta, \nabla\alpha(\zeta))$ and $\beta(\zeta) = g_1(\zeta, \nabla\beta(\zeta))$ and then simplifying, we obtain

$$2\vartheta_\mu(\zeta)(1 - f^2) = \vartheta_\mu(\zeta)(1 - f^2).$$

It follows that $f^2 = 1$.

The linear mapping $F : \mathfrak{z} \to \mathfrak{z}$ is symmetric and positive definite with respect to the metric g_1 on $\mathfrak{z}$. Thus F has exactly two positive real eigenvalues, which we denote by λ and Λ. If both eigenvalues are greater than (resp., less than) 1, then $g_2(Z, Z) > g_1(Z, Z)$ (resp., $g_2(Z, Z) < g_1(Z, Z)$) for all $Z \in S(\mathfrak{z})$, which contradicts Lemma 2.13 since at least one geodesic in each manifold must have a central component. We may therefore assume that

$$0 < \lambda \leq 1 \leq \Lambda.$$

Let e be a g_1-unit eigenvector with eigenvalue λ and let E be a g_1-unit eigenvector with eigenvalue Λ. We prove that $\lambda = \Lambda$ by contradiction.

Assume $\lambda < \Lambda$. If $\lambda = 1$ or $\Lambda = 1$, then there are exactly two vectors in $S(\mathfrak{z})$ that satisfy $1 = f^2 = g_2(F^{-1}\zeta, F^{-1}\zeta) = g_1(\zeta, \zeta)$. If $\lambda = 1$ then $\zeta = \pm e$ and if $\Lambda = 1$ then $\zeta = \pm E$. By the definition of ζ as the vector at which $|\alpha(Z)|$ is no longer maximal, this implies that $|\alpha(Z)|$ is maximal on a half-plane of $\mathfrak{z}$. This is a contradiction because $\alpha(Z) = g_1(Z, A)$ and we have eliminated Euclidean factors; that is, maximal eigenvalue curves are positive.

We may thus assume that $\lambda < 1 < \Lambda$. Hence, there are exactly four vectors in $S(\mathfrak{z})$ that satisfy $1 = f^2 = g_2(F^{-1}\zeta, F^{-1}\zeta) = g_1(\zeta, \zeta)$. In particular,

$$\zeta = \pm\sqrt{\frac{\lambda(\Lambda - 1)}{\Lambda - \lambda}}\, e \pm \sqrt{\frac{\Lambda(1 - \lambda)}{\Lambda - \lambda}}\, E.$$

Also, there are exactly four vectors $\hat{\zeta}$ in $S(\mathfrak{z})$ that are also in $S'(\mathfrak{z})$ (i.e., that are unit vectors in both metrics). In particular,

$$\hat{\zeta} = \pm\sqrt{\frac{\Lambda - 1}{\Lambda - \lambda}}\, e \pm \sqrt{\frac{1 - \lambda}{\Lambda - \lambda}}\, E.$$

Observe that, since $\mathfrak{z}$ is two-dimensional, these four possible values of $\hat{\zeta}$ divide $\mathfrak{z}$ into four regions. On each region, either $g_2(Z, Z) > g_1(Z, Z)$ for all Z and we call the region *increasing,* or $g_2(Z, Z) < g_1(Z, Z)$ for all Z and we call the region *decreasing.* Note that the increasing regions correspond to the regions containing $\pm E$, while the decreasing regions correspond to those containing $\pm e$. Furthermore, $|\alpha(Z)|$ is the maximal eigenvalue curve in two of the analogous regions determined by the four possible values of ζ.

We now show that the eigenvalue $\beta(Z)$ is also a linear eigenvalue. Note first that if an eigenvalue curve $\beta(Z)$ is maximal on an open neighborhood $\mathcal{O}$ and if $\beta(Z)$ is not linear on $\mathcal{O}$, then $\nabla\beta(Z)$ is not constant on $\mathcal{O}$. By Lemma 2.15, it follows that $\mathcal{O} \cap \mathcal{D} \neq \emptyset$. Thus, every $Z \in \mathcal{O}$ is arbitrarily close to an element of $\mathcal{D}$.

