Characterization of Convex Domains with Noncompact Automorphism Group

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Introduction

The conformal mapping theorem of Riemann asserts that a simply connected domain in $\mathbb{C}$, different from $\mathbb{C}$, is biholomorphically equivalent to the open unit disc $U = \{ z \in \mathbb{C} : |z| < 1 \}$. Many authors have been interested in the generalization of this result in several complex variables (cf. [2; 3; 4; 9; 14]). The situation is quite different there: a small C^2 perturbation of the unit ball $\mathbb{B}^{n+1}$ in $\mathbb{C}^{n+1}$ can be nonequivalent to $\mathbb{B}^{n+1}$, even if it is simply connected. This shows that a domain in $\mathbb{C}^{n+1}$ is not completely described by its topological properties. Thus one must study the automorphism group of a domain to find a polynomial representation of it, that is, a rigid polynomial domain and a biholomorphic equivalence between our original domain and this rigid polynomial domain.

From now on, we consider pseudoconvex domains with noncompact automorphism group. More precisely, we assume that there exists a family $(h_\nu)_\nu$ of automorphisms of Ω, a point p in Ω, and a point p_∞ in $\partial\Omega$ such that

$$\lim_{\nu\to\infty} h_\nu(p) = p_\infty.$$

We say that p_∞ is an *accumulating point* for an orbit of $\mathrm{Aut}(\Omega)$.

A very useful tool for constructing a biholomorphism from Ω to a rigid polynomial model domain is the scaling method introduced by Pinchuk [13]. We will describe the scaling for our problem in Section 2. This method allowed Bedford and Pinchuk [2] to prove that if a domain is bounded in $\mathbb{C}^2$, pseudoconvex, and real analytic of finite type $2m$ in the sense of d'Angelo [6] with noncompact automorphism group, then it is biholomorphic to the ellipsoid $E_m = \{ (z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^{2m} < 1 \}$, where $m \geq 1$. The scaling gives moreover a nice short proof of the Wong–Rosay theorem [14] when p_∞ is a point of strict pseudoconvexity.

Bedford and Pinchuk [3] used this method and an analysis of vector fields to prove that if a domain Ω is bounded in $\mathbb{C}^{n+1}$, smooth, convex, and of finite type, and if $\mathrm{Aut}(\Omega)$ is noncompact, then Ω is biholomorphically equivalent to a weighted homogeneous convex rigid polynomial domain $D = \{ (z_0, z') \in \mathbb{C} \times \mathbb{C}^n : \mathrm{Re}\, z_0 + P(z') < 0 \}$. In the bounded case, the noncompactness of the automorphism group is equivalent to the existence of an accumulating point, and it seems relevant that the domain could be characterized by its geometry near this

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point. This remark has been the starting point for the work of Berteloot [4] and for our generalization of the result of [3] without any assumption of global convexity. This generalization is based on the construction of polydiscs on a convex domain in $\mathbb{C}^{n+1}$, given by McNeal in [12].

The main result of this paper can be stated as follows.

THEOREM 1. *Let Ω be a domain in $\mathbb{C}^{n+1}$, and let p_∞ be a point of $\partial\Omega$. Assume that p_∞ is an accumulating point for a sequence of automorphisms of Ω. If $\partial\Omega$ is smooth, convex, and of finite type $2m$ near p_∞, then Ω is biholomorphically equivalent to a rigid polynomial domain*

$$D = \{ (z_0, z') \in \mathbb{C} \times \mathbb{C}^n : \operatorname{Re} z_0 + P(z') < 0 \},$$

where P is a real nondegenerate convex polynomial of degree less than or equal to $2m$.

By "domain" we mean a connected open set, not necessarily bounded; the nondegenerancy of P is given by the condition "$\{P = 0\}$ without nontrivial analytic set".

As a corollary of this result, we obtain a generalization of the Wong–Rosay theorem for unbounded domains.

THEOREM 2. *Let Ω be a domain in $\mathbb{C}^{n+1}$, and let p_∞ be a point of $\partial\Omega$. Assume that p_∞ is an accumulating point for a sequence of automorphisms of Ω. If $\partial\Omega$ is smooth and strictly pseudoconvex near p_∞, then Ω is biholomorphically equivalent to the unit ball in $\mathbb{C}^{n+1}$.*

This second theorem was obtained independently by Efimov by a different method.

The paper is organized as follows. In the first section, we describe the construction of the polydiscs around points near the boundary of a convex domain, and give some of their properties. In Section 2 we are in the hypothesis of Theorem 1. If q^ν is the point $h_\nu(p)$ where p belongs to Ω and $(h_\nu)_\nu$ is a noncompact sequence of automorphisms of Ω accumulating to p_∞, we localize the polydiscs centered at the points q^ν. This allows us to rescale the domain Ω by a dilation we define there. We show then that the scaled domains converge to a rigid polynomial domain. At the end of this section, we study more precisely the scaled domains when the sequence $(q^\nu)_\nu$ converges normally to the point $p_\infty = 0$, that is, when $q^\nu = (-\varepsilon_\nu, 0, \dots, 0)$ in a fixed coordinate system $(z_0, \dots, z_n)$ centered at 0. We show that the limit domain is the homogeneous representation of Ω near 0 in the coordinates $(z_0, \dots, z_n)$. In Section 3, we prove the biholomorphic equivalence between Ω and the limit rigid polynomial domain given by the two theorems.

