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The Pointwise Convergence of Möbius Maps

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1. Introduction

In 1957, Piranian and Thron [6] classified the possible limits of a pointwise convergent sequence of Möbius maps acting in the extended complex plane. Here we consider the problem for Möbius maps acting in higher dimensions.

Let g_n be any sequence of Möbius maps of the extended complex plane $\mathbb{C}_\infty$ onto itself. Let C be the set of points z at which the sequence $g_n(z)$ converges and, for z in C, let $g(z) = \lim_n g_n(z)$.

Theorem A [6]. *Suppose that $C \neq \emptyset$. Then one of the following possibilities occurs*:

(a) $C = \mathbb{C}_\infty$, *and g is a Möbius map*;
(b) $C = \mathbb{C}_\infty$, *and g is constant on the complement of one point but not on* $\mathbb{C}_\infty$;
(c) $C = \{z_1, z_2\}$ *and* $g(z_1) \neq g(z_2)$; *or*
(d) *g is constant on C.*

It is clear that other possibilities can arise in higher dimensions; for example, the sequence of iterates of a nontrivial rotation in $\mathbb{R}^3$ converges on, and only on, the axis of the rotation and at ∞. Here, we establish the corresponding result in higher dimension, and we replace the arguments about the coefficients of the g_n used in [6] by geometric arguments. We shall see that, even in two dimensions, the two cases in which g takes precisely two values play very different roles in the discussion; in fact, (c) is closer to (a) than to (b). The following similar result for quasiconformal mappings is known [8, pp. 69–73].

Theorem B. *Let D be a subdomain of $\mathbb{R}^{k+1}$, and let f_n be a sequence of K-quasiconformal mappings of D into $\mathbb{R}^{k+1}$ that converges pointwise on D to a function f. Then one of the following possibilities occurs*:

(a) *f is a K-quasiconformal map of D onto some domain D'*;
(b) *f takes precisely two values on D, one of which is taken at one point only*; *or*
(c) *f is constant on D.*

However, Theorem B does not subsume Theorem A, for C need not be a domain; indeed, the problem of characterizing the possible sets C in Theorem A is not

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solved in [6] (see [4]). Nonetheless, we identify the largest open set on which *locally uniform convergence* occurs, and often this is more important than pointwise convergence. We shall prove the following result, where $\mathbb{R}^k_\infty$ is the usual compactification of $\mathbb{R}^k$.

THEOREM 1.1. *Let g_n be a sequence of Möbius maps acting on $\mathbb{R}^k_\infty$ that converges pointwise on C (and nowhere else) to the function g. If $C \neq \emptyset$, then one of the following occurs:*

(a) *g is the restriction of some Möbius map to C and $C = h(V \cup \{\infty\})$ for some Möbius map h, where V is either $\{0\}$ or a nontrivial subspace of $\mathbb{R}^k$ not of dimension $k-1$;*
(b) *$C = \mathbb{R}^k_\infty$, and g is constant on the complement of a single point in $\mathbb{R}^k_\infty$ but not on $\mathbb{R}^k_\infty$; or*
(c) *g is constant on C.*

To recapture Theorem A, put $k = 2$. Then V in (a) is $\{0\}$ or $\mathbb{C}$, so that C is $\mathbb{C}_\infty$ or a doubleton. Note that (a) in Theorem 1.1 includes both the cases (a) and (c) in Theorem A.

2. Möbius Maps in Higher Dimensions

The Möbius group $\mathcal{M}_n$ consists of those transformations of $\mathbb{R}^n_\infty$ onto itself that are the composition of an even number of inversions in $(n-1)$-dimensional Euclidean spheres and hyperplanes. We embed $\mathbb{R}^k_\infty$ in $\mathbb{R}^{k+1}_\infty$ by identifying $(x_1, \dots, x_k)$ with $(x_1, \dots, x_k, 0)$. Then $\mathbb{R}^k_\infty$ is the boundary of the upper half-space $\mathbb{H}^{k+1}$ given by $x_{k+1} > 0$, and this is a model of $(k+1)$-dimensional hyperbolic space with the hyperbolic metric ρ_{k+1} derived from the line element $|dx|/x_{k+1}$. The elements of $\mathcal{M}_k$ extend naturally to elements of $\mathcal{M}_{k+1}$ that preserve $\mathbb{H}^{k+1}$, and these extensions constitute the group of conformal isometries of the hyperbolic space $(\mathbb{H}^{k+1}, \rho_{k+1})$. Because $\mathcal{M}_k$ acts as the group of isometries of the metric space $(\mathbb{H}^{k+1}, \rho_{k+1})$, it is generally more profitable to study $\mathcal{M}_k$ through its action on $\mathbb{H}^{k+1}$ rather than its action on $\mathbb{R}^k_\infty$. For more details see [1; 2; 5; 7].