Recall from Lemma 2.13 that $g_1(Z, Z) \leq g_2(Z, Z)$ for all $Z \in \mathcal{D}$. Therefore, the region on which $|\beta|$ is maximal must be entirely contained in the two increasing regions of $\mathfrak{z}$. Suppose another eigenvalue curve, call it δ, is maximal at E, and suppose that δ is nonlinear in a neighborhood of E. Because E is an eigenvector of F, it follows that $\delta^*(E) = \delta(FE) = \Lambda\delta(E)$ and $\alpha^*(E) = \Lambda\alpha(E)$. Hence, δ maximal near E implies that δ^* is maximal near E. Since δ^* is also not linear in a sufficiently small neighborhood of E (in the g_2-metric), Lemma 2.13 implies that $g_2(E, E) \leq g_1(E, E)$. Thus $g_1(E, E) = g_2(E, E)$, a contradiction. Therefore, δ

is linear near E. By an argument analogous to that showing A is a rational direction, we conclude that $\nabla^*\delta^*$ is a rational direction. To establish $\delta(Z) = |\beta(Z)|$, recall that there are only four possible vectors ζ at which eigenvalue curves can intersect when one of them is linear. Since $|\alpha|$ is maximal on two disjoint regions and $|\alpha|$ is linear on these regions, it follows that $\delta = |\beta|$ and that β is linear.

We have reduced to the case where we have exactly two linear eigenvalue curves, α and β, that intersect at the four possible values of ζ. We now denote

$$\alpha(Z) = |\langle A, Z\rangle| \quad \text{and} \quad \beta(Z) = |\langle B, Z\rangle|$$

for all $Z \in \mathfrak{z}$, where $A = \nabla\alpha$ and $B = \nabla\beta$. Note that A and B are linearly independent, rational directions. Hence there exist $a_0, b_0 \in \mathbb{R}$ such that $A_0 = a_0 A$, $B_0 = b_0 B$, and $\log\Gamma \cap \mathfrak{z} = \operatorname{span}_{\mathbb{Z}}\{A_0, B_0\}$.

Because A and B are rational directions, there exists $V + \hat{Z} \in \log\Gamma$ such that $V \in \ker j(B^\perp)$. To see this, consider the submersion $N \to \bar{N} = N/\exp(\mathbb{R}B)$. On the Lie algebra level, the corresponding map is $\mathfrak{n} \to \bar{\mathfrak{n}} = \mathfrak{n}/\mathbb{R}B$. This is a Riemannian submersion if we view $\bar{\mathfrak{n}}$ as $\mathfrak{v} \oplus \mathbb{R}B^\perp$, where $B^\perp$ is a unit vector in $\mathfrak{z}$ that is orthogonal to B. That is, we keep the same inner product on $\mathfrak{v}$, and vectors in the center are orthogonally projected onto $\operatorname{span}\{B^\perp\}$. For use of Riemannian submersions in this context, see [GtM2]. The center of $\bar{\mathfrak{n}}$ is then $\mathbb{R}B^\perp \oplus \ker j(B^\perp)$. Since B is a rational direction, it follows that $\bar{\Gamma}$, the image of Γ under this submersion, is also a cocompact, discrete subgroup of $\bar{N}$. (See [CG, Lemma 5.1.4] for more details.) Given this information about $\bar{\Gamma}$ and given that the center of $\bar{N}$ is a rational subgroup with respect to the lattice $\bar{\Gamma}$, there exists a $\bar{\gamma} \in \bar{\Gamma}$ such that $\log(\bar{\gamma}) = V + \bar{Z}$, where $V \in \ker j(B^\perp)$. Any preimage $\gamma \in \Gamma$ of $\bar{\gamma}$ will thus satisfy $\log\gamma = V + \hat{Z}$ where $V \in \ker j(B^\perp)$.

