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Bounds on the Average Bending of the Convex Hull Boundary of a Kleinian Group

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1. Introduction

In this paper we consider hyperbolic manifolds with incompressible convex core boundary. We show that total bending along a geodesic arc on the boundary of the convex core is bounded above by a function of its length. Integrating this function over the unit tangent bundle of the boundary of the convex core, we obtain a new universal upper bound on the total bending of the convex core boundary. Furthermore, we produce a new universal upper bound on the Lipschitz constant for the map from the convex core boundary to the hyperbolic structure at infinity. These results improve on earlier bounds of Bridgeman and Canary.

Let $N = \mathbf{H}^3/\Gamma$ be an orientable hyperbolic manifold with domain of discontinuity $\Omega(\Gamma)$ and limit set L_Γ. In this paper we restrict ourselves to the case when all the components of $\Omega(\Gamma)$ are simply connected. This is a natural restriction to make and includes the set of quasi-Fuchsian groups. Let $CH(L_\Gamma)$ be the convex hull of Γ and let β_Γ be the bending lamination on $\partial CH(L_\Gamma)$. Let $C(N) = CH(L_\Gamma)/\Gamma$ be the convex core and let β_N be the bending lamination on $\partial C(N)$. Then we observe that $\partial C(N)$ is incompressible if and only if the components of $\Omega(\Gamma)$ are all simply connected.

If α is a geodesic arc in $CH(L_\Gamma)$ then the average bending $B(\alpha)$ is defined to be the bending per unit length, or specifically

$$B(\alpha) = \frac{i(\alpha, \beta_\Gamma)}{l(\alpha)},$$

where i is the intersection number and $l(\alpha)$ is the length of α (see [2]).

In [2], Bridgeman considers bounds on the average bending for quasi-Fuchsian groups and proves that, for a quasi-Fuchsian group Γ, if $l(\alpha) \leq \log 3$ then $i(\alpha, \beta_\Gamma) \leq 2\pi$. In [3], the geometry of the convex core boundary $\partial C(N)$ is compared with the geometry of the domain of discontinuity Ω/Γ for a general Kleinian group. One outcome is an improvement of the bound just described on intersection number to prove that, for a Kleinian group Γ such that the components of $\Omega(\Gamma)$ are simply connected, if $l(\alpha) \leq 2\sinh^{-1} 1$ then $i(\alpha, \beta_\Gamma) \leq 2\pi$.

Both these bounds on the intersection number give universal upper bounds for the average bending of geodesic arcs of a given fixed length. By considering geodesics α of length $l(\alpha) = 2\sinh^{-1} 1$, we obtain $B(\alpha) \leq \pi/\sinh^{-1} 1$.

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Bounds on the average bending imply a surprising number of results about the geometry of the convex hull boundary. In particular, Bridgeman and Canary prove the following.

THEOREM 1.1 [3; 4]. *Let* $K = \pi/\sinh^{-1} 1 \approx 3.5644$, *and let* Γ *be a Kleinian group such that the components of* $\Omega(\Gamma)$ *are simply connected. Then*:

1. *if* $l(\beta_N)$ *is the length of the bending lamination* β_N, *then*
$$l(\beta_N) \le K \cdot \pi^2 |\chi(\partial C(N))|;$$
2. *if* α *is a closed geodesic in the boundary of the convex core* $\partial C(N)$, *then*
$$B(\alpha) = \frac{i(\alpha, \beta_N)}{l(\alpha)} \le K;$$
3. *there exists a* $(1+K)$ *Lipschitz map* $s\colon \partial C(N) \to \Omega(\Gamma)/\Gamma$ *that is a homotopy inverse of the retract map* $r\colon \Omega(\Gamma)/\Gamma \to \partial C(N)$.

Epstein, Marden, and Markovic [6] consider convex pleated planes in $\mathbf{H}^3$ and prove a number of important results. One part of their paper defines the *roundedness* of a convex pleated plane. Given a convex pleated plane P with bending lamination β_P, the roundedness of P is defined to be the supremum of $i(\alpha, \beta_P)$ over all geodesics α of length 1. Epstein, Marden, and Markovic define C_1 to be the supremum of roundedness over all embedded convex pleated planes, and they note that the upper bound on the intersection number in [2] applies in the absence of a group structure and hence $i(\alpha, \beta_P) \le 2\pi$ for $l(\alpha) \le \log 3$. Because $1 < \log 3$, this implies that $C_1 \le 2\pi$ and, giving an example of an embedded convex pleated plane with roundedness of $\pi + 1$, the authors therefore prove that $\pi + 1 \le C_1 \le 2\pi$.

The main result of this paper is the following theorem.

MAIN THEOREM. *There exists a monotonically increasing function*
$$F\colon [0, 2\sinh^{-1} 1] \to [\pi, 2\pi]$$
such that, if Γ *is a Kleinian group* (*where the components of* $\Omega(\Gamma)$ *are simply connected*) *and if* α *is a geodesic arc in* $\partial CH(L_\Gamma)$ *of length* $l(\alpha) \le 2\sinh^{-1} 1$, *then*
$$i(\alpha, \beta_\Gamma) \le F(l(\alpha)).$$

In this paper we give an explicit formula for F and use it to demonstrate the following improvement on Theorem 1.1.

THEOREM 1.2. *There exist constants* $K_0, K_1 < K$ *with* $K_0 < 2.8396$ *and* $K_1 < 3.4502$ *such that, if* Γ *is a Kleinian group where the components of* $\Omega(\Gamma)$ *are simply connected, then*:

1. *if* $l(\beta_N)$ *is the length of the bending lamination* β_N, *then*
$$l(\beta_N) \le K_0 \cdot \pi^2 |\chi(\partial C(N))|;$$

2. *if α is a closed geodesic in the boundary of the convex core $\partial C(N)$, then*

$$B(\alpha) = \frac{i(\alpha, \beta_N)}{l(\alpha)} \leq K_1;$$

3. *there exists a* $(1 + K_1)$ *Lipschitz map* $s: \partial C(N) \to \Omega(\Gamma)/\Gamma$ *that is a homotopy inverse of the retract map* $r: \Omega(\Gamma)/\Gamma \to \partial C(N)$.

