Hypersurfaces of $\bar{\mathbb{E}}^4$ Satisfying $\Delta\vec{H} = \lambda\vec{H}$

Filip Defever

1. Introduction

Minimal submanifolds of Euclidean spaces are contained not only in larger classes of submanifolds—for example, in the class of submanifolds of finite type—but also in the class of submanifolds satisfying $\Delta\vec{H} = \lambda\vec{H}$, $\lambda \in \bar{\mathbb{R}}$. The study of submanifolds satisfying $\Delta\vec{H} = \lambda\vec{H}$ was initiated by B.-Y. Chen in 1988, and arose in the context of his theory of submanifolds of finite type. For a survey of recent results on submanifolds of finite type and various related topics, see for example [8].

Let M^n be an n-dimensional connected submanifold of the Euclidean space $\bar{\mathbb{E}}^m$. Denote by $\vec{x}$, $\vec{H}$, and Δ (respectively) the position vector field of M^n, the mean curvature vector field of M^n, and the Laplace operator on M^n, with respect to the Riemannian metric g on M^n, induced from the Euclidean metric of the ambient space $\bar{\mathbb{E}}^m$. Then, as is well known (see e.g. [1]),

$$\Delta\vec{x} = -n\vec{H}. \tag{1}$$

This shows, in particular, that M^n is a minimal submanifold of $\bar{\mathbb{E}}^m$ if and only if its coordinate functions are harmonic (i.e., if they are eigenfunctions of Δ with eigenvalue 0):

$$\vec{H} = \vec{0} \iff \Delta\vec{x} = \vec{0}. \tag{2}$$

Condition (2) can be generalized in several directions. Takahashi [14] studied and classified submanifolds in Euclidean space for which

$$\Delta\vec{x} = \lambda\vec{x}, \quad \lambda \in \bar{\mathbb{R}}, \tag{3}$$

that is, submanifolds for which all coordinate functions are eigenfunctions of Δ with the same eigenvalue λ. Rephrased in terms of Chen's theory of submanifolds of finite type, Takahashi's condition (3) characterizes the 1-type submanifolds of $\bar{\mathbb{E}}^m$. In particular, for hypersurfaces of $\bar{\mathbb{E}}^m$, Takahashi's result [14] shows that the only 1-type hypersurfaces of $\bar{\mathbb{E}}^m$ are the minimal hypersurfaces ($\lambda = 0$) and (the open parts of) a hypersphere S^{m-1} of $\bar{\mathbb{E}}^m$ ($\lambda > 0$).

Condition (2) was generalized in another direction by Chen, who in 1985 initiated the study of submanifolds of $\bar{\mathbb{E}}^m$ satisfying

Received August 7, 1996.
Postdoctoraal Onderzoeker N.F.W.O., Belgium.
Michigan Math. J. 44 (1997).

$$\Delta\vec{H} = \vec{0} \iff \Delta^2\vec{x} = \vec{0}. \tag{4}$$

A submanifold M^n of $\bar{\mathbb{E}}^m$ satisfying condition (4) is said to have *harmonic mean curvature vector field.* Equivalently, submanifolds satisfying (4) are also called *biharmonic submanifolds.*

Conditions (3) and (4) may be generalized and combined into the condition

$$\Delta\vec{H} = \lambda\vec{H}, \quad \lambda \in \bar{\mathbb{R}}. \tag{5}$$

The study of submanifolds of $\bar{\mathbb{E}}^m$ satisfying (5) was initiated by Chen in 1988. In [3] it was proved that a submanifold M^n of a Euclidean space satisfies the condition $\Delta\vec{H} = \lambda\vec{H}$ for some $\lambda \in \bar{\mathbb{R}}$ if and only if M^n is biharmonic ($\lambda = 0$) or of 1-type or of null 2-type. For various results on submanifolds satisfying (5) in Euclidean spaces and other (also indefinite) space forms, see [5; 6; 8] and the references therein. However, a complete classification of the surfaces satisfying condition (5) has been achieved (in [2]) only for surfaces in $\bar{\mathbb{E}}^3$.

Concerning hypersurfaces of $\bar{\mathbb{E}}^4$, [11] classifies the hypersurfaces satisfying (5) with the supplementary condition of conformal flatness. From this and other partial results (see further), one may remark that all known examples of hypersurfaces of $\bar{\mathbb{E}}^4$ that satisfy (5) have constant mean curvature. The aim of the present paper is to prove, without any additional assumptions, that this property holds in general. More precisely, we prove the following.