Note that $V \in \ker j(B^\perp)$ implies

$$[V, \mathfrak{n}] \subset \operatorname{span}_{\mathbb{R}}\{B\}.$$

By Proposition 1.15, $V + \hat{Z} + kA_0 \in \log\Gamma$ for all k. Let $\gamma_k = \exp(V + \hat{Z} + kA_0)$. We claim that, in this context, the minimal period ω_k for γ_k satisfies

$$\omega_k^2 = |V|^2 + \frac{4\pi|(\hat{Z} + kA_0)_V^\perp|^2\big(\alpha((\hat{Z} + kA_0)_V^\perp) - \pi\big)}{\alpha((\hat{Z} + kA_0)_V^\perp)^2}.$$

To see this, let $\sigma(s)$ be a unit speed geodesic in N such that $\sigma(0) = e$, $\sigma'(0) = X_0 + Z_0$, and $\gamma_k\sigma(s) = \sigma(s + \omega_k)$ for all s. See Theorem 1.19 for a description of the periods of γ_k. By Theorem 1.18, $V \in \ker j(Z_0)$. Since $\mathfrak{z}$ is two-dimensional and $V \in \ker j(B^\perp)$, it follows that $Z_0 \in \operatorname{span}_{\mathbb{R}}\{B^\perp\}$. Using $[V, \mathfrak{n}] \subset \operatorname{span}_{\mathbb{R}}\{B\}$, we obtain $K_V^\perp(X_2 + Z_0) \in \operatorname{span}_{\mathbb{R}}\{B^\perp\}$ and $(\hat{Z} + kA_0)_V^\perp \in \operatorname{span}_{\mathbb{R}}\{B^\perp\}$. Hence, $K_V^\perp(X_2 + Z_0) \in \operatorname{span}_{\mathbb{R}}\{Z_0\}$ and $\cos\beta = 1$. By a straightforward calculation, $\vartheta_m(\bar{Z}_0) = \vartheta_m((\hat{Z} + kA_0)_V^\perp)/|(\hat{Z} + kA_0)_V^\perp|$, and the minimal period takes the desired form.

Since γ_k must have the same minimal period in both metrics, we have

$$|V|^2 + \frac{4\pi|(\hat{Z}+kA_0)^{\perp}_{V}|^2(\alpha((\hat{Z}+kA_0)^{\perp}_{V}) - \pi)}{\alpha((\hat{Z}+kA_0)^{\perp}_{V})^2} = \|V\|^2 + \frac{4\pi\|(\hat{Z}+kA_0)^{\perp'}_{V}\|^2(\alpha^*((\hat{Z}+kA_0)^{\perp'}_{V}) - \pi)}{\alpha^*((\hat{Z}+kA_0)^{\perp'}_{V})^2},$$

where $|\cdot|$ to the g_1-norm and $\|\cdot\|$ refers to the g_2-norm. Also, $\perp$ refers to the orthogonal complement of $[V,\mathfrak{n}]$ in $\mathfrak{z}$ in the g_1-norm while $\perp'$ refers to the orthogonal complement of $[V,\mathfrak{n}]$ in $\mathfrak{z}$ in the g_2-norm (see Definition 1.17). Now, since V has the same length in both the g_1-norm and g_2-norm, we have

$$\frac{|(\hat{Z}+kA_0)^{\perp}_{V}|^2\big(\alpha((\hat{Z}+kA_0)^{\perp}_{V}) - \pi\big)}{\alpha((\hat{Z}+kA_0)^{\perp}_{V})^2} = \frac{\|(\hat{Z}+kA_0)^{\perp'}_{V}\|^2\big(\alpha^*((\hat{Z}+kA_0)^{\perp'}_{V}) - \pi\big)}{\alpha^*((\hat{Z}+kA_0)^{\perp'}_{V})^2}.$$