We define the constant B_N by

$$B_N = \frac{l(\beta_N)}{\pi^2 |\chi(\partial C(N)|}.$$

Then B_N can be interpreted as the average bending of the manifold N. Thus, Theorem 1.2 gives that $B_N \leq 2.8396$.

Evaluating F at 1, we obtain an improved upper bound on the constant C_1.

THEOREM 1.3. *The supremum C_1 of roundedness over embedded convex pleated planes satisfies*

$$C_1 \leq F(1) = 2\pi - 2\sin^{-1}\left(\frac{1}{\cosh 1}\right) = 4.8731.$$

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2. Background

An orientable hyperbolic 3-manifold $\mathbf{H}^3/\Gamma$ is the quotient of hyperbolic 3-space $\mathbf{H}^3$ by a discrete torsion-free subgroup of the group $\mathrm{Isom}_+(\mathbf{H}^3)$ of orientation-preserving isometries of $\mathbf{H}^3$. We may identify $\mathrm{Isom}_+(\mathbf{H}^3)$ with the group $\mathrm{PSL}_2(\mathbf{C})$ of Möbius transformations of $\hat{\mathbf{C}}$. The *domain of discontinuity* $\Omega(\Gamma)$ is the largest open set in $\hat{\mathbf{C}}$ on which Γ acts properly discontinuously, and the limit set L_Γ is its complement. In this paper we will consider only Kleinian groups Γ such that the components of $\Omega(\Gamma)$ are simply connected. We note that, in particular, if Γ is quasi-Fuchsian then $\Omega(\Gamma)$ has two simply connected components.

The main object of interest in this paper is the convex hull of a Kleinian group. The *convex hull* $CH(L_\Gamma)$ of L_Γ is the smallest convex subset of $\mathbf{H}^3$ such that all geodesics with both limit points in L_Γ are contained in $CH(L_\Gamma)$. The *convex core* $C(N)$ of $N = \mathbf{H}^3/\Gamma$ is the quotient of $CH(L_\Gamma)$ by Γ, and it is the smallest convex submanifold of N such that the inclusion map is a homotopy equivalence. Each component of the boundary $\partial C(N)$ of the convex core is a pleated surface; in other words, there is a pathwise isometry $f: S \to \partial C(N)$ from a hyperbolic surface S onto N that is totally geodesic in the complement of a disjoint collection β_N of geodesics known as the *pleating locus.* For a complete description of the geometry of the convex hull, see Epstein and Marden [5].

The pleating locus β_N inherits a measure on arcs transverse to β_N that records the total amount of bending along any transverse arc, so β_N is a measured lamination. A *measured lamination* on a finite-area hyperbolic surface S consists of a closed subset λ of S that is the disjoint union of geodesics, together with an invariant measure (with respect to projection along λ) on arcs transverse to λ. The set of measured laminations whose support is a finite collection of simple closed geodesics is dense in the space $ML(S)$ of all measured laminations on S (see [7]).

3. Hyperbolic Geometry

We now state some elementary facts about hyperbolic geometry. For a reference see either Thurston [9] or Beardon [1]. In the following we compactify $\mathbf{H}^n$ using the sphere at infinity, $\mathbf{S}^{n-1}_\infty$.

Let g_1, g_2 be two geodesics in $\mathbf{H}^n$. Then g_1, g_2 are *parallel* if $g_1 \cap g_2 = \emptyset$. Furthermore, g_1, g_2 are *ultraparallel* if $\bar{g}_1 \cap \bar{g}_2 = \emptyset$. Note that g_1, g_2 have a unique commmon perpendicular if and only if they are ultraparallel.

The following lemma describes the shortest curve between a geodesic and a ray in the hyperbolic plane. The proof is an elementary exercise and is omitted for the sake of brevity.

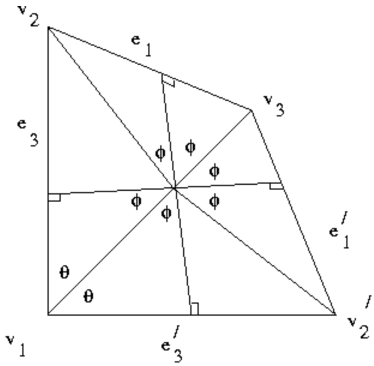
LEMMA 3.1. *Let g be a geodesic and let r be a ray in $\mathbf{H}^2$, with r having finite endpoint x, such that $\bar{g} \cap \bar{r} = \emptyset$. Let g_r be the unique geodesic such that $r \subset g_r$. If g and g_r are ultraparallel, let p be the unique perpendicular between g and g_r. If p exists and if $p \cap r \neq \emptyset$, then the shortest curve from g to r is p; otherwise, the shortest curve from g to r is the unique perpendicular from x to g.*

Let T be a hyperbolic triangle with vertices v_1, v_2, v_3 and edges e_1, e_2, e_3 such that e_i is opposite v_i. A curve α in T *joins e_1 to e_3 via e_2* if α has endpoints on e_1 and e_3 (respectively) and contains a point of e_2.

LEMMA 3.2. *Let T have angle θ at v_1 and let v_2 and v_3 both be ideal vertices. Then the shortest curve in T that joins e_1 to e_3 via e_2 has length $L(\theta)$, given by*

$$L(\theta) = \begin{cases} \cosh^{-1}\left(\dfrac{2}{\sqrt{3-\sec\theta}}\right) + \cosh^{-1}\left(\dfrac{2\cos\theta}{\sqrt{3-\sec\theta}}\right), & \theta < \dfrac{\pi}{3}, \\ \cosh^{-1}\left(\dfrac{1}{\sin(\theta/2)}\right), & \theta \geq \dfrac{\pi}{3}. \end{cases} \tag{1}$$

Proof. We reflect T in edge e_2 to obtain triangle T' with vertices v_i' and edges e_i'. Because we reflected in e_2, we have $e_2' = e_2$ as well as $v_1' = v_1$ and $v_3' = v_3$. We consider the quadrilateral $Q = T \cup T'$. The geodesic e_1 is opposite the ray e_3' in the quadrilateral Q. Let α be the shortest curve from e_1 to e_3', as described in Lemma 3.1. If $\alpha \subset Q$ then α must intersect the diagonal e_2 of Q. Therefore,



Quadrilateral Q

by reflecting T' back onto T, we obtain the shortest curve in T that joins e_1 to e_3 via e_2. We will show that α is indeed always in Q and that the formula for L is correct.