The chordal cross-ratio $[x_1, x_2, x_3, x_4]$ of four distinct points x_j in $\mathbb{R}^k_\infty$ is defined by

$$[x_1, x_2, x_3, x_4] = \frac{\sigma(x_1, x_3)\sigma(x_2, x_4)}{\sigma(x_1, x_4)\sigma(x_2, x_3)},$$

where σ is the chordal metric defined on $\mathbb{R}^n_\infty$ by

$$\sigma(x, y) = \frac{2|x-y|}{\sqrt{1+|x|^2}\sqrt{1+|y|^2}}.$$

It is known (see [2, Thm. 3.2.7]) that a map of $\mathbb{R}^k_\infty$ into itself is Möbius if and only if it preserves chordal cross-ratios. Providing we make the usual conventions about ∞, it follows that

$$[x_1, x_2, x_3, x_4] = \frac{|x_1 - x_3||x_2 - x_4|}{|x_1 - x_4||x_2 - x_3|}.$$

Finally, to minimize the notation when using cross-ratios we shall often write $g_n x$ for $g_n(x)$, and similarly for g_n^{-1}.

3. Some Preliminary Results

We are given Möbius maps g_n acting on $\mathbb{R}^k_\infty$ such that $g(x)$ exists if and only if $x \in C$. Suppose first that there are x_1 and x_2 such that

$$\lim_n g_n(x_1) = \lim_n g_n(x_2) = \alpha, \quad x_1 \neq x_2. \tag{3.1}$$

We may assume that $\alpha \neq \infty$, and we choose a point ζ on the geodesic ℓ in $\mathbb{H}^{k+1}$ with endpoints x_1 and x_2. Then $g_n(\zeta) \to \alpha$ in the Euclidean topology of $\mathbb{R}^{k+1}_\infty$. Now let x be any point in $\mathbb{H}^{k+1}$. Then $\rho_{k+1}(g_n(x), g_n(\zeta)) = \rho_{k+1}(x, \zeta)$ and so $g_n(x) \to \alpha$. The local uniform convergence on $\mathbb{H}^{k+1}$ follows easily from the geometry, and we have proved the next result.

Lemma 3.1. *Suppose that a sequence g_n of Möbius maps converges at two distinct points of $\mathbb{R}^k_\infty$ to the same value α. Then $g_n \to \alpha$ locally uniformly on $\mathbb{H}^{k+1}$.*

Condition (3.1) is a special case of what is known as the *general convergence* of a sequence of Möbius maps; see [3] for more details. Suppose now that (3.1) holds and that x_1, x_2, x_3, x_4 are distinct points in $\mathbb{R}^k_\infty$. Choose a point ζ' on the geodesic with endpoints x_3 and x_4; then, by Lemma 3.1, $g_n(\zeta') \to \alpha$. It is now evident that there must be a subsequence of $g_n(x_3)$ or of $g_n(x_4)$ that converges to α, and we have proved the next result.

Lemma 3.2. *Suppose that* (3.1) *holds. Then there is at most one x in $\mathbb{R}^k_\infty$ such that the sequence $g_n(x)$ does not accumulate at α. In particular, g is constant on C or on the complement of one point of C.*

The main result in this section is as follows.