Let $\hat{Z} = z_a A_0 + z_b B_0$. By a routine calculation, we obtain

$$(A_0)^{\perp}_{V} = A_0 - \frac{g_1(A_0,B)}{g_1(B,B)}B,$$

and

$$(\hat{Z}+kA_0)^{\perp}_{V} = (z_a+k)(A_0)^{\perp}_{V}.$$

Similarly,

$$(\hat{Z}+kA_0)^{\perp'}_{V} = (z_a+k)(A_0)^{\perp'}_{V}.$$

Using the definition of α,α^* and reducing, we obtain:

$$(z_a+k)\frac{|(A_0)^{\perp}_{V}|^2}{g_1((A_0)^{\perp}_{V},A)} - \pi\frac{|(A_0)^{\perp}_{V}|^2}{g_1((A_0)^{\perp}_{V},A)^2} = (z_a+k)\frac{\|(A_0)^{\perp'}_{V}\|^2}{g_2((A_0)^{\perp'}_{V},A)} - \pi\frac{\|(A_0)^{\perp'}_{V}\|^2}{g_2((A_0)^{\perp'}_{V},A)^2}.$$

Since this must be true for all values of k, we conclude that

$$\frac{|(A_0)^{\perp}_{V}|^2}{g_1((A_0)^{\perp}_{V},A)} = \frac{\|(A_0)^{\perp'}_{V}\|^2}{g_2((A_0)^{\perp'}_{V},A)}$$

and

$$\frac{|(A_0)^{\perp}_{V}|^2}{g_1((A_0)^{\perp}_{V},A)^2} = \frac{\|(A_0)^{\perp'}_{V}\|^2}{g_2((A_0)^{\perp'}_{V},A)^2}.$$

We thus have

$$g_1((A_0)^{\perp}_{V},A) = g_2((A_0)^{\perp'}_{V},A).$$

Now,

$$g_1((A_0)^{\perp}_{V},A) = g_1(A_0,A) - \frac{g_1(A_0,B)g_1(A,B)}{g_1(B,B)} = a_0\left(|A|^2 - \frac{g_1(A,B)^2}{|B|^2}\right).$$

Therefore,

$$|A|^2 - \frac{g_1(A,B)^2}{|B|^2} = \|A\|^2 - \frac{g_2(A,B)^2}{\|B\|^2}.$$

Repeating the same argument for β, β^* and B, we obtain, after interchanging the roles of A and B,

$$|B|^2 - \frac{g_1(A,B)^2}{|A|^2} = \|B\|^2 - \frac{g_2(A,B)^2}{\|A\|^2}.$$

Note that both sides are nonzero by the linear independence of A and B. Let $A = a_1 e + a_2 E$ and $B = b_1 e + b_2 E$. Then $\|A\|^2 = \lambda a_1^2 + \Lambda a_2^2$, $\|B\|^2 = \lambda b_1^2 + \Lambda b_2^2$, and $g_2(A,B) = \lambda a_1 b_1 + \Lambda a_2 b_2$. By a straightforward computation, we obtain

$$\frac{\|B\|^2}{|B|^2} = \frac{\|A\|^2}{|A|^2} = \frac{\|A\|^2\|B\|^2 - g_2(A,B)^2}{|A|^2|B|^2 - g_1(A,B)^2} = \lambda\Lambda.$$

Thus

$$\frac{a_1^2\lambda + a_2^2\Lambda}{a_1^2 + a_2^2} = \lambda\Lambda = \frac{b_1^2\lambda + b_2^2\Lambda}{b_1^2 + b_2^2}.$$

Recall we are assuming that $\lambda < 1 < \Lambda$. Then, since A and B are linearly independent and nonzero, it follows that

$$a_2 = \pm\sqrt{\frac{\lambda(\Lambda - 1)}{\Lambda(1-\lambda)}}\,a_1,$$

$$b_2 = \mp\sqrt{\frac{\lambda(\Lambda - 1)}{\Lambda(1-\lambda)}}\,b_1.$$

Finally, $g_1(A,\zeta) = g_1(B,\zeta) \neq 0$ by definition of ζ, where

$$\zeta = \pm\sqrt{\frac{\lambda(\Lambda - 1)}{\Lambda - \lambda}}\,e \pm \sqrt{\frac{\Lambda(1-\lambda)}{\Lambda - \lambda}}\,E.$$

By plugging these values in, we obtain either $g_1(A,\zeta) = 0$ or $g_2(B,\zeta) = 0$, a contradiction.