We let p_{v_1} be the perpendicular from e_1 to v_1. Then p_{v_1} bisects T and meets e_3' in an angle $3\theta/2$. Let E_3' be the geodesic containing the ray e_3' and let p be the perpendicular from e_1 to E_3', if it exists. If p exists then both p_{v_1} and p are perpendicular to e_1. Therefore, if p exists then either $p = p_{v_1}$ or p does not intersect with p_{v_1}. Since p_{v_1} makes an angle $3\theta/2$ with e_3', it follows that $p = p_{v_1}$ if and only if $3\theta/2 = \pi/2$. Furthermore, p intersects the interior of e_3' if and only if $3\theta/2 < \pi/2$. Thus for $\theta < \pi/3$ we have that p intersects the interior of e_3' and $\alpha = p$. Otherwise, if $\theta \geq \pi/3$ then $\alpha = p_{v_1}$. Therefore, by hyperbolic trigonometry we have

$$L(\theta) = \cosh^{-1}\left(\frac{1}{\sin(\theta/2)}\right) \quad \text{for } \theta \geq \frac{\pi}{3}.$$

We now consider $\theta < \pi/3$. Then $\alpha = p$ and intersects e_3' in an interior point. Since Q is convex, α intersects the diagonal e_2 in an interior point c. We join c to each vertex of Q and then drop a perpendicular from c to each side of Q. This decomposes Q into eight hyperbolic right-angled triangles. Let ϕ be the angle at c between α and e_2 in this decomposition. By symmetry, all but two of the angles at c are equal to ϕ. Hence, the other angles are both $\pi - 3\phi$ (see figure).

We let l_1 be the length of α in T and l_2 the length of α in T'. Then in T we have a right-angled triangle with one ideal vertex having one angle equal to ϕ and one side of length l_1. Thus, by hyperbolic trigonometry we have

$$\cosh(l_1) = \frac{1}{\sin(\phi)}.$$

Also we have a right-angled triangle with one ideal vertex having one angle equal to $\pi - 3\phi$ and one side equal to l_2. Therefore,

$$\cosh(l_2) = \frac{1}{\sin(\pi - 3\phi)} = \frac{1}{\sin(3\phi)}.$$

As a result,

$$L(\theta) = \cosh^{-1}\left(\frac{1}{\sin(\phi)}\right) + \cosh^{-1}\left(\frac{1}{\sin(3\phi)}\right). \tag{2}$$

To relate ϕ to θ, we note that we have a right-angled hyperbolic triangle with angles θ and ϕ and with side of length l_2 opposite angle θ. Then it follows that

$$\cosh(l_2) = \frac{\cos\theta}{\sin\phi}.$$

Substituting in for $\cosh(l_2)$, we obtain

$$\cos\theta = \frac{\sin\phi}{\sin(3\phi)} = \frac{1}{2\cos(2\phi)+1}.$$

Solving for ϕ in terms of θ, we obtain

$$\cos 2\phi = \frac{1-\cos\theta}{2\cos\theta} = \frac{1}{2}(\sec\theta - 1).$$

Thus

$$\sin^2\phi = \frac{1}{2}(1-\cos 2\phi) = \frac{3-\sec\theta}{4}.$$

Therefore, we finally have the form of L for $\theta < \pi/3$ given by

$$L(\theta) = \cosh^{-1}\left(\frac{2}{\sqrt{3-\sec\theta}}\right) + \cosh^{-1}\left(\frac{2\cos\theta}{\sqrt{3-\sec\theta}}\right). \qquad \square$$

We now describe the behavior of the function L.

LEMMA 3.3. *The function $L\colon [0,\pi] \to \mathbf{R}$ is continuous and monotonically decreasing.*

Proof. By definition, L is a smooth function on each of the intervals $[0,\pi/3)$ and $[\pi/3,\pi]$. For $\theta = \pi/3$ we have $L(\pi/3) = \cosh^{-1}(2)$. Also, $\lim_{\theta\to(\pi/3)^-} L(\theta) = \cosh^{-1}(2) + \cosh^{-1}(1) = \cosh^{-1}(2)$. Thus L is continuous.

To prove the remainder of the lemma, we consider $L'(\theta)$ restricted to the intervals $[0,\pi/3)$ and $[\pi/3,\pi]$ separately. We note that if $f(x) = \cosh^{-1}\left(\frac{1}{\sin x}\right)$ then the derivative satisfies

$$f'(x) = \frac{-|\tan x|}{\tan x \sin x} = \frac{\pm 1}{\sin x},$$

where the sign is determined by the sign of $-\tan x$.

On the interval $[\pi/3,\pi]$ we have

$$L'(\theta) = \frac{-1}{2\sin(\theta/2)}.$$

Thus L is monotonically decreasing on the interval $[\pi/3,\pi]$.

We now consider the monotonicity of L on the interval $[0,\pi/3)$. Since $\phi \in (\pi/6,\pi/4]$, it follows by equation (2) that

$$L'(\phi) = \frac{-1}{\sin\phi} + \frac{3}{\sin 3\phi}.$$

Since $\sin 3\phi = \sin\phi(2\cos 2\phi + 1)$ we have

$$L'(\phi) = \frac{3 - (2\cos 2\phi + 1)}{\sin\phi(2\cos 2\phi + 1)} = \frac{2(1 - \cos 2\phi)}{\sin\phi(2\cos 2\phi + 1)},$$

and since $\sin^2\phi = \frac{1}{2}(1 - \cos 2\phi)$ we have

$$L'(\phi) = \frac{4\sin^2\phi}{\sin\phi(2\cos 2\phi + 1)} = \frac{4\sin\phi}{(2\cos 2\phi + 1)}.$$

As $\phi \in (\pi/6, \pi/4]$, both the numerator and denominator are greater than zero and hence $L'(\phi) > 0$. Since ϕ is monotonicaly decreasing as a function of θ, we conclude that L is monotonically decreasing on $[0, \pi/3)$.

Finally we observe that, since L is continuous on $[0, \pi]$ and monotonically decreasing on both $[0, \pi/3)$ and $[\pi/3, \pi]$, it follows that L is monotonically decreasing on the interval $[0, \pi]$. □

Evaluating the endpoints yields

$$L(0) = 2\cosh^{-1}\left(\sqrt{2}\right) = 2\sinh^{-1} 1, \qquad L(\pi) = 0.$$

Thus, L maps the interval $[0, \pi]$ on to the interval $[0, 2\sinh^{-1} 1]$.