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1. Polydiscs of McNeal

The coordinates in $\mathbb{C}^{n+1}$ are denoted $z = (z_0, z')$, where z_0 is an element in $\mathbb{C}$ and z' an element in $\mathbb{C}^n$.

Let Ω be a domain in $\mathbb{C}^{n+1}$. Assume that $\partial\Omega$ is convex of finite type $2m$ near a point p_∞ of $\partial\Omega$. There exists a neighborhood V of p_∞ in $\mathbb{C}^{n+1}$ such that $\Omega \cap V$ is convex and is defined by a convex function r of the form

$$r(z_0, z') = \operatorname{Re} z_0 + \varphi(\operatorname{Im} z_0, z'),$$

where φ is a function of class $\mathcal{C}^\infty$.

We may also assume that there exists a real positive number ε_0 such that, for every $-\varepsilon_0 < \varepsilon < \varepsilon_0$, the level sets $\{r(z) = \varepsilon\}$ are convex.

REMARK 1.1. The fact that the type of $\partial\Omega$ at p_∞ is even is given by the convexity of Ω near this point.

The construction in [12] is local and so it is locally valid for unbounded domains.

To every point q in $\Omega \cap V$ and every sufficiently small positive constant ε we associate

(1) a holomorphic coordinate system $(z_0^{q,\varepsilon}, \dots, z_n^{q,\varepsilon})$ centered at q and preserving orthogonality,
(2) points $p_0^{q,\varepsilon}, \dots, p_n^{q,\varepsilon}$ on the hypersurface $S_\varepsilon = \{r(z) = r(q) + \varepsilon\}$, and
(3) real positive numbers $\tau_0^{q,\varepsilon}, \dots, \tau_n^{q,\varepsilon}$.

The construction proceeds as follows. We first compute the distance from q to S_ε. Working with sufficiently small ε, there is a unique point $p_0^{q,\varepsilon}$ in S_ε where this distance is achieved. The corresponding complex line is called $z_0^{q,\varepsilon}$, and $p_0^{q,\varepsilon}$ lies on the real positive axis $x_0^{q,\varepsilon}$; we set $\tau_0^{q,\varepsilon} = d(q, p_0^{q,\varepsilon})$. Then we consider the orthogonal complement of the complex line $z_0^{q,\varepsilon}$ in $\mathbb{C}^{n+1}$. We compute the distance from q to S_ε on each complex line in this complement. Because of the assumption of finite type, the largest such distance is finite and is achieved at a point $p_1^{q,\varepsilon}$ on the real positive axis $x_1^{q,\varepsilon}$ of the complex line $z_1^{q,\varepsilon}$. We set $\tau_1^{q,\varepsilon} = d(q, p_1^{q,\varepsilon})$. Repeating this process, we obtain the construction.

One must be aware of the dependence of all the coordinates, points, and numbers on q and ε.

We can now define the polydiscs $P_\varepsilon(q)$ in the new coordinates $(z_0^{q,\varepsilon}, \dots, z_n^{q,\varepsilon})$ centered at q:

$$P_\varepsilon(q) = \{ z \in \mathbb{C}^{n+1} : |z_0^{q,\varepsilon}| < \tau_0^{q,\varepsilon}, \dots, |z_n^{q,\varepsilon}| < \tau_n^{q,\varepsilon} \}.$$

Using the convexity of the level sets $\{r(z) = \varepsilon\}$, we obtain a complete localization of the polydiscs, given by the following two lemmas.

LEMMA 1.1. *There exists a positive constant C such that, for all q in $\Omega \cap V$ and sufficiently small ε,*

$$CP_{q,\varepsilon} \subset \{r(z) < r(q) + \varepsilon\}.$$

Proof. This inclusion is given in [12]. □

LEMMA 1.2. *For all q in $\Omega \cap V$ and sufficiently small ε, we obtain the inclusion*

$$CP_{q,\varepsilon} \subset \{r(z) > r(q) - \varepsilon\}.$$

This localization is not studied by McNeal. However, it is of crucial importance in our problem; indeed, we will use this argument to control the defining functions after our scaling.

Proof. The defining function r has the form $r(z) = \operatorname{Re} z_0 + \varphi(\operatorname{Im}(z_0), z')$, where φ is a function of class $\mathcal{C}^\infty$. The hypersurface S_ε (resp. $S_{-\varepsilon}$) is the image of the hypersurface $S_0 = \{ z \in \Omega \cap V : r(z) = r(q) \}$ under the translation of the vector $(\varepsilon, 0')$ (resp. $(-\varepsilon, 0')$). The lemma is proved by contradiction. Assume that there exists a point $q_{-\varepsilon}$ on $S_{-\varepsilon} \cap CP_\varepsilon(q)$. We denote by L_q the real line from q to $q_{-\varepsilon}$, by q_ε the intersection point between S_ε and L_q, and by q^0 and q^1 the points $q^0 = q + (\varepsilon, 0')$ and $q^1 = q - (\varepsilon, 0')$ on the hypersurfaces S_ε and $S_{-\varepsilon}$. We define the real tangent space $T_q^{\mathbb{R}}$ (respectively $T_{q^0}^{\mathbb{R}}$ and $T_{q^1}^{\mathbb{R}}$) to S_0 (respectively S_ε and $S_{-\varepsilon}$) at q (respectively q^0 and q^1). The spaces $T_{q^0}^{\mathbb{R}}$ and $T_{q^1}^{\mathbb{R}}$ are the images of $T_q^{\mathbb{R}}$ under the translations of the vectors $(\varepsilon, 0')$ and $(-\varepsilon, 0')$. The convexity hypothesis implies that S_ε, S_0, and $S_{-\varepsilon}$ are on the same side of these spaces. As the distance from q to $T_{q^0}^{\mathbb{R}}$ and $T_{q^1}^{\mathbb{R}}$ on L_q are equal, it follows that $d(q, q_{-\varepsilon}) > d(q, q_\varepsilon)$. Then q_ε is a point of the polydisc $P_\varepsilon(q)$. This contradicts Lemma 1.1. □

The hypothesis on $\partial\Omega$ near p_∞ implies uniform estimates on the numbers $\tau_j^{q,\varepsilon}$.