THEOREM. *A hypersurface of $\bar{\mathbb{E}}^4$ satisfying $\Delta\vec{H} = \lambda\vec{H}$ must necessarily have constant mean curvature.*

2. Preliminaries

Let M^3 be a hypersurface of the Euclidean space $\bar{\mathbb{E}}^4$. Denote by ∇ and $\tilde{\nabla}$ the Levi–Civita connections of M^3 and $\bar{\mathbb{E}}^4$, respectively. For any vector fields X, Y tangent to M^3, the formula of Gauss is given by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y)\xi, \tag{6}$$

where h is the scalar-valued second fundamental form and ξ is a unit normal vector. Denote by S the shape operator of ξ. Then the formula of Weingarten is given by

$$\tilde{\nabla}_X \xi = -S(X), \tag{7}$$

where $\langle S(X), Y\rangle = h(X,Y)$. The mean curvature vector $\vec{H} = H\vec{\xi}$, with $H = \frac{1}{3}\operatorname{trace} S$, is a well-defined normal vector field to M^3 in $\bar{\mathbb{E}}^4$. The equation of Codazzi is given by

$$(\nabla_X S)Y = (\nabla_Y S)X. \tag{8}$$

The Gauss equation reads

$$R(X,Y)Z = S(X)\langle S(Y), Z\rangle - S(Y)\langle S(X), Z\rangle. \tag{9}$$

We now consider a hypersurface M^3 of $\bar{\mathbb{E}}^4$ satisfying the condition (5)

$$\Delta\vec{H} = \lambda\vec{H}, \quad \lambda \in \bar{\mathbb{R}}.$$

Introducing a local orthonormal frame $\{e_i\}_{i=1}^3$, the Laplace operator Δ acting on a vector-valued function $\vec{V}$ is given by

$$\Delta\vec{V} = \sum_{i=1}^{3} \left[\tilde{\nabla}_{\nabla_{e_i} e_i}\vec{V} - \tilde{\nabla}_{e_i}\tilde{\nabla}_{e_i}\vec{V}\right]. \tag{10}$$

With (10), we find the following necessary and sufficient conditions for a hypersurface M^3 of $\bar{\mathbb{E}}^4$ to satisfy (5):

$$S(\nabla H) = -\frac{3H}{2}(\nabla H), \tag{11}$$

$$\Delta H + H \operatorname{tr} S^2 = \lambda H, \quad \lambda \in \bar{\mathbb{R}}, \tag{12}$$

where the Laplace operator Δ acting on a scalar-valued function f is given by

$$\Delta f = \sum_{i=1}^{3} [\nabla_{e_i} e_i f - e_i e_i f]. \tag{13}$$

3. Hypersurfaces of $\bar{\mathbb{E}}^4$ satisfying $\Delta\vec{H} = \lambda\vec{H}$

In the present section, we prove that the mean curvature of hypersurfaces of $\bar{\mathbb{E}}^4$ satisfying condition (5) has to be constant. We first remark that following a result of Chen [3], a submanifold M^n of $\bar{\mathbb{E}}^m$ satisfies (5) if and only if:

(i) M^n is biharmonic ($\lambda = 0$); or
(ii) M^n is of 1-type; or
(iii) M^n is of null 2-type.

Concerning (i), in [12] and (independently, by another method) in [9] it is proved that the only biharmonic hypersurfaces M^3 of $\bar{\mathbb{E}}^4$ are the minimal ones. Concerning (ii), we note that Takahashi's result [14] completely identifies the 1-type hypersurfaces of $\bar{\mathbb{E}}^4$: besides the minimal ones ($\lambda = 0$), the only 1-type hypersurfaces of $\bar{\mathbb{E}}^4$ are the hyperspheres S^3 of $\bar{\mathbb{E}}^4$ ($\lambda > 0$). Concerning (iii), we remark that a result by Ferrández and Lucas [10] implies the classification of all null 2-type hypersurfaces of $\bar{\mathbb{E}}^4$ with at most 2 distinct principal curvatures: a hypersurface M^3 of $\bar{\mathbb{E}}^4$ of null 2-type and having at most two distinct principal curvatures is locally isometric to a generalized cylinder $\bar{\mathbb{R}} \times S^2$ or $\bar{\mathbb{R}}^2 \times S^1$.