We define Θ to be the inverse function of L. Because L when restricted to $[\pi/3, \pi]$ has a simple inverse function, we have

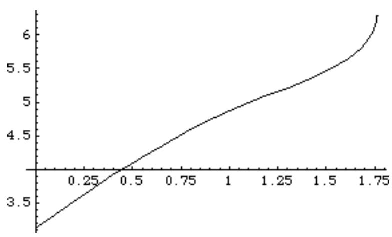
$$\Theta(x) = 2\sin^{-1}\left(\frac{1}{\cosh x}\right) \quad \text{for } \ 0 \le x \le \cosh^{-1} 2 = 1.3169.$$

We define the function $F\colon [0, 2\sinh^{-1} 1] \to [\pi, 2\pi]$ by $F(x) = 2\pi - \Theta(x)$. In particular, we note that $1 \le \cosh^{-1} 2$ entails

$$F(1) = 2\pi - 2\sin^{-1}\left(\frac{1}{\cosh 1}\right) = 4.8731.$$

A direct corollary of the description of L is the following description of F.

COROLLARY 3.4. *The function* $F\colon [0, 2\sinh^{-1} 1] \to [\pi, 2\pi]$ *is continuous and monotonically increasing.*



Graph of function F

We now consider a configuration of planes in $\mathbf{H}^3$. Let H_1, H_2, H_3 be three closed half-spaces in $\mathbf{H}^3$ and set $P_i = \partial H_i$. We consider the convex set $C = \mathbf{H}^3 - \bigcup H_i^o$ obtained by taking the complements of the interiors of the half-spaces H_i. A curve $\alpha\colon [0,1] \to C$ *joins* P_1 to P_3 via P_2 if $\alpha(0) \in P_1$, $\alpha(1) \in P_3$, and $\alpha(t) \in P_2$ for some $t \in [0,1]$.

LEMMA 3.5 [3]. *Let H_1, H_2, H_3 be disjoint half-spaces in $\mathbf{H}^3$ and let $\alpha\colon [0,1] \to C$ be a curve joining P_1 to P_3 via P_2. Then $l(\alpha) \geq 2\sinh^{-1} 1$. Furthermore, if $l(\alpha) = 2\sinh^{-1} 1$ then $\bar{H}_1 \cap \bar{H}_2 = \{a\}$, $\bar{H}_2 \cap \bar{H}_3 = \{b\}$, and $\bar{H}_1 \cap \bar{H}_3 = \{c\}$, where a, b, c are three distinct points in $\mathbf{S}^2_\infty$.*

We now consider a configuration that arises in the proof of the main theorem.

LEMMA 3.6. *Let H_1, H_2, H_3 be half-spaces such that $H_1 \cap H_2 = \emptyset$, $H_1 \cap H_3 = \emptyset$, and $\bar{H}_1 \cap \bar{H}_2 = \{a\}$ for $a \in \mathbf{S}^2_\infty$. If there exists a curve $\alpha\colon [0,1] \to C$ of length $l \leq 2\sinh^{-1} 1$ joining P_1 to P_3 via P_2, then the interior dihedral angle θ between H_2 and H_3 satisfies*

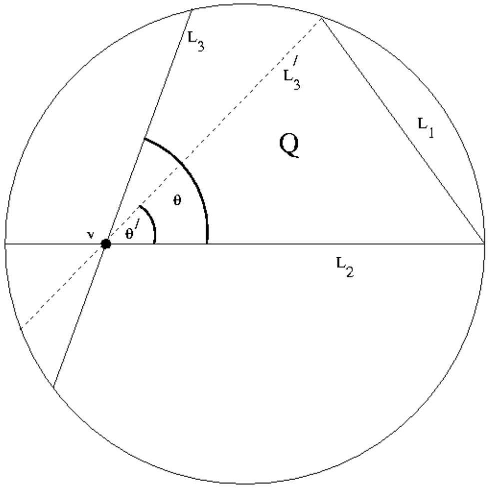
$$\theta \geq \Theta(l).$$

Proof. Since $l \leq 2\sinh^{-1} 1$, by Lemma 3.5 we can assume that $\bar{H}_2 \cap \bar{H}_3 \neq \emptyset$ and hence the interior dihedral angle is well-defined. Take P to be the unique plane perpendicular to P_1, P_2, and P_3; hence P must pass through the point a. We let $L_i = P_i \cap P$ and let $Q = C \cap P$. Then Q is a (possibly infinite-area) quadrilateral with vertex v given by $v = L_2 \cap L_3$. Because P is perpendicular to P_2 and P_3, the angle at v in Q is the dihedral angle between H_2 and H_3. Orthogonal projection maps α onto the region Q and decreases distance. Therefore, projecting α onto Q yields a curve $\alpha'\colon [0,1] \to Q$ of length $l' \leq l$ that joins L_1 to L_3 via L_2. Let $\alpha'(t) \in L_2$; then we let g be the geodesic arc joining $\alpha'(0)$ to $\alpha'(t)$. We replace the arc $\alpha'([0,t])$ by g to obtain $\alpha'' = g \cup \alpha'([t,1])$, and hence length l'' of α'' satisfies $l'' \leq l' \leq l$. We truncate Q to form a finite-area triangle T by letting L_3' be the diagonal in Q containing v. Triangle T is bounded by L_1, L_2, and L_3'. The angle θ' at v in T satisfies $\theta' \leq \theta$. By definition of g, we have $g \subset T$. Therefore, as L_3' separates L_3 from L_1 in Q, a subarc of α'' must join L_1 to L_3' via L_2. We thus have $l \geq l'' \geq L(\theta')$. Since $\theta' \leq \theta$ and L is monotonically decreasing, $L(\theta') \geq L(\theta)$. Therefore, $l \geq L(\theta)$ and again, as L is monotonically decreasing, $L^{-1}(l) \leq \theta$. Thus, finally, $\theta \geq \Theta(l)$. □

4. Support Planes

We first need to recall some background material on convex hulls. For a full description of convex hulls, see [5].