Theorem 3.3. *Let g_n be a sequence of maps in $\mathcal{M}_k$, and suppose that there exist distinct points x_1, x_2, x_3 in $\mathbb{R}^k_\infty$ such that*

$$\lim_n g_n(x_1) = \lim_n g_n(x_2) = \alpha, \qquad \lim_n g_n(x_3) = \beta, \tag{3.2}$$

where $\alpha \neq \beta$. Then $g_n \to \alpha$ locally uniformly on $\mathbb{R}^k_\infty \backslash \{x_3\}$.

Proof. Lemma 3.2 implies that, for every x in $\mathbb{R}^k_\infty \backslash \{x_3\}$, the sequence $g_n(x)$ accumulates at α. If one such sequence does not converge to α, then we can pass to a subsequence that converges to some point other than α, and this violates Lemma 3.2. Thus $g_n \to \alpha$ pointwise on $\mathbb{R}^k_\infty \backslash \{x_3\}$.

We shall now show that $g_n \to \alpha$ uniformly on each compact subset of $\mathbb{R}^k_\infty \backslash \{x_3\}$. Since each Möbius map is a Lipschitz map of $\mathbb{R}^k_\infty$ onto itself, there is no loss of generality in assuming that $x_3 = \infty = \beta$ and $\alpha = 0$; thus (3.2) becomes

$$\lim_n g_n(x_1) = \lim_n g_n(x_2) = 0, \qquad \lim_n g_n(\infty) = \infty,$$

and we have to show that $g_n \to 0$ locally uniformly on $\mathbb{R}^k$. Let K be a compact subset of $\mathbb{R}^k$, and suppose that $y \in K$. Then

$$\frac{\sigma(x_1,\infty)\sigma(x_2,y)}{\sigma(x_1,x_2)\sigma(y,\infty)} = \frac{\sigma(g_nx_1,g_n\infty)\sigma(g_nx_2,g_ny)}{\sigma(g_nx_1,g_nx_2)\sigma(g_ny,g_n\infty)},$$

so that

$$\sigma(g_nx_2,g_ny) \le \frac{8\sigma(g_nx_1,g_nx_2)}{\sigma(x_1,x_2)\sigma(y,\infty)\sigma(g_nx_1,g_n\infty)}.$$

The denominator is bounded below independently of y, and this implies uniform convergence on K. □

4. Proof of Theorem 1.1

We are given Möbius maps g_n acting on $\mathbb{R}^k_\infty$ such that $g(x)$ exists if and only if $x \in C$. The case when g is constant (which includes the case when C is a singleton) is case (c), and the case when C is a doubleton and g is not constant is case (a) with $V = \{0\}$. Thus, from now on we may assume that C has at least three points and that g is not constant on C. Suppose first that g is not injective. Then (3.2) holds for some x_j, and Theorem 3.3 implies that (b) in Theorem 1.1 holds.

In the remainder of the proof we may assume that there exist distinct points x_1, x_2, x_3 and distinct points y_1, y_2, y_3 such that

$$g_n(x_1) \to y_1, \quad g_n(x_2) \to y_2, \quad g_n(x_3) \to y_3. \tag{4.1}$$

It is known [2, Thm. 3.6.5] that (4.1) implies that a subsequence of the g_n converges uniformly on $\mathbb{R}^k_\infty$ to some Möbius transformation; and, as $g_n \to g$ on C, we see that g extends from C to a Möbius map (which we continue to call g) on $\mathbb{R}^k$. This is the first assertion in (a), and to complete the proof of Theorem 1.1 we need to show that $C = h(V \cup \{\infty\})$, where h is Möbius and V is a subspace of $\mathbb{R}^k$. The idea is to show that the given situation can be reduced to the case in which $g_n(0) = 0$ and $g_n(\infty) = \infty$. Then g_n is a linear map of $\mathbb{R}^k$ onto itself, and the convergence set of a sequence of linear maps is a subspace of $\mathbb{R}^k$.

It is convenient to write $g_n \hookrightarrow (C, g)$ to mean that C is the set of convergence of g_n and that $g_n \to g$ on (and only on) C. We need the following two preliminary results, which will be proved after we have completed the proof of Theorem 1.1.