In view of these results, in order to prove our statement it is sufficient to analyze the content of condition (5) with $\lambda \neq 0$ for hypersurfaces M^3 of $\bar{\mathbb{E}}^4$ with exactly three different principal curvatures. We therefore suppose that M^3 is a hypersurface of $\bar{\mathbb{E}}^4$ that does *not* have constant mean curvature, and then show that this assumption entails a contradiction.

We supposed that M^3 does not have constant mean curvature H. It follows that $\nabla H \neq \vec{0}$, and (11) shows that ∇H is an eigenvector of S with corresponding eigenvalue $\lambda_1 = -3H/2$. We now choose a local orthonormal frame $\{e_i\}_{i=1}^3$ consisting of eigenvectors of S and such that e_1 is a unit vector in the direction of ∇H.

With respect to this local frame, S is diagonal, and its matrix representation takes the form

$$\begin{pmatrix} -3H/2 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix}. \tag{14}$$

Moreover, we also have that

$$e_1(H) \neq 0, \quad e_2(H) = 0, \quad e_3(H) = 0. \tag{15}$$

Since we could confine ourselves to the case where, in (14), all three principal curvatures are mutually different, we have that

$$3H/2 + \lambda_2 \neq 0, \quad 3H/2 + \lambda_3 \neq 0, \quad \lambda_3 - \lambda_2 \neq 0. \tag{16}$$

Writing $\nabla_{e_i} e_j = \omega_i^k(e_j)e_k$, the Codazzi equations (8) for $\langle(\nabla_{e_1} S)e_2, e_1\rangle$ and $\langle(\nabla_{e_1} S)e_3, e_1\rangle$ show that

$$\omega_1^1(e_2) = 0, \quad \omega_1^1(e_3) = 0. \tag{17}$$

Next, the Codazzi equations (8) for $\langle(\nabla_{e_1} S)e_2, e_2\rangle$ and $\langle(\nabla_{e_1} S)e_3, e_3\rangle$ readily give that

$$\omega_2^2(e_1) = -\frac{e_1(\lambda_2)}{3H/2 + \lambda_2}, \qquad \omega_3^3(e_1) = -\frac{e_1(\lambda_3)}{3H/2 + \lambda_3}. \tag{18}$$

Similarly, the Codazzi equations (8) for $\langle(\nabla_{e_2} S)e_3, e_2\rangle$ and $\langle(\nabla_{e_2} S)e_3, e_3\rangle$ imply that

$$\omega_2^2(e_3) = \frac{e_3(\lambda_2)}{\lambda_3 - \lambda_2}, \qquad \omega_3^3(e_2) = \frac{e_2(\lambda_3)}{\lambda_2 - \lambda_3}. \tag{19}$$

Finally, in view of (15), we have that $[e_2, e_3](H) = 0$. Together with the Codazzi equations (8) for $\langle(\nabla_{e_1} S)e_2, e_3\rangle$ and $\langle(\nabla_{e_1} S)e_3, e_2\rangle$, it follows that

$$\omega_1^2(e_3) = 0, \quad \omega_2^3(e_1) = 0, \quad \omega_3^1(e_2) = 0. \tag{20}$$

Now, the Gauss equations (9) for $\langle R(e_2, e_3)e_1, e_2\rangle$ and $\langle R(e_2, e_3)e_1, e_3\rangle$ show that

$$e_3\left(\frac{e_1(\lambda_2)}{3H/2 + \lambda_2}\right) = \frac{e_3(\lambda_2)}{\lambda_3 - \lambda_2}\left(\frac{e_1(\lambda_3)}{3H/2 + \lambda_3} - \frac{e_1(\lambda_2)}{3H/2 + \lambda_2}\right), \tag{21}$$

$$e_2\left(\frac{e_1(\lambda_3)}{3H/2 + \lambda_3}\right) = \frac{e_2(\lambda_3)}{\lambda_2 - \lambda_3}\left(\frac{e_1(\lambda_3)}{3H/2 + \lambda_3} - \frac{e_1(\lambda_2)}{3H/2 + \lambda_2}\right). \tag{22}$$

On the other hand, in view of (13) and (15), equation (12) takes the form

$$e_1 e_1(H) - \left(\frac{e_1(\lambda_2)}{3H/2 + \lambda_2} + \frac{e_1(\lambda_3)}{3H/2 + \lambda_3}\right)e_1(H) - H\left(\frac{45H^2}{2} - 2\lambda_2\lambda_3\right) + \lambda H = 0. \tag{23}$$