Therefore, our assumption that $\lambda < 1 < \Lambda$ is false, and we thus conclude that $\Lambda = \lambda = 1$. It follows that g_2 and g_1 induce the same metric on $\mathfrak{z}$. □

References

[A] P. Anselone, *Collectively compact operator approximation theory and applications to integral equations,* Prentice-Hall, Englewood Cliffs, NJ, 1971.

[Bd] P. Bérard, *Spectral geometry: Direct and inverse problems,* Lecture Notes in Math., 1207, Springer-Verlag, New York, 1986.

[BGM] M. Berger, P. Gauduchon, and E. Mazet, *Le spectre d'une variété Riemannienne,* Lecture Notes in Math., 194, Springer-Verlag, New York, 1971.

[Bs] A. L. Besse, *Manifolds all of whose geodesics are closed,* Ergeb. Math. Grenzgeb. (3), 93, Springer–Verlag, New York, 1978.

[Bu] P. Buser, *Geometry and spectra of compact Riemann surfaces,* Progr. Math., 106, Birkhäuser, Boston, 1992.

[C] I. Chavel, *Eigenvalues in Riemannian geometry,* Academic Press, Orlando, FL, 1984.

[CdV] Y. Colin de Verdière, *Spectre du Laplaican et longueur des géodesiques periodique I, II,* Compositio Math. 27 (1973), 83–106, 159–184.

[CG] L. Corwin and F. P. Greenleaf, *Representations of nilpotent Lie groups and their applications; Part 1: Basic theory and examples,* Cambridge Stud. Adv. Math., 18, Cambridge Univ. Press, New York, 1990.

[Cr] C. Croke, *Rigidity for surfaces of non-positive curvature,* Comment. Math. Helv. 65 (1990), 150–169.

[De] L. DeMeyer, *Closed geodesics in compact nilmanifolds,* Manuscripta Math. 105 (2001), 283–310.

[DG] D. DeTurck and C. S. Gordon, *Isospectral deformations. I. Riemannian structures on two-step nilspaces,* Comm. Pure Appl. Math. 40 (1987), 367–387.

[DGu] J. J. Duistermaat and V. W. Guillemin, *The spectrum of positive elliptic operators and period bicharacteristics,* Invent. Math. 29 (1975), 39–79.

[E1] P. Eberlein, *Geometry of two-step nilpotent groups with a left invariant metric,* Ann. Sci. École Norm. Sup. (4) 27 (1994), 611–660.

[E2] ———, Personal communication.

[G1] C. S. Gordon, *The Laplace spectra versus the length spectra of Riemannian manifolds,* Nonlinear problems in geometry (Mobile, AL, 1985), Contemp. Math., 51, pp. 63–80, Amer. Math. Soc., Providence, RI, 1986.

[G2] ———, *Isospectral closed Riemannian manifolds which are not locally isometric,* J. Differential Geom. 37 (1993), 639–649.

[G3] ———, *Isospectral closed Riemannian manifolds which are not locally isometric: II,* Contemporary mathematics: Geometry of the spectrum (R. Brooks, C. S. Gordon, P. Perry, eds.), Contemp. Math., 173, pp. 121–131, Amer. Math. Soc., Providence, RI, 1994.

[GM] C. S. Gordon and Y. Mao, *Comparisons of Laplace spectra, length spectra and geodesic flows of some Riemannian manifolds,* Math. Res. Lett. 1 (1994), 677–688.

[GMS] C. S. Gordon, Y. Mao, and D. Schueth, *Symplectic rigidity of geodesic flows on two-step nilmanifolds,* Ann. Sci. École Norm. Sup. (4) 30 (1997), 417–427.