LEMMA 1.3. *There exists a positive constant c such that*
(i) $\tau_0^{q,\varepsilon} \le c\varepsilon$ *and*
(ii) *for every* $j \ge 1$, $\tau_j^{q,\varepsilon} \le c\varepsilon^{1/2m}$.

These estimates are obtained by McNeal but we give a complete proof of them here.

Proof. Let $\delta_X(z, S_\varepsilon)$ be the distance from the point z to the hypersurface S_ε along the complex line generated by the vector X. We need to show that there exists a positive constant c and a neighborhood U of p_∞ such that for every z in $\Omega \cap U$, every sufficiently small ε, and every X in $\mathbb{C}^{n+1}$,

$$\delta_X(z, S_\varepsilon) \le c d(z, S_\varepsilon)^{1/2m}.$$

Because of the definition of the hypersurface S_ε, it is sufficient to show the result for the hypersurface $\partial\Omega$.

Since Ω is pseudoconvex of finite type $2m$ near p_∞, Lemma 3.5 of [15] implies that there exists a neighborhood U of p_∞ in $\mathbb{C}^{n+1}$ and a positive constant C_1 such that, for every z in $\Omega \cap U$ and X in $\mathbb{C}^{n+1}$, we have

$$K_{\Omega \cap U}(z, X) \ge C_1 \frac{|X|}{d(z, \partial\Omega)^{1/2m}}, \tag{1}$$

where $K_{\Omega \cap U}(z, X)$ is the infinitesimal Kobayashi pseudometric at the point z and at the vector X.

If U is chosen relatively compact in V and such that the domain $\Omega \cap U$ is convex, then we have the following estimate on the Kobayashi pseudometric for z in $\Omega \cap U$ and X in $\mathbb{C}^{n+1}$ (see e.g. [3]):

$$K_{\Omega\cap U}(z, X) \leq \frac{|X|}{\delta_X(z, \partial\Omega)}. \tag{2}$$

Setting $c = 1/C_1$, conditions (1) and (2) give the following inequality for z in $\Omega \cap U$ and X in $\mathbb{C}^{n+1}$:

$$\delta_X(z, \partial\Omega) \leq cd(z, \partial\Omega)^{1/2m}. \qquad \square$$

The change of coordinates from the canonical system to the system $(z_0^{q,\varepsilon}, \ldots, z_n^{q,\varepsilon})$ is the composition of a translation $T_{q,\varepsilon}$ and of a unitary transform $A_{q,\varepsilon}$. Besides the change of coordinates, $(A_{q,\varepsilon} \circ T_{q,\varepsilon})^{-1}$ is defined in a fixed neighborhood of the origin. The corresponding defining function $r_{q,\varepsilon}$ is defined by

$$r_{q,\varepsilon} = r \circ (A_{q,\varepsilon} \circ T_{q,\varepsilon})^{-1}.$$

McNeal gives the following uniform estimates on the derivatives of the functions $r_{q,\varepsilon}$.

LEMMA 1.4.

(i) *For every* $j \leq n$, $\dfrac{\partial r_{q,\varepsilon}}{\partial z_j^{q,\varepsilon}}(p_j^{q,\varepsilon})$ *is real.*

(ii) *There exists a positive constant* c' *such that, for every* $j \leq n$,
$$\left|\frac{\partial r_{q,\varepsilon}}{\partial z_j^{q,\varepsilon}}(p_j^{q,\varepsilon})\right| \geq c' \frac{\tau_0^{q,\varepsilon}}{\tau_j^{q,\varepsilon}}.$$

(iii) *If* $j \leq n-1$ *then, for all* $k > j$, $\dfrac{\partial r_{q,\varepsilon}}{\partial z_k^{q,\varepsilon}}(p_j^{q,\varepsilon}) = 0$.

Proof. In the coordinates $(z_0^{q,\varepsilon}, \ldots, z_n^{q,\varepsilon})$, $x_0^{q,\varepsilon}$, which is the real normal axis to S_ε at $p_0^{q,\varepsilon}$, is a small perturbation of the normal axis x_0 to $\partial\Omega$ at p. Restricting U if necessary and using the form of the function r, we may assume that for all q in U and all sufficiently small ε,

$$\frac{1}{2} \leq \left|\frac{\partial r_{q,\varepsilon}}{\partial x_0^{q,\varepsilon}}(p_0^{q,\varepsilon})\right| \leq 2.$$

This proves parts (i) and (ii) for $j = 0$.