LEMMA 4.1. *Suppose that* $g_n \hookrightarrow (C, g)$. *Then*:

(a) *for any Möbius map* f, $g_n f \hookrightarrow (f^{-1}(C), gf)$; *and*
(b) *if* h_n *are Möbius maps that converge uniformly to the Möbius map* h, *then* $h_n g_n \hookrightarrow (C, hg)$.

LEMMA 4.2. *Suppose that* $a_n \to 0$ *and* $b_n \to \infty$. *Then there exist Möbius maps* F_n *that converge uniformly to the identity map* I *on* $\mathbb{R}^k_\infty$, *with* $F_n(a_n) = 0$ *and* $F_n(b_n) = \infty$.

We continue with the proof of Theorem 1.1. Because (4.1) holds, we can find a Möbius map f such that $f(0) = x_1$, $f(\infty) = x_2$, and $f(1) = x_3$. Then, by Lemma 4.1 and with $G_n = f^{-1}g^{-1}g_n f$,

$$G_n \hookrightarrow (f^{-1}(C), I), \qquad G_n(0) \to 0, \quad G_n(1) \to 1, \quad G_n(\infty) \to \infty. \tag{4.2}$$

We now apply Lemma 4.2 with $a_n = G_n(0)$ and $b_n = G_n(\infty)$ to obtain the F_n, and then we apply Lemma 4.1(b) with $h_n = F_n$. This gives

$$F_n G_n \hookrightarrow (f^{-1}(C), I), \qquad F_n G_n(0) = 0, \quad F_n G_n(\infty) = \infty. \tag{4.3}$$

Since $F_n G_n$ is Möbius and fixes both 0 and ∞, we see that $F_n G_n(x) = \lambda_n A_n(x)$, where $\lambda_n > 0$ and A_n is an orthogonal matrix with determinant 1 (see [2, Thm. 3.5.1]). It is obvious that if a sequence of linear maps converges at two points then it also converges at any linear combination of these two points; thus its convergence set is a subspace of $\mathbb{R}^k$. We conclude that the convergence set of $F_n G_n$ is, say, $V \cup \{\infty\}$ for some subspace V of $\mathbb{R}^k$. It now follows from (4.3) that $C = f(V \cup \{\infty\})$ as required.

It remains to show that if $k \geq 2$ then $\dim(V) \neq k - 1$. We suppose then that $k \geq 2$ and $\dim(V) \geq k - 1$. Now, for any nonzero x in V (and such x do exist), we have

$$|x| = \lim_n |F_n G_n(x)| = \lim_n \lambda_n |A_n(x)| = \lim_n \lambda_n |x|,$$

so that $\lambda_n \to 1$. This means that the orthogonal maps A_n converge to the identity on V and hence on a subspace V_0 of dimension $k - 1$. Now select an orthonormal basis $e_1, \dots, e_k$ of $\mathbb{R}^k$ such that $e_1, \dots, e_{k-1}$ is a basis of V_0. Then the matrices of the maps A_n with respect to the basis $\{e_i\}$ converge to the diagonal matrix (m_{ij}) with $m_{11} = \cdots = m_{k-1,k-1} = 1$. Since $\det(m_{ij}) = 1$, it follows that $m_{k,k} = 1$ so that the maps A_n converge to I throughout $\mathbb{R}^k$. It is for this reason that the convergence set cannot be a subspace of dimension $k - 1$ (unless $k - 1 = 0$). The proof of Theorem 1.1 is complete. $\square$

We now prove Lemmas 4.1 and 4.2.

Proof of Lemma 4.1. We omit the (easy) proof of (a). To prove (b) we must show that (i) if $x \in C$ then $h_n g_n(x) \to hg(x)$, and (ii) if $h_n g_n(u)$ converges then $u \in C$. For any Möbius f_0 and g_0, we write

$$\hat{\sigma}(f_0, g_0) = \sup_x \sigma(f_0(x), g_0(x)).$$

Then $\hat{\sigma}$ is the metric of uniform convergence on $\mathbb{R}^k_\infty$, and $\mathcal{M}_k$ is a topological group with respect to this metric. Suppose $x \in C$. Then

$$\begin{aligned} \sigma(h_n g_n(x), hg(x)) &\leq \sigma(h_n g_n(x), hg_n(x)) + \sigma(hg_n(x), hg(x)) \\ &\leq \hat{\sigma}(h_n, h) + L(h)\sigma(g_n(x), g(x)), \end{aligned}$$

where $L(h)$ is the Lipschitz constant for h. It follows that $h_n g_n(x) \to hg(x)$, which proves (i).