If Γ is a Kleinian group with convex hull $CH(L_\Gamma)$, then a *support plane* to $CH(L_\Gamma)$ is a hyperbolic plane P in H^3 that bounds a half-space H_P such that $H_P \cap \partial CH(L_\Gamma) \subseteq P$. The half-space H_P is considered to be implicit, so P is naturally oriented by taking the normal to point toward the interior of H_P.



Intersection of H_1, H_2, H_3 with unique perpendicular plane

Thus, a support plane P to a convex hull $CH(L_\Gamma)$ does not pass through $\partial CH(L_\Gamma)$ but does have a glancing intersection with it. In general, the intersection of P and $\partial CH(L_\Gamma)$ can either be a single geodesic, called a *bending line,* or a flat piece of the convex hull boundary bounded by a set of disjoint geodesics, called a *flat*. If P_1 and P_2 are support planes with $P_1 \cap P_2 \neq \emptyset$ and $P_1 \neq P_2$, then the line $r = P_1 \cap P_2$ is called a *ridge line.*

If $x \in \partial CH(L_\Gamma)$ then either x lies in the interior of a flat or x is on some bending line. If x is in the interior of a flat then there is a unique support plane P containing x. If $x \in b$, where b is a bending line, let $\Sigma(b)$ be the set of support planes to b. The set $S(b)$ of oriented planes containing b is a circle and $\Sigma(b) \subseteq S(b)$. Since $\Sigma(b)$ is connected, it is either a closed arc or a point. We let P_1 and P_2 be the two extreme planes of $\Sigma(b)$. If b is oriented then we can refer to the extreme planes as the left and right extreme planes. The *bending angle* at b is defined to be the angle between P_1 and P_2. Thus, the bending angle is the exterior dihedral angle between the extreme planes at b. If x is on a bending line b, we define $\beta(x)$ to be the bending angle at b; otherwise we define $\beta(x) = 0$.

The union of the bending lines in $\partial CH(L_\Gamma)$ is denoted β_Γ and is called the *bending lamination*. Thurston defined a *transverse measure* on β_Γ that assigns, to every arc α transverse to β_Γ, a value $i(\alpha, \beta_\Gamma)$ that corresponds to the amount of bending along α (see [8]). Therefore, β_Γ is a measured lamination. In particular, the bending measure is a countable additive measure on the set of transverse arcs (see [5]); that is, if α is subdivided into subarcs $\{\alpha_1, \dots, \alpha_n\}$ transverse to β_Γ, then

$$i(\alpha, \beta_\Gamma) = \sum_{i=1}^{n} i(\alpha_i, \beta_\Gamma).$$

If the arc α is a closed arc with endpoints x, y whose interior α^o is transverse to β_Γ, then we define

$$i(\alpha, \beta_\Gamma) = \beta(x) + i(\alpha^o, \beta_\Gamma) + \beta(y).$$

The bending lamination β_Γ on $\partial CH(L_\Gamma)$ projects to the pleating locus β_N of $\partial C(N)$.

In [3], the definition of the intersection form is modified to allow the subarcs to have endpoints on β_Γ and keep track of support planes. Let P and Q be support planes at x and y, respectively. If α intersects a bending line b, then an orientation on α gives an orientation on the bending line b. Thus we orient α from x to y, and we let $\bar{P}$ be the rightmost support plane at x and $\bar{Q}$ the leftmost support plane at y. Let θ_P be the exterior dihedral angle between P and $\bar{P}$, and let θ_Q be the exterior dihedral angle between the support planes $\bar{Q}$ and Q. Then we define

$$i(\alpha, \beta_\Gamma)_P^Q = \theta_P + i(\alpha^o, \beta_\Gamma) + \theta_Q.$$

Observe that if α has unique support planes at its endpoints then $i(\alpha, \beta_\Gamma)_P^Q = i(\alpha, \beta_\Gamma)$.

Let $\alpha\colon [0, 1] \to \partial CH(L_\Gamma)$ be a path whose interior is transverse to β_Γ and let $\{0 = t_0 < t_1 < \cdots < t_n = 1\}$ be a subdivision of $[0, 1]$. Let α_i be the closed subarc obtained by restricting α to the interval $[t_{i-1}, t_i]$. Let P_i be a support plane at $\alpha(t_i)$ with $P_0 = P$ and $P_n = Q$. Then it follows from the additivity of the standard intersection number that

$$i(\alpha, \beta_\Gamma)_P^Q = \sum_{i=1}^{n} i(\alpha_i, \beta_\Gamma)_{P_{i-1}}^{P_i}.$$

This is the key additivity property for our modified intersection number.

Let $\alpha\colon [0, 1] \to \partial CH(L_\Gamma)$ be a path whose interior is transverse to β_Γ and let P and Q be support planes to $\partial CH(L_\Gamma)$ at $\alpha(0)$ and $\alpha(1)$. We travel along α to obtain a continuous one-parameter family of support planes $\{P_t \mid t \in [0, k]\}$ along α from P to Q (see [3] for a full description). Since a point on α may not have a unique support plane, there is a continuous monotonically increasing (piecewise linear) function $s\colon [0, k] \to [0, 1]$ such that P_t is a support plane to $\alpha(s(t))$.

We say that (P, Q) is a *roof* over α if, for all $t \in [0, k]$, $P \cap P_t \neq \emptyset$ and the interiors of the half-spaces H_P and H_{P_t} also intersect. Furthermore, we say that (P, Q) is a π*-roof* if (P, P_t) is a roof over $\alpha([0, s(t)])$ for all $0 \leq t < k$ but (P, Q) is not a roof over α. We will see that if (P, Q) is a π-roof then $\bar{H}_P \cap \bar{H}_Q = \{a\}$ where $a \in \mathbf{S}^2_\infty$.

We now define monotonicity for geodesics in the hyperbolic plane. Let $\{g_t\}$ be a continuous family of geodesics in a hyperbolic plane that is indexed by an interval J. We say that the family is *monotonic* on J if, given $a, b \in J$ such that $a < b$ and $g_a \cap g_b \neq \emptyset$, we have $g_t = g_a$ for all $t \in [a, b]$.

The following lemma allows us to estimate the intersection number along a geodesic on $\partial CH(L_\Gamma)$ by using support planes.