Let $\delta(z) = \sum_{j=0}^{n} |z_j^{q,\varepsilon}|^2$ and, for every $j \geq 1$, $\delta_j(z) = \sum_{k=0}^{j-1} |z_k^{q,\varepsilon}|^2$. By construction, for every $j \geq 1$ the point $p_j^{q,\varepsilon}$ satisfies

$$\delta(p_j^{q,\varepsilon}) = \max_{z \in S_\varepsilon \cap H_j} \delta(z),$$

where $H_j = \{z \in \mathbb{C}^{n+1} : z_0^{q,\varepsilon} = \cdots = z_{j-1}^{q,\varepsilon} = 0\}$. Using Lagrange multipliers, we obtain that there exist two real numbers $\lambda_{0,j}$ and $\lambda_{1,j}$ such that

$$d\delta(p_j^{q,\varepsilon}) = \lambda_{0,j} d\delta_j(p_j^{q,\varepsilon}) + \lambda_{1,j} dr_{q,\varepsilon}(p_j^{q,\varepsilon}).$$

This gives directly parts (i) and (iii) of the lemma since all the terms are real. For part (ii) see [12]. $\square$

REMARK 1.2. It is important to note that c' does not depend on the variables q and ε.

2. Scaling of $\Omega \cap U$

From now on we assume that p_∞ is an accumulating point for a sequence of automorphisms of Ω. Then there exists a family $(h_\nu)_{\nu \geq 0}$ of automorphisms of Ω and a point p in Ω such that

$$\lim_{\nu \to \infty} h_\nu(p) = p_\infty.$$

For convenience we use the following notation:

$$\begin{aligned} q^\nu &= h_\nu(p), \\ \varepsilon_\nu &= -r(q^\nu), \\ A_\nu \circ T_\nu &= A_{q^\nu, \varepsilon_\nu} \circ T_{q^\nu, \varepsilon_\nu}, \\ r_\nu &= r_{q^\nu, \varepsilon_\nu}. \end{aligned}$$

The positive numbers $\tau_{\nu,0}, \ldots, \tau_{\nu,n}$ and the points $p_0^\nu, \ldots, p_n^\nu$ are those associated with q^ν and ε_ν.

The function r_ν is given in a fixed neighborhood of 0 by

$$r_\nu(z) = -\varepsilon_\nu + \mathrm{Re}\left(\sum_{j=0}^{n} a_j^\nu z_j\right) + \sum_{2 \leq |\alpha| + |\beta| \leq 2m} C_{\alpha\beta}^\nu z^\alpha \bar{z}^\beta + O(|z|^{2m+1}),$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, and $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$. We note that $O(|z|^{2m+1})$ is independent of ν.

Let $r \circ A$ be the limit of r_ν when ν goes to infinity. A is a unitary transform, and the convergence is a $\mathcal{C}^\infty$ convergence on a fixed compact neighborhood of p_∞. Then, for every j less than or equal to n and for every multi-index α and β satisfying $2 \leq |\alpha| + |\beta| \leq 2m$, there exist two complex numbers a_j and $C_{\alpha\beta}$ such that

$$\lim_{\nu \to \infty} a_j^\nu = a_j \quad \text{and} \quad \lim_{\nu \to \infty} C_{\alpha\beta}^\nu = C_{\alpha\beta}.$$

The scaling that people used before is not precise enough in our case. Indeed, this provides a rigid polynomial domain, but it is necessary to study the geometry of this domain to obtain the normality of holomorphic maps. The situation is quite different with the new scaling we use here. It is more convenient because we assign a weight to every complex line z_j. Such an assignment allows us to control the points on every axis and then we know how the scaled domains converge. In the convex case, this scaling is more natural because it induces the normality of the holomorphic maps we consider. There seems to be a similarity between the scaling we use here and the one used by Frankel [8].

We must be aware of the dependence of the axis $(z_0, \ldots, z_n)$ on the variable ν. For convenience we do not state this explicitly.

Let us consider the dilation

$$\Lambda_\nu(z) = (\tau_{\nu,0} z_0, \ldots, \tau_{\nu,n} z_n)$$

and the function

$$\tilde{r}^\nu = \frac{1}{\varepsilon_\nu} r_\nu \circ \Lambda_\nu.$$

The function $\tilde{r}^\nu$ has the following form:

$$\tilde{r}^\nu(z) = -1 + \frac{1}{\varepsilon_\nu}\operatorname{Re}\left(\sum_{j=0}^{n} a_j^\nu \tau_{\nu,j} z_j\right) + \frac{1}{\varepsilon_\nu}\sum_{2\le|\alpha|+|\beta|\le 2m} C_{\alpha\beta}^\nu \tau_\nu^{\alpha+\beta} z^\alpha \bar{z}^\beta + O\big((\varepsilon_\nu)^{1/2m}|z|^{2m+1}\big),$$

where $\tau_\nu^{\alpha+\beta} = \tau_{\nu,1}^{\alpha_1+\beta_1}\cdots\tau_{\nu,n}^{\alpha_n+\beta_n}$.

PROPOSITION 2.1. *The functions $\tilde{r}^\nu$ are smooth and convex, and there exists a subsequence of $(\tilde{r}^\nu)_\nu$ that converges uniformly on compact subsets of $\mathbb{C}^{n+1}$ to a smooth convex function $\tilde{r}$ of the form*

$$\tilde{r}(z) = -1 + \operatorname{Re}\left(\sum_{j\ge 0} b_j z_j\right) + P(z'),$$

where P is a real convex polynomial of degree less than or equal to $2m$.