Acting with e_2 and with e_3 on (23), and combining with the expressions (21) and (22), we have

$$e_2\left(\frac{e_1(\lambda_2)}{3H/2 + \lambda_2}\right) = -\frac{e_2(\lambda_3)}{\lambda_2 - \lambda_3}\left(\frac{e_1(\lambda_2)}{3H/2 + \lambda_2} - \frac{e_1(\lambda_3)}{3H/2 + \lambda_3}\right) + \frac{2H}{e_1(H)}(\lambda_2 - \lambda_3)^2 \frac{e_2(\lambda_3)}{\lambda_2 - \lambda_3}, \tag{24}$$

$$e_3\left(\frac{e_1(\lambda_3)}{3H/2+\lambda_3}\right) = -\frac{e_3(\lambda_2)}{\lambda_3-\lambda_2}\left(\frac{e_1(\lambda_3)}{3H/2+\lambda_3} - \frac{e_1(\lambda_2)}{3H/2+\lambda_2}\right) + \frac{2H}{e_1(H)}(\lambda_3-\lambda_2)^2\frac{e_3(\lambda_2)}{\lambda_3-\lambda_2}. \tag{25}$$

We also have the Gauss equations (9) for $\langle R(e_1,e_2)e_1,e_2\rangle$ and $\langle R(e_3,e_1)e_1,e_3\rangle$, which yield the following relations:

$$e_1\left(\frac{e_1(\lambda_2)}{3H/2+\lambda_2}\right) - \left(\frac{e_1(\lambda_2)}{3H/2+\lambda_2}\right)^2 = -\frac{3}{2}H\lambda_2, \tag{26}$$

$$e_1\left(\frac{e_1(\lambda_3)}{3H/2+\lambda_3}\right) - \left(\frac{e_1(\lambda_3)}{3H/2+\lambda_3}\right)^2 = -\frac{3}{2}H\lambda_3. \tag{27}$$

Using (17)–(20), we find that

$$[e_1,e_2] = \frac{e_1(\lambda_2)}{3H/2+\lambda_2}e_2. \tag{28}$$

In addition, we take into account the relation

$$e_1\left(\frac{e_2(\lambda_3)}{\lambda_2-\lambda_3}\right) = \frac{e_1(\lambda_3)e_2(\lambda_3)}{(3H/2+\lambda_3)(\lambda_2-\lambda_3)}, \tag{29}$$

which follows from the Gauss equation (9) for $\langle R(e_3,e_1)e_2,e_3\rangle$.

Applying both sides of (28) on $\frac{e_1(\lambda_2)}{(3H/2+\lambda_2)}$, and using (24), (26), (27), and (29), we deduce that

$$\begin{aligned} 0 = \frac{e_2(\lambda_3)}{\lambda_2-\lambda_3}\Bigg[&\left(\frac{e_1(\lambda_2)}{3H/2+\lambda_2} - \frac{e_1(\lambda_3)}{3H/2+\lambda_3}\right)^2 + e_1\left(\frac{H}{e_1(H)}\right)(\lambda_2-\lambda_3)^2 \\ &- \frac{H}{e_1(H)}\left(\left(3\frac{e_1(\lambda_2)}{3H/2+\lambda_2} - \frac{e_1(\lambda_3)}{3H/2+\lambda_3}\right)(\lambda_2-\lambda_3)^2 \right.\\ &\left. - 2(\lambda_2-\lambda_3)e_1(\lambda_2-\lambda_3)\right)\Bigg]. \end{aligned} \tag{30}$$

Equation (30) shows that at least one of the factors $e_2(\lambda_3)$, or the expression between square brackets, must vanish.

We now prove that $e_2(\lambda_3)$ must necessarily be zero, since the assumption that $e_2(\lambda_3) \neq 0$ runs into contradiction. Indeed, suppose that $e_2(\lambda_3) \neq 0$. Then the factor between square brackets must vanish:

$$e_1\left(\frac{H}{e_1(H)}\right) = \frac{H}{e_1(H)}\left(\left(3\frac{e_1(\lambda_2)}{3H/2+\lambda_2} - \frac{e_1(\lambda_3)}{3H/2+\lambda_3}\right) - 2\frac{e_1(\lambda_2-\lambda_3)}{(\lambda_2-\lambda_3)}\right) \tag{31}$$

$$- \frac{1}{(\lambda_2-\lambda_3)^2}\left(\frac{e_1(\lambda_2)}{3H/2+\lambda_2} - \frac{e_1(\lambda_3)}{3H/2+\lambda_3}\right)^2. \tag{32}$$