LEMMA 4.1 [3]. *Let $\alpha\colon [0,1] \to \partial CH(L_\Gamma)$ be a parameterized geodesic arc, let (P, Q) be a roof over α, and let $\{P_t \mid t \in [0,k]\}$ be the continuous one-parameter family of support planes over α joining P to Q. Then*:

1. *we have*
$$i(\alpha, \beta_\Gamma)_P^Q \le \theta \le \pi,$$
where θ is the exterior dihedral angle between P and Q; and
2. *there is a $\bar{t} \in [0,k]$ such that $P_t = P$ if $t \in [0, \bar{t}\,]$ and the ridge lines $\{r_t = P \cap P_t \mid t > \bar{t}\}$ exist and form a monotonic family of geodesics on P.*

The following corollary follows immediately from Lemma 4.1 by continuity.

COROLLARY 4.2 [3]. *If (P, Q) is a π-roof over α, then $i(\alpha, \beta_\Gamma)_P^Q \le \pi$ and $\bar{H}_P \cap \bar{H}_Q = \{a\}$ where $a \in \Omega(\Gamma)$.*

We now restate the main theorem before proving it.

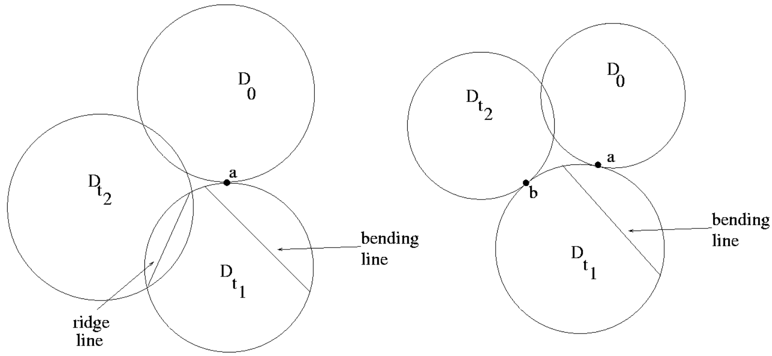
MAIN THEOREM. *There exists a monotonically increasing function*
$$F\colon [0, 2\sinh^{-1} 1] \to [\pi, 2\pi]$$
such that, if Γ is a Kleinian group (*where the components of $\Omega(\Gamma)$ are simply connected*) *and if α is a geodesic arc in $\partial CH(L_\Gamma)$ of length $l(\alpha) \le 2\sinh^{-1} 1$, then*
$$i(\alpha, \beta_\Gamma) \le F(l(\alpha)).$$

Proof. Let $\alpha\colon [0,1] \to \partial CH(L_\Gamma)$ be a parameterized geodesic arc on the boundary of the convex hull of Γ. We let K be the corresponding component of $\Omega(\Gamma)$. Hence by hypothesis K is open and simply connected. Also, by the description of the convex hull, any bending line that α intersects has endpoints in ∂K (see [5]).

Let P be the leftmost support plane at $\alpha(0)$ and Q the rightmost support plane at $\alpha(1)$. Then by definition we have $i(\alpha, \beta_\Gamma) = i(\alpha, \beta_\Gamma)_P^Q$. Let $\{P_t \mid t \in [0,k]\}$ be the continuous one-parameter family of support planes to α joining P to Q, let H_t be the associated support plane of P_t, and let D_t be the closed disk in $\mathbf{S}^2_\infty$ given by $D_t = \bar{H}_t \cap \mathbf{S}^2_\infty$. In particular, $P_0 = P$ and $P_k = Q$. We will make use of the fact that $D_t^o \subset K$, where D_t^o is the interior of D_t.

If (P, Q) is a roof over α, then (by Lemma 4.1) the exterior angle of intersection θ of P and Q is an upper bound for $i(\alpha, \beta_\Gamma)_P^Q$. Therefore, $i(\alpha, \beta_\Gamma)_P^Q \le \theta \le \pi \le F(l(\alpha))$.

Otherwise, we let t_1 be the smallest value of $t > 0$ such that (P, P_t) is not a roof over $\alpha([0, s(t)])$. We let $s(t_1) = s_1$ and $\alpha_1 = \alpha|_{[0,s_1]}$. Then (P, P_{t_1}) is a π-roof over α_1 and so, by Corollary 4.2, $i(\alpha_1, \beta_\Gamma)_P^{P_{t_1}} \le \pi$ and $\bar{H}_0 \cap \bar{H}_{t_1} = \{a\}$ where $a \in \mathbf{S}^2_\infty$. If (P_{t_1}, Q) is a roof over $\alpha([s_1, 1])$, we let $\alpha_2 = \alpha|_{[s_1,1]}$. Hence the exterior angle of intersection θ_1 of P_{t_1} and Q is an upper bound for $i(\alpha_2, \beta_\Gamma)_{P_{t_1}}^Q$. Thus we have
$$i(\alpha, \beta_\Gamma)_P^Q = i(\alpha_1, \beta_\Gamma)_P^{P_{t_1}} + i(\alpha_2, \beta_\Gamma)_{P_{t_1}}^Q \le \pi + \theta_1.$$



Case 1 (left) and Case 2 (right)

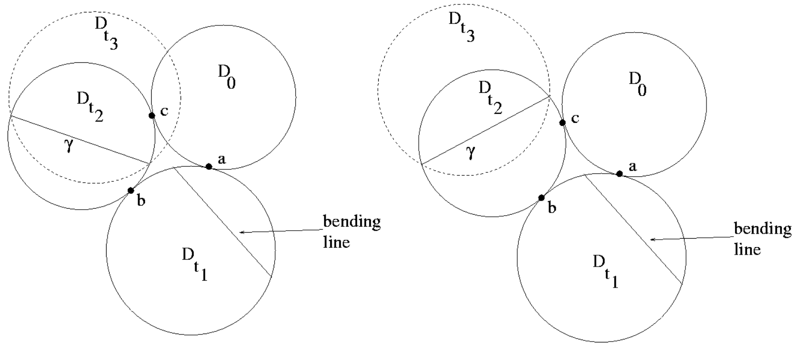
If $Q \cap P \neq \emptyset$ then we consider the set $S = \mathbf{S}^2_\infty - (D^o_0 \cup D^o_{t_1} \cup D^o_k)$. Then we have that $\partial K \subset S$. Therefore $S = T_1 \cup T_2$, where T_i are spherical triangles. Also we have that $T_1 \cap T_2 = \{a\}$, where $\bar{H}_0 \cap \bar{H}_{t_1} = \{a\}$. Since (P_0, P_{t_1}) is a π-roof, it follows that $a \in K$. Also, by monotonicity of ridge lines, the bending line on P_{t_1} that α intersects has one endpoint in T_1 and the other in T_2 (Case 1; see figure). Thus ∂K is disconnected, contradicting the fact that K is simply connected. We therefore have that $Q \cap P = \emptyset$ and that the support planes P, P_{t_1}, Q have the configuration described in Lemma 3.6. Hence the interior dihedral angle $\bar{\theta}_1 = \pi - \theta_1$ between P_{t_1} and Q satisfies $\Theta(l) \leq \bar{\theta}_1$, so

$$i(\alpha, \beta_\Gamma)^Q_P \leq \pi + \theta_1 \leq 2\pi - \Theta(l) = F(l).$$