Proof. The functions $\tilde{r}^\nu$ are smooth by construction, and are obtained as affine transformations of the convex function r. Hence they are convex, and the function $\tilde{r}$ will be smooth and convex as the limit of smooth convex functions. Let us study the convergence of the functions $\tilde{r}^\nu$. Because $O\big((\varepsilon_\nu)^{1/2m}|z|^{2m+1}\big)$ converges to zero on compact subsets of $\mathbb{C}^{n+1}$ when ν goes to infinity, we must study a convergence in the space of polynomials of degree less than or equal to $2m$. Since this space is of finite dimension, all the norms are equivalent and there exists a positive constant d_1 such that, for every $\nu \ge 1$,

$$\sup_{j,\alpha,\beta}\left\{|a_j^\nu|\tau_{\nu,j},\, |C_{\alpha\beta}^\nu|\tau_\nu^{\alpha+\beta}\right\} \le d_1 \sup_{|\omega|\le C}\left|\operatorname{Re}\left(\sum_j a_j^\nu \tau_{\nu,j}\omega_j\right) + \sum_{\alpha,\beta} C_{\alpha\beta}^\nu \tau_\nu^{\alpha+\beta}\omega^\alpha\bar{\omega}^\beta\right|.$$

This implies that there exists a positive constant d_2 such that, for every $\nu \ge 1$,

$$\sup_{j,\alpha,\beta}\left\{|a_j^\nu|\tau_{\nu,j},\, |C_{\alpha\beta}^\nu|\tau_\nu^{\alpha+\beta}\right\} \le d_2 \sup_{z\in CP_{\varepsilon_\nu}(q^\nu)}\left|\operatorname{Re}\left(\sum_j a_j^\nu z_j\right) + \sum_{\alpha,\beta} C_{\alpha\beta}^\nu z^\alpha\bar{z}^\beta\right|.$$

Using Lemmas 1.1 and 1.2, we obtain

$$\sup_{z\in CP_{\varepsilon_\nu}(q_\nu)} |r(z)| \le 2\varepsilon_\nu.$$

On the other hand, using Lemma 1.3 and the definition of the polydiscs $P_{\varepsilon_\nu}(q_\nu)$ yields

$$\sup_{z\in CP_{\varepsilon_\nu}(q_\nu)} O(|z|^{2m+1}) \le \varepsilon_\nu.$$

All these estimates provide a constant d_3 independent of ν such that

$$\sup_{j,\alpha,\beta}\left\{|a_j^\nu|\tau_{\nu,j},\, |C_{\alpha\beta}^\nu|\tau_\nu^{\alpha+\beta}\right\} \le d_3\varepsilon_\nu.$$

Consequently, we can extract from the sequence $(\tilde{r}^\nu)_\nu$ a subsequence that converges to the function $\tilde{r}$ given by Proposition 2.1, where b_j are complex numbers. □

Let Ω_ν be the image of $\Omega \cap U$ under the change $\Lambda_\nu^{-1} \circ A_\nu \circ T_\nu$. Proposition 2.1 implies that the family Ω_ν converges to the domain $\tilde{D} = \{\tilde{r}(z) < 0\}$ in the sense of the local Hausdorff convergence.

We conclude this section with the particular case when the sequence $(q^\nu)_\nu$ converges normally to the point $p_\infty = 0$. We denote the multitype of $\partial\Omega$ at 0 by $\mathcal{M}(\partial\Omega, 0) = (1, m_1, \dots, m_n)$; the points q^ν will be $(-\varepsilon_\nu, 0, \dots, 0)$ in the coordinates (z_0, z') defined by Bedford and Pinchuk in [3] and independently by Yu in [16]. Thus we may assume that, for all $\nu \geq 1$, $\tau_{0,\nu} = \varepsilon_\nu$. The function r is defined in a fixed neighborhood of 0 by

$$r(z_0, z') = \operatorname{Re} z_0 + P_0(z') + R(z),$$

where P_0 is a nondegenerate weighted homogeneous polynomial of degree 1 with respect to the weights $\mathcal{M}(\partial\Omega, 0)$, and R denotes terms of degree > 1. The domain

$$\Omega_{\text{hom}} = \{\, (z_0, z') \in \mathbb{C}^{n+1} : \operatorname{Re} z_0 + P_0(z') < 0 \,\}$$

is called the *homogeneous representation* of the domain Ω at 0.

REMARK 2.1. Using the same scaling as in [3]—that is, without moving the coordinates—it is possible to show that in the case of a normal convergence for a convex domain, the domain Ω is biholomorphic to the domain Ω_{hom}.

We now prove that Ω_{hom} is also the limit domain obtained after our scaling.

PROPOSITION 2.2. *The sequence of functions $(\tilde{r}_\nu)_\nu$ converges uniformly on compact subsets of $\mathbb{C}^{n+1}$ to the function $\tilde{r}$ defined by*

$$\tilde{r}(z) = -1 + \operatorname{Re} z_0 + P_0(z').$$

If $z = (z_0, z')$ is a point in $\mathbb{C}^{n+1}$ and if t is a real positive number, then P_0 and R satisfy the relations

$$P_0(t^{1/m_1} z_1, \dots, t^{1/m_n} z_n) = t P_0(z');$$

$$R(z) = o\left(|z_0| + \sum_{j=1}^{n} |z_j|^{m_j} \right).$$

Let e_j^ν be the unitary vector such that $p_j^\nu = q^\nu + \tau_{\nu,j} e_j^\nu$. In the coordinates (z_0, z') associated with $\mathcal{M}(\partial\Omega, 0)$, the point e_j^ν will be written $e_j^\nu = (0, e_{j,1}^\nu, \dots, e_{j,n}^\nu)$. As in [16], the multitype $\mathcal{M}(\partial\Omega, 0)$ will be denoted as

$$\mathcal{M}(\partial\Omega, 0) = (1, m_1, \dots, m_1, \dots, m_k, \dots, m_k).$$

This means that we take account of the multiplicity of each integer m_j. For every $j \leq k$, we assume that there are exactly s_j coordinates with weight m_j, that is,

$$2m = m_1 \geq \cdots \geq m_k \geq 2;$$

we set $s_0 = 0$.