Acting with e_2 on both sides of (32), in view of (24), (22), and (28) we have

$$2\left(\frac{e_1(\lambda_2)}{3H/2+\lambda_2} - \frac{e_1(\lambda_3)}{3H/2+\lambda_3}\right) = \frac{H}{e_1(H)}(\lambda_2-\lambda_3)^2. \tag{33}$$

Moreover, applying e_2 on (33) yields

$$\left(\frac{e_1(\lambda_2)}{3H/2+\lambda_2}-\frac{e_1(\lambda_3)}{3H/2+\lambda_3}\right)=\frac{2H}{e_1(H)}(\lambda_2-\lambda_3)^2. \tag{34}$$

Now, (33) and (34) form a set of equations from which it would follow that $\lambda_2 = \lambda_3$; this contradicts (16). Hence, we conclude that $e_2(\lambda_3) = 0$.

Analogously, from (17) and (18) we have

$$[e_3, e_1] = -\frac{e_1(\lambda_3)}{3H/2+\lambda_3}e_3, \tag{35}$$

which we apply on $\frac{e_1(\lambda_3)}{3H/2+\lambda_3}$. In view of the Gauss equation (9) for $\langle R(e_1, e_2)e_3, e_2\rangle$,

$$e_1\left(\frac{e_3(\lambda_2)}{\lambda_3-\lambda_2}\right)=\frac{e_1(\lambda_2)e_3(\lambda_2)}{(3H/2+\lambda_2)(\lambda_3-\lambda_2)}; \tag{36}$$

using (25)–(27), we deduce that

$$\begin{aligned} 0=\frac{e_3(\lambda_2)}{\lambda_3-\lambda_2}\Bigg[&\left(\frac{e_1(\lambda_3)}{3H/2+\lambda_3}-\frac{e_1(\lambda_2)}{3H/2+\lambda_2}\right)^2+e_1\left(\frac{H}{e_1(H)}\right)(\lambda_3-\lambda_2)^2\\ &-\frac{H}{e_1(H)}\left(\left(3\frac{e_1(\lambda_3)}{3H/2+\lambda_3}-\frac{e_1(\lambda_2)}{3H/2+\lambda_2}\right)(\lambda_3-\lambda_2)^2\right.\\ &\left.-2(\lambda_3-\lambda_2)e_1(\lambda_3-\lambda_2)\right)\Bigg]. \end{aligned} \tag{37}$$

In a similar fashion, one can show that $e_3(\lambda_2)$ must necessarily vanish. Indeed, following the same line of proof, the assumption that $e_3(\lambda_2) \neq 0$ runs into contradiction.

In summary, we have proved independently that $e_2(\lambda_3)$ and $e_3(\lambda_2)$ must vanish separately. Hence, we conclude both that

$$e_2(\lambda_3) = 0 \quad \text{and} \quad e_3(\lambda_2) = 0. \tag{38}$$

In view of (38), the Gauss equation (9) for $\langle R(e_2, e_3)e_2, e_3\rangle$ gives the following relation:

$$\frac{e_1(\lambda_2)e_1(\lambda_3)}{(3H/2+\lambda_2)(3H/2+\lambda_3)}+\lambda_2\lambda_3=0. \tag{39}$$

Calculating $e_1e_1(H)$ from (26) and (27) and combining with (23), using (39) gives the conditions

$$\left(\frac{e_1(\lambda_2)}{3H/2+\lambda_2}+\frac{e_1(\lambda_3)}{3H/2+\lambda_3}\right)e_1(H)=\frac{189}{8}H^3-\frac{9}{2}H\lambda_2\lambda_3-\frac{3}{4}\lambda H, \tag{40}$$

$$e_1e_1(H)=\frac{369}{8}H^3-\frac{13}{2}H\lambda_2\lambda_3-\frac{7}{4}\lambda H. \tag{41}$$

Acting with e_1 on both sides of (40) and using (26), (27), and (39), we calculate the next expression:

$$\begin{aligned}\left(\frac{e_1(\lambda_2)}{3H/2+\lambda_2}+\frac{e_1(\lambda_3)}{3H/2+\lambda_3}\right)&(441H^2-26\lambda_2\lambda_3-10\lambda)H\\ &=(432H^2-26\lambda_2\lambda_3-3\lambda)e_1(H). \end{aligned} \tag{42}$$