Now let t_2 be the smallest value of $t \in [t_1, k]$ such that (P_{t_1}, P_t) is not a roof over $\alpha([s_1, s(t)])$, and let $s(t_2) = s_2$. Because (P_{t_1}, P_{t_2}) is a π-roof, we have $\bar{H}_{t_1} \cap \bar{H}_{t_2} = \{b\}$. Since $l(\alpha) < 2\sinh^{-1} 1$, by Lemma 3.5 it follows that $\bar{H}_0 \cap \bar{H}_{t_2} \neq \emptyset$. Then, letting $S = \mathbf{S}^2_\infty - (D^o_0 \cup D^o_{t_1} \cup D^o_{t_2})$, we have $\partial K \subset S$. As before, $S = T_1 \cup T_2$, where T_i are spherical triangles. Also as before, $a, b \in K$ and the bending line on P_{t_1} that α intersects has one endpoint in T_1 and the other in T_2. If $H_0 \cap H_{t_2} \neq \emptyset$, then $T_1 \cap T_2 = \{a, b\}$ (Case 2). Hence ∂K is disconnected, giving a contradiction to K being simply connected.

We can therefore assume that $\bar{H}_0 \cap \bar{H}_{t_2} = \{c\}$ where $c \in \mathbf{S}^2_\infty$. Then $T_1 \cap T_2 = \{a, b, c\}$. Also, by Lemma 3.5, $l(\alpha([0, s_2])) \geq 2\sinh^{-1} 1$. Since $l(\alpha) \leq 2\sinh^{-1} 1$ we have $l(\alpha([0, s_2])) = 2\sinh^{-1} 1$ and $s_2 = 1$. Hence the support planes P_t $(t_2 \leq t \leq k)$ intersect P_{t_2} along a bending line γ with $\alpha(1) \in \gamma$. Thus the planes P_t $(t_2 \leq t \leq k)$ are obtained by rotating P_{t_2} about γ. Because $b \in K$, we know that b is not an endpoint of γ. Also, by monotonicity of ridge lines at the point b, we obtain P_t $(t > t_2)$ by rotating P_{t_2} away from P_{t_1}.

We first consider the case when the geodesic γ in P_{t_2} separates the points b and c on the boundary of P_{t_2}. If γ does separate b and c then, rotating P_{t_2} about γ, we see that for $t > t_2$ either $b \in D^o_t$ or $c \in D^o_t$. As P_{t_2} is rotated away from P_{t_1}, there



Case 3 (left) and Case 4 (right)

is a $t_3 > t_2$ such that $b \notin D^o_{t_3}$ (Case 3); hence $c \in D^o_{t_3}$ and $c \in K$. Therefore ∂K is disconnected, contradicting the fact that K is simply connected.

If γ does not separate b and c then, as P_{t_2} is rotated away from P_{t_1}, we can choose a $t_3 > t_2$ such that $\bar{H}_{t_1} \cap \bar{H}_{t_3} = \emptyset$ and $H_0 \cap H_{t_3} = \emptyset$ (Case 4). Note that we cannot assume $\bar{H}_0 \cap \bar{H}_{t_3} = \emptyset$, since the point c may be an endpoint of γ. It follows that the three half-spaces H_0, H_{t_1}, H_{t_3} are disjoint, with a geodesic arc of length $2\sinh^{-1} 1$ joining P_0 to P_{t_3} via P_{t_1}. Then, by Lemma 3.5, the closures of the half-spaces H_0, H_{t_1}, H_{t_3} intersect pairwise in a point on $\mathbf{S}^2_\infty$. This contradicts the fact that $\bar{H}_{t_1} \cap \bar{H}_{t_3} = \emptyset$. $\square$

5. The Bending Lamination

Bridgeman and Canary [4] have shown that the length of the measured lamination β on a finite-area hyperbolic surface S can be evaluated by an integral over the unit tangent bundle. For $p \in T_1(S)$ we let $\alpha_p\colon (0, L) \to S$ be the parameterized geodesic arc of length L given by $\alpha_p(t) = g_t(p)$, where $g_t\colon T_1(S) \to S$ is time-t geodesic flow. Then

$$l(\beta) = \frac{1}{4L}\int_{T_1(S)} i(\alpha_p, \beta)\, d\Omega.$$

Let β_N be the bending lamination on $\partial C(N)$, fix $L = 2\sinh^{-1} 1$, and let $p \in T_1(\partial C(N))$. Then, if α_p does not intersect β_N, we let $d(p) = L$; otherwise, we define $d(p)$ to be the minimum number such that $\alpha_p(d(p)) \in \beta_N$. Then α_p intersects β_N only for length at most $L - d(p)$. Therefore,

$$i(\alpha_p, \beta_N) \le F(L - d(p)).$$

Thus we have that

$$l(\beta_N) \le \frac{1}{4L}\int_{T_1(S)} F(L - d(p))\, d\Omega.$$

To perform the integration, we decompose the complement of β_N in $\partial C(N)$ into ideal triangles by adding geodesics to β_N to obtain a geodesic lamination $\tilde{\beta}_N$ such that $\beta_N \subset \tilde{\beta}_N$. If we let $\tilde{d}(p)$ be the minimum number such that $\alpha_p(\tilde{d}(p)) \in \tilde{\beta}_N$, then $\tilde{d}(p) \le d(p)$. Therefore, since F is monotonically increasing, $F(L-d(p)) \le F(L-\tilde{d}(p))$. Thus

$$l(\beta_N) \le \frac{1}{4L}\int_{T_1(S)} F(L-\tilde{d}(p))\, d\Omega.$$

The right-hand side of this integral is the same over the unit tangent bundle of each ideal triangle. Since the area of $\partial C(N)$ is $2\pi|\chi(\partial C(N)|$, it follows that $\partial C(N) - \tilde{\beta}_N$ consists of $2|\chi(\partial C(N)|$ ideal triangles. We therefore let $U \subset \mathbf{H}^2$ be an ideal hyperbolic triangle and, for each $p \in T_1(U)$, define $D(p)$ to be the minimum number such that $\alpha_p(D(p)) \in \partial U$. Then

$$l(\beta_N) \le \frac{2|\chi(\partial C(N)|}{4L}\int_{T_1(U)} F(L-D(p))\, d\Omega.$$