For convenience we write $A_\nu \simeq B_\nu$ if there exist two positive constants C_1 and C_2 independent of ν such that, for every ν,

$$C_1 B_\nu \le A_\nu \le C_2 B_\nu.$$

For every $\nu \ge 1$ and $1 \le j \le n$, e_j^ν is a unitary vector of the tangent space $T_0^{\mathbb{C}}(\partial\Omega)$. If L_j^ν is the complex line generated by e_j^ν then the construction of e_j^ν implies that, for every $1 \le p \le k$ and every $s_0 + \cdots + s_{p-1} + 1 \le j \le s_0 + \cdots + s_p$, the order of contact between L_j^ν and $\partial\Omega$ is exactly m_p. We may also assume that the family $(e_1^\nu, \ldots, e_n^\nu)_\nu$ converges to a unitary system $(e_1, \ldots, e_n)$ in $T_0^{\mathbb{C}}(\partial\Omega)$. The order of contact between the complex line associated with e_j and $\partial\Omega$ at 0 is finite and, according to Corollary 6 of [16], e_j is associated with m_p. If x_j^ν is the real axis such that e_j^ν is a point on $\mathbb{R}x_j^\nu$ then there exists a positive constant c_1 such that for every $\nu \ge 1$, every $1 \le p \le k$, and every $s_0 + \cdots + s_{p-1} + 1 \le j \le s_0 + \cdots + s_p$,

$$\left|\frac{\partial^{m_p} P_0}{(\partial x_j^\nu)^{m_p}}(0)\right| \ge c_1.$$

Consequently we have the condition

$$\varepsilon_\nu \simeq (\tau_{\nu,j})^{m_p}. \tag{3}$$

Let $e_j = (e_{j,1}, \ldots, e_{j,n})$. Since e_j is associated with m_p, there exists a positive constant c_2 and a number l_j in $\{s_0 + \cdots + s_{p-1} + 1, \ldots, s_0 + \cdots + s_p\}$ such that

$$|e_{j,l_j}| \ge c_2. \tag{4}$$

Consider now the defining function

$$\tilde{r}^\nu = \frac{1}{\varepsilon_\nu} r \circ (A_\nu \circ T_\nu)^{-1} \circ \Lambda_\nu.$$

Because $\lim_{\nu\to\infty} \frac{1}{\varepsilon_\nu} R \circ (A_\nu \circ T_\nu)^{-1} \circ \Lambda_\nu = 0$ uniformly on compact subsets of $\mathbb{C}^{n+1}$, we are interested in the convergence of the function

$$\tilde{r}^\nu - 1 - \operatorname{Re}(z_0) - \frac{1}{\varepsilon_\nu} R \circ (A_\nu \circ T_\nu)^{-1} \circ \Lambda_\nu.$$

This function can be written as $P_0 \circ B_\nu$, where B_ν is the invertible matrix

$$B_\nu = \left(\frac{\overline{e_{i,j}^\nu}\,\tau_{\nu,j}}{(\varepsilon_\nu)^{1/m_i}}\right)_{i,j}.$$

To prove the convergence of $P_0 \circ B_\nu$, we need some information on the polynomial P_0. Let $\tilde{S}^1 = \{ z \in \mathbb{C}^n : \sum_j |z_j|^{m_j} = 1 \}$. Then we have the following lemma.

LEMMA 2.3. *There exists a positive constant c_3 such that*

$$\inf_L \max_{z \in L \cap \tilde{S}^1} P_0(z) \ge c_3,$$

where the infimum is taken over each complex line L in $\mathbb{C}^n$.

Proof. This is done by contradiction. Assume that there exists a sequence $(L_n)_n$ of complex lines such that

$$\max_{z \in L_n \cap \tilde{S}^1} P_0(z) < \frac{1}{n}.$$

We may assume that the sequence $(L_n)_n$ converges to a complex line L. Each point z in $L \cap \tilde{S}^1$ is the limit of a sequence of points $(z^n)_n$ in $L_n \cap \tilde{S}^1$, so $P_0(z) = 0$. Then P_0 is identically 0 on $L \cap \tilde{S}^1$ and, as it is a convex polynomial, it is identically 0 on L. This contradicts the nondegenerancy of P_0. □

LEMMA 2.4. *The sequence $(B_\nu)_\nu$ converges to a matrix B in* $\mathrm{GL}(n, \mathbb{C})$.

Proof. Assume that there exists a sequence of points $(z^\nu)_\nu$ such that

$$\forall \nu \geq 1, \quad \|z^\nu\| = 1;$$
$$\lim_{\nu \to \infty} \|B_\nu(z^\nu)\| = +\infty.$$

Set $q^\nu = B_\nu(z^\nu) = (q_1^\nu, \ldots, q_n^\nu)$, and let L_ν be the complex line such that the point

$$\tilde{q}^\nu = \left(\frac{q_1^\nu}{\left(\sum_j |q_j^\nu|^{m_j}\right)^{1/m_1}}, \ldots, \frac{q_n^\nu}{\left(\sum_j |q_j^\nu|^{m_j}\right)^{1/m_n}} \right)$$

is on $L_\nu \cap \tilde{S}^1$.