[GW1] C. S. Gordon and E. N. Wilson, *Isospectral deformations of compact solvmanifolds,* J. Differential Geom. 19 (1984), 241–256.

[GW2] ———, *The spectrum of the Laplacian on Riemannian Heisenberg manifolds,* Michigan Math. J. 33 (1986), 253–271.

[GW3] ———, *Continuous families of isospectral Riemannian manifolds which are not locally isometric,* J. Differential Geom. 47 (1997), 504–529.

[Gt1] R. Gornet, *The length spectrum and representation theory on two and three-step nilpotent Lie groups,* Contemporary mathematics: Geometry of the spectrum (R. Brooks, C. S. Gordon, P. Perry, eds.), Contemp. Math., 173, pp. 133–156, Amer. Math. Soc., Providence, RI, 1994.

[Gt2] ———, *A new construction of isospectral Riemannian nilmanifolds with examples,* Michigan Math. J. 43 (1996), 159–188.

[Gt3] ———, *The marked length spectrum vs. the p-form spectrum of Riemannian nilmanifolds,* Comment. Math. Helv. 71 (1996), 297–329.

[GtM1] R. Gornet and M. Mast, *The length spectrum of Riemannian two-step nilmanifolds,* Ann. Sci. École Norm. Sup. (4) 33 (2000), 181–209.

[GtM2] ———, *Length minimizing geodesics and the length spectrum of Riemannian two-step nilmanifolds,* J. Geom. Anal. 13 (2003), 107–143.

[GtM3] ———, Notes in preparation.

[K] A. Kaplan, *Riemannian nilmanifolds attached to Clifford modules,* Geom. Dedicata 11 (1981), 127–136.

[Kt] T. Kato, *A short introduction to perturbation theory for linear operators,* Springer-Verlag, New York, 1982.

[LP] K. B. Lee and K. Park, *Smoothly closed geodesics in 2-step nilmanifolds,* Indiana Univ. Math. J. 45 (1996), 1–14.

[M] M. Mast, *Closed geodesics in* 2*-step nilmanifolds,* Indiana Univ. Math. J. 43 (1994), 885–911.

[MR] R. J. Miatello and J. P. Rosetti, *Length spectra and p-spectra of compact flat manifolds,* J. Geom. Anal. 13 (2003), 631–657.

[Ot1] J. Otal, *Le spectre marqué des longeurs des surfaces à courbure négative,* Ann. of Math. (2) 131 (1990), 151–162.

[Ot2] ———, *Sur les longueurs des géodesiques d'une metrique à courbure négative dans le disque,* Comment. Math. Helv. 65 (1990), 334–347.

[P1] H. Pesce, *Calcul du spectre d'une nilvariété de rang deux et applications,* Trans. Amer. Math. Soc. 339 (1993), 433–461.

[P2] ———, *Une formule de Poisson pour les variétés de Heisenberg,* Duke Math. J. 73 (1994), 79–95.

[R] M. S. Raghunathan, *Discrete subgroups of Lie groups,* Ergeb. Math. Grenzgeb. (3), 68, Springer–Verlag, New York, 1972.

[S] A. Selberg, *Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series,* J. Indian Math. Soc. (N.S.) 20 (1956), 47–87.

[V] V. S. Varadarajan, *Lie groups, Lie algebras, and their representations,* Grad. Texts in Math., 102, Springer-Verlag, New York, 1984.

[W] G. Walschap, *Cut and conjugate loci in two-step nilpotent Lie groups,* J. Geom. Anal. 7 (1997), 343–355.

R. Gornet
University of Texas – Arlington
Arlington, TX 76019-0408

rgornet@uta.edu

M. B. Mast
University of Massachusetts – Boston
Boston, MA 02125-3393

mmast@math.umb.edu

Current address
University of Notre Dame
Notre Dame, IN 46556-4618

mmast@nd.edu