To perform the integration, we work in the upper half-space model for $\mathbf{H}^2$ and let

$$U = \left\{(x, y) \mid -1 \le x \le -1,\ y \ge \sqrt{1-x^2}\right\}.$$

We denote the three sides of U by e_1, e_2, e_3, where $e_1 = \{(-1, t) \mid t > 0\}$, $e_2 = \{(1, t) \mid t > 0\}$, and $e_3 = \left\{\left(t, \sqrt{1-t^2}\right) \mid -1 < t < 1\right\}$.

Let $p \in T_1(U)$, where p has basepoint (x, y) and tangent vector v. We drop perpendiculars from (x, y) to each of the sides e_1, e_2, e_3 and label them as P_1, P_2, P_3 respectively. Let $d_i(x, y)$ denote the length of P_i. We have

$$\tanh d_1(x, y) = \frac{1+x}{\sqrt{(1+x)^2+y^2}},$$
$$\tanh d_2(x, y) = \frac{1-x}{\sqrt{(1-x)^2+y^2}},$$
$$\tanh d_3(x, y) = \frac{x^2+y^2-1}{\sqrt{(x_2+y^2-1)^2+4y^2}}.$$

The geodesic ray in the direction p intersects at most one side of U. Let the ray intersect side e_i and make an angle θ with P_i. Then we have a right-angled triangle with angle θ, hypotenuse of length $D(p)$, and adjacent side of length $d_i(x, y)$. Therefore, $D(p)$ satisfies

$$\tanh D(p) = \frac{\tanh d_i(x, y)}{\cos\theta}.$$

Since $F(L - D(p)) = 0$ for $D(p) \ge L$, it follows that the domain over which we integrate satisfies

$$\cos\theta \ge \frac{\tanh d_i(x, y)}{\tanh L}.$$

Thus we split the integral over $T_1(U)$ and obtain

$$\int_{T_1(U)} F(L - D(p))\, d\Omega$$
$$= \int_U \frac{dx\,dy}{y^2}\left(\sum_{i=1}^{3}\int_{-\cos^{-1}\left(\frac{\tanh d_i(x,y)}{\tanh L}\right)}^{\cos^{-1}\left(\frac{\tanh d_i(x,y)}{\tanh L}\right)} F\left(L - \tanh^{-1}\left(\frac{\tanh d_i(x,y)}{\cos\theta}\right)\right) d\theta\right).$$

We define the constant K_0 by

$$K_0$$
$$= \frac{1}{2\pi^2 L}\int_U \frac{dx\,dy}{y^2}\left(\sum_{i=1}^{3}\int_{-\cos^{-1}\left(\frac{\tanh d_i(x,y)}{\tanh L}\right)}^{\cos^{-1}\left(\frac{\tanh d_i(x,y)}{\tanh L}\right)} F\left(L - \tanh^{-1}\left(\frac{\tanh d_i(x,y)}{\cos\theta}\right)\right) d\theta\right).$$

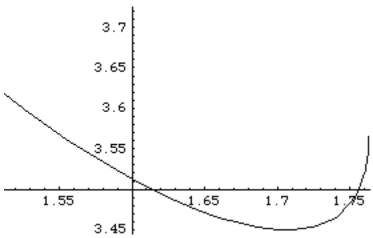
We then perform the integration using Mathematica and, rounding up to four decimal places, obtain $K_0 < 2.8396$. We thus have the following improvement on the bound on the length of the bending lamination.

THEOREM 1.2, PART 1. *If Γ is a Kleinian group such that the components of $\Omega(\Gamma)$ are simply connected, then*

$$l(\beta_N) \le K_0 \cdot \pi^2 |\chi(\partial C(N)|.$$

6. The Average Bending Function

In [2], Thurston's description of the minimal Lipschitz constant between two hyperbolic surfaces (see [10]) is applied to prove the following: If the average bending satisfies $B(\alpha) \le k$ for all geodesic arcs α of a fixed length l, then there is a $(1+k)$ Lipschitz map that is a homotopy inverse of the retract map $r\colon \Omega(\Gamma)/\Gamma \to C(N)$. In particular, by using $l = 2\sinh^{-1} 1$ we can choose $k = K = \pi/\sinh^{-1} 1$ (see [3]).



Graph of $F(x)/x$ for x near $2\sinh^{-1} 1$

The Main Theorem states that for $l(\alpha) \le 2\sinh^{-1} 1$ we have $B(\alpha) \le F(l(\alpha))/l(\alpha)$. Hence we consider the function $\mathcal{K}(x) = F(x)/x$ (see figure); the minimum value of $\mathcal{K}$ on the interval $[0, 2\sinh^{-1} 1]$ gives a better bound than K in Theorem 1.2. We let K_1 be the minimum value of $\mathcal{K}$. Graphing $\mathcal{K}(x)$, we see that the minimum of $\mathcal{K}(x)$ is obtained at approximately $x = 1.7063$. Evaluating at

$x = 1.7063$ then yields $K_1 \le \mathcal{K}(1.7063) \le 3.4502$. Thus we obtain the final two parts of Theorem 1.2.

THEOREM 1.2, PART 2. *If α is a closed geodesic on $\partial C(N)$, then*

$$B(\alpha) \le K_1.$$

THEOREM 1.2, PART 3. *The retract map $r\colon \Omega(\Gamma)/\Gamma \to \partial C(N)$ has a homotopy inverse $s\colon \partial C(N) \to \Omega(\Gamma)/\Gamma$ with Lipschitz constant $(1 + K_1)$.*

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