We may choose θ_ν such that the point $e^{i\theta_\nu}\tilde{q}^\nu$ satisfies

$$P_0(e^{i\theta_\nu}\tilde{q}^\nu) = \max_{z \in L_\nu \cap \tilde{S}^1} P_0(z).$$

Then we obtain the identities

$$\begin{aligned}(P_0 \circ B_\nu)(e^{i\theta_\nu} z^\nu) = P_0(e^{i\theta_\nu} q^\nu) &= \sum_j |q_j^\nu|^{m_j} P_0(e^{i\theta_\nu}\tilde{q}^\nu) \\ &\geq c_3 \sum_j |q_j^\nu|^{m_j} \to_{\nu \to \infty} \infty.\end{aligned}$$

However, by Proposition 2.1 the sequence $(P_0 \circ B_\nu)_\nu$ converges to a polynomial Q. This contradicts the preceding inequality, and the sequence $(B_\nu)_\nu$ converges to a matrix B. Moreover, using condition (3), the convergence of $(e_1^\nu, \ldots, e_n^\nu)_\nu$, and condition (4), we obtain that B is a lower triangular matrix and that each diagonal bloc is an invertible bloc of dimension s_j. Then B is an element of $\mathrm{GL}(n, \mathbb{C})$. □

We can now prove Proposition 2.2.

Proof of Proposition 2.2. We showed that the sequence $(\tilde{r}^\nu)_\nu$ converges to the function $\tilde{r}$ defined on $\mathbb{C}^{n+1}$ by

$$\tilde{r}(z) = -1 + \mathrm{Re}(z_0) + (P_0 \circ B)(z).$$

Considering the change of coordinates

$$\tilde{z}_0 = z_0 - 1 \quad \text{and} \quad \tilde{z}' = B(z'),$$

we obtain the proposition. □

REMARK 2.2. This result shows that the normal convergence of the points q^ν implies that the scaled domains converge to the rigid homogeneous polynomial representation of Ω.

3. Final Result

In this section we give the proofs of Theorems 1 and 2 by showing the normality of certain holomorphic maps.

Let us consider the mapping f_ν from $h_\nu^{-1}(\Omega \cap U)$ to Ω_ν defined by

$$f_\nu = \Lambda_\nu^{-1} \circ A_\nu \circ T_\nu \circ h_\nu.$$

We know that $\lim_{\nu\to\infty} h_\nu^{-1}(\Omega \cap U) = \Omega$, and we showed in Section 2 that $\lim_{\nu\to\infty} \Omega_\nu = \tilde{D}$.

LEMMA 3.1. *The family $(f_\nu)_\nu$ is a normal family.*

Proof. Let $e_j = \Lambda_\nu^{-1}(p_j^\nu)$ for every $\nu \geq 1$ and $j \geq 0$.

The coordinates we consider now are those defined in Section 1 and thus they depend on ν. In such coordinates $(z_0, \dots, z_n)$, $p_j^\nu = (0, \dots 0, \tau_{\nu,j}, 0, \dots, 0)$ and so $e_j = (0, \dots, 0, 1, 0, \dots, 0)$. Moreover,

$$\frac{\partial \tilde{r}^\nu}{\partial z_j}(e_j) = \frac{\tau_{\nu,j}}{\varepsilon_\nu} \frac{\partial r^\nu}{\partial z_j}(p_j^\nu);$$

using part (ii) of Lemma 1.4, we obtain a positive constant d_4 such that

$$\left| \frac{\partial \tilde{r}^\nu}{\partial z_j}(e_j) \right| \geq d_4 \frac{\tau_{\nu,0}}{\varepsilon_\nu}.$$

We conclude then that there exists a positive constant d_5 such that, for all sufficiently large ν,

$$\left| \frac{\partial \tilde{r}^\nu}{\partial z_j}(e_j) \right| \geq d_5. \tag{5}$$

Part (iii) of Lemma 1.4 then implies that, for every $k > j$ and all sufficiently large ν,

$$\frac{\partial \tilde{r}^\nu}{\partial z_k}(e_j) = 0.$$

Using the estimate (5), we can show that the family $(f_\nu^0)_\nu$ of the first components of the maps f_ν is a normal family. Indeed, for every $\nu \geq 1$, e_0 is a point in $\partial\Omega_\nu$. According to Lemma 1.4, the real tangent space to $\partial\Omega_\nu$ at e_0 is given by

$$T_{e_0}^{\mathbb{R}}(\partial\Omega_\nu) = \left\{ z \in \mathbb{C}^{n+1} : \left(\frac{\partial \tilde{r}^\nu}{\partial x_0}(e_0) \right) (\operatorname{Re} z_0 - 1) = 0 \right\}.$$

Since Ω_ν is convex, we may assume that for every z in Ω_ν,

$$\frac{\partial \tilde{r}^\nu}{\partial x_0}(e_0)(\operatorname{Re} z_0 - 1) \leq 0.$$

Let K be a compact subset of Ω. For all sufficiently large ν, K is a compact subset of $h_\nu^{-1}(\Omega \cap U)$ and so $f_\nu(K)$ is a compact subset of Ω_ν. Then every point ω in K satisfies the inequality

$$\frac{\partial \tilde{r}^\nu}{\partial x_0}(e_0)(\mathrm{Re}(f_\nu^0(\omega)) - 1) \leq 0;$$

using condition (4), we may assume that

$$\mathrm{Re}(f_\nu^0(\omega)) \leq 1.$$

Consequently, the family $(f_\nu^0)_\nu$ is normal. However, for every $\nu \geq 1$ we have the equality $f_\nu(p) = 0$. Then we may extract from $(f_\nu^0)_\nu$ a subsequence, still called $(f_\nu^0)_\nu$, that converges to a holomorphic mapping f^0 from Ω to $\mathbb{C}$.