Now suppose that $h_n g_n(u) \to v$ for some u and v. Then

$$\begin{aligned}\sigma(g_n(u), h^{-1}(v)) &= \sigma(h^{-1}hg_n(u), h^{-1}(v)) \\ &\le L(h^{-1})\sigma(hg_n(u), v) \\ &\le L(h^{-1})[\sigma(hg_n(u), h_n g_n(u)) + \sigma(h_n g_n(u), v)] \\ &\le L(h^{-1})[\hat{\sigma}(h, h_n) + \sigma(h_n g_n(u), v)] \\ &\to 0\end{aligned}$$

as $n \to \infty$, so that $u \in C$. This completes the proof of Lemma 4.1. □

Proof of Lemma 4.2. Let $\beta(x) = x/|x|^2$ (this in inversion in the unit sphere), and let

$$f_n(x) = x - \beta(b_n), \quad g_n(x) = x - \beta f_n \beta(a_n), \quad F_n(x) = g_n \beta f_n \beta(x).$$

We note that

$$\beta(b_n) \to 0, \qquad \beta f_n \beta(a_n) = \beta(\beta(a_n) - \beta(b_n)) \to \beta(\infty) = 0;$$

hence f_n, g_n, and F_n are Möbius maps (i.e., their coefficients are finite) for all sufficiently large n. Next, it is clear that if $t(x) = x + \tau_n$ and $\tau_n \to 0$ then $t_n \to I$ uniformly on $\mathbb{R}^k_\infty$. Thus $f_n \to I$ and $g_n \to I$, so $F_n = g_n \beta f_n \beta \to I\beta I\beta = I$. Finally, $F_n(a_n) = 0$ and $F_n(b_n) = \infty$ because

$$\begin{aligned}F_n^{-1}(0) &= \beta f_n^{-1} \beta g_n^{-1}(0) = \beta f_n^{-1} \beta(\beta f_n \beta(a_n)) = a_n, \\ F_n^{-1}(\infty) &= \beta f_n^{-1} \beta g_n^{-1}(\infty) = \beta f_n^{-1} \beta(\infty) = \beta f_n^{-1}(0) = \beta\beta(b_n) = b_n.\end{aligned}$$ □

It is easy to see that any subspace V of $\mathbb{R}^k$, with ∞ attached, can arise as the set C of convergence of some sequence g_n of Möbius maps, provided that $\dim(V) \neq k-1$ (unless $k = 1$). We simply write $\mathbb{R}^k = V \oplus W$ and define a sequence of orthogonal maps (a) that leave V and W invariant and (b) whose restrictions to V and W are the orthogonal maps I (the identity) and A_n, respectively. Provided that $\dim(W) \neq 1$, we can clearly choose the A_n so that the only point of W at which convergence occurs is the origin. Thus the convergence set of the Möbius g_n is $V \cup \{\infty\}$.

5. Locally Uniform Convergence

Given (3.1), namely, $\lim_n g_n(x_1) = \lim_n g_n(x_2) = \alpha$ where $x_1 \neq x_2$, we may ask: *where does g_n converge locally uniformly to α?* For each w in $\mathbb{R}^k_\infty$, let $\Lambda(w)$ be the set of accumulation points of the sequence $g_n^{-1}(w)$. Then we have the following result [3, Thm. 9.6].

THEOREM 5.1. *Suppose that a sequence g_n of Möbius maps acting on $\mathbb{R}^k_\infty$ satisfies* (3.1). *If $w \neq \alpha$, then $g_n \to \alpha$ locally uniformly on $\mathbb{R}^k_\infty \backslash \Lambda(w)$ and on no larger open subset of $\mathbb{R}^k_\infty$. In particular, $\Lambda(w)$ is independent of w in $\mathbb{R}^k_\infty \backslash \{\alpha\}$.*

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