Let us show now that $(f_\nu^1)_\nu$ is a normal family. The real tangent space to Ω_ν at e_1 is given by

$$T_{e_1}^{\mathbb{R}}(\partial\Omega_\nu) = \left\{ z \in \mathbb{C}^{n+1} : \mathrm{Re}\left(\frac{\partial \tilde{r}^\nu}{\partial z_0}(e_1) z_0\right) + \frac{\partial \tilde{r}^\nu}{\partial x_1}(e_1)(\mathrm{Re}\, z_1 - 1) = 0 \right\}.$$

However,

$$\lim_{\nu\to\infty} \frac{\partial \tilde{r}^\nu}{\partial x_1}(e_1) = \frac{\partial \tilde{r}}{\partial x_1}(e_1);$$

$$\lim_{\nu\to\infty} \frac{\partial \tilde{r}^\nu}{\partial z_0}(e_1) f_\nu^0(\omega) = \frac{\partial \tilde{r}}{\partial z_0}(e_1) f^0(\omega).$$

Using condition (5), we may assume after translation that, for all ω in K and sufficiently large ν,

$$\mathrm{Re}\, f_\nu^1(\omega) \leq 1.$$

Then the family $(f_\nu^1)_\nu$ is normal and there exists a subsequence that converges to a holomorphic mapping from Ω to $\mathbb{C}$. Repeating this process, we obtain that, after extraction, the family $(f_\nu)_\nu$ converges to a mapping f from Ω to $\bar{\tilde{D}}$. □

Let us study now the behavior of the family $(f_\nu^{-1})_\nu$.

LEMMA 3.2. *The domain Ω is taut.*

Proof. Since Ω is convex of finite type near the point p_∞, there exists a local peak function for Ω. By this we mean a holomorphic function defined on a neighborhood U of p_∞ in $\bar{\Omega}$, equal to unity at the point p_∞ and of modulus < 1 elsewhere. Using Theorem 2.3 of [7], we obtain an estimate on the Kobayashi infinitesimal pseudometric near p_∞ that implies the tautness of $\Omega \cap U$. Because an orbit of $\mathrm{Aut}(\Omega)$ accumulates there, Proposition 2.1 of [4] shows that Ω is taut. □

REMARK 3.1. With the assumptions on Ω near p_∞, we can show in fact that Ω is complete hyperbolic and consequently taut.

LEMMA 3.3. *f is a biholomorphic mapping from Ω to $\tilde{D}$.*

Proof. By Lemma 3.2, the family $(f_\nu^{-1})_\nu$ is a normal family and, since $f_\nu^{-1}(0) = p \in \Omega$, this converges to a mapping g from $\tilde{D}$ to $\bar{\Omega}$. We can now apply Theorem 1 of [9, Sec. 6]. This theorem, given for bounded domains, can be extended to unbounded domains (following the original proof step-by-step), and proves that f is a biholomorphism from Ω to $\tilde{D}$ with $f^{-1} = g$. □

REMARK 3.2. We need not show that the domain $\tilde{D} = \{\tilde{r}(z) < 0\}$ is taut in order to prove Theorem 1. We will obtain this result as a consequence of the biholomorphism between Ω and $\tilde{D}$. However, the tautness of $\tilde{D}$ may be proved by using the proof of Lemma 3.1 and (more precisely) the localization of all the tangent spaces.

Proof of Theorem 1. Using Lemma 3.3, we know that Ω is biholomorphically equivalent to the domain $\tilde{D} = \{ (z_0, z') \in \mathbb{C}^{n+1} : -1+\mathrm{Re}\big(\sum_{j=0}^n b_j z_j\big)+P(z') < 0 \}$. Using condition (5), we can see that the constant b_0 is different from 0. Then by an affine change of coordinates, $\tilde{D}$ is equivalent to the convex domain $D = \{ (z_0, z') \in \mathbb{C}^{n+1} : \mathrm{Re}\, z_0 + P(z') < 0 \}$. Since Ω is hyperbolic, D is hyperbolic and by a result of Barth [1], D contains no nontrivial complex affine line. Then there is no complex line in ∂D and, according to [11, Thm. 1.1], D is of finite type and P is nondegenerate. □

Proof of Theorem 2. Because p_∞ is a point of strict pseudoconvexity, a classical result gives that Ω is locally biholomorphic to a strictly convex domain. Hence there is a neighborhood V' of p_∞ in $\mathbb{C}^{n+1}$ and a biholomorphism ψ from V' to $\psi(V')$ such that $\psi(\Omega \cap V')$ is strictly convex. Since ψ is a holomorphic mapping, the boundary of $\psi(\Omega \cap V')$ is of class C^∞. We are then in the situation of Theorem 1, replacing $\Omega \cap V$ by $\psi(\Omega \cap V')$. The limit function $\tilde{r}$ has the following form:

$$\tilde{r}(z) = -1 + \mathrm{Re}\left(\sum_{j=0}^{n} b_j z_j\right) + P(z'),$$

where P is a convex positive homogeneous polynomial of degree 2. Since P is a positive definite Hermitian form, $P(z') = \sum_{j=1}^n |z_j|^2$ after a holomorphic change of coordinates. Then Ω is biholomorphically equivalent to the unit ball in $\mathbb{C}^{n+1}$. □

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