We now apply e_1 on (42). A calculation using (40)–(42) yields the following algebraic relation between H, $\lambda_2\lambda_3$, and λ:

$$
\begin{aligned}
0 = {} & -6{,}863{,}560{,}515H^8 + 1{,}143{,}208{,}323H^6\lambda - 42{,}663{,}996H^4\lambda^2 \\
& + 401{,}436H^2\lambda^3 + 2{,}160\lambda^4 + 485{,}815{,}806H^6(\lambda_2\lambda_3) \\
& - 226{,}624{,}230H^4\lambda(\lambda_2\lambda_3) + 4{,}883{,}760H^2\lambda^2(\lambda_2\lambda_3) + 23{,}784\lambda^3(\lambda_2\lambda_3) \\
& + 59{,}355{,}504H^4(\lambda_2\lambda_3)^2 + 11{,}332{,}152H^2\lambda(\lambda_2\lambda_3)^2 + 26{,}416\lambda^2(\lambda_2\lambda_3)^2 \\
& - 6{,}157{,}008H^2(\lambda_2\lambda_3)^3 - 135{,}200\lambda(\lambda_2\lambda_3)^3 + 140{,}608(\lambda_2\lambda_3)^4. \qquad (43)
\end{aligned}
$$

From (43), acting twice with e_1 and using (40)–(42), we obtain a second independent algebraic relation between H, $\lambda_2\lambda_3$, and λ:

$$
\begin{aligned}
0 = {} & -34{,}834{,}534{,}938{,}767{,}774{,}385H^{14} + 5{,}808{,}891{,}661{,}968{,}730{,}851H^{12}\lambda \\
& - 379{,}857{,}621{,}779{,}657{,}460H^{10}\lambda^2 + 11{,}998{,}169{,}284{,}652{,}424H^8\lambda^3 \\
& - 182{,}885{,}218{,}372{,}368H^6\lambda^4 + 1{,}132{,}797{,}170{,}736H^4\lambda^5 \\
& - 1{,}452{,}914{,}496H^2\lambda^6 + 12{,}530{,}840{,}824{,}018{,}014{,}192H^{12}(\lambda_2\lambda_3) \\
& - 2{,}010{,}178{,}653{,}285{,}140{,}784H^{10}\lambda(\lambda_2\lambda_3) + 110{,}824{,}454{,}432{,}277{,}468H^8\lambda^2(\lambda_2\lambda_3) \\
& - 2{,}543{,}059{,}734{,}076{,}368H^6\lambda^3(\lambda_2\lambda_3) + 22{,}341{,}826{,}156{,}032H^4\lambda^4(\lambda_2\lambda_3) \\
& - 45{,}136{,}605{,}888H^2\lambda^5(\lambda_2\lambda_3) - 5{,}137{,}344\lambda^6(\lambda_2\lambda_3) \\
& - 1{,}367{,}090{,}289{,}917{,}896{,}788H^{10}(\lambda_2\lambda_3)^2 \\
& + 256{,}854{,}808{,}101{,}239{,}388H^8\lambda(\lambda_2\lambda_3)^2 - 11{,}422{,}286{,}566{,}141{,}896H^6\lambda^2(\lambda_2\lambda_3)^2 \\
& + 16{,}193{,}713{,}852{,}912H^4\lambda^3(\lambda_2\lambda_3)^2 - 535{,}601{,}789{,}856H^2\lambda^4(\lambda_2\lambda_3)^2 \\
& - 114{,}156{,}288\lambda^5(\lambda_2\lambda_3)^2 + 21{,}787{,}456{,}623{,}001{,}056H^8(\lambda_2\lambda_3)^3 \\
& - 15{,}563{,}208{,}968{,}133{,}840H^6\lambda(\lambda_2\lambda_3)^3 + 501{,}571{,}375{,}583{,}616H^4\lambda^2(\lambda_2\lambda_3)^3 \\
& - 3{,}048{,}670{,}842{,}048H^2\lambda^3(\lambda_2\lambda_3)^3 - 836{,}379{,}648\lambda^4(\lambda_2\lambda_3)^3 \\
& + 5{,}888{,}457{,}235{,}428{,}768H^6(\lambda_2\lambda_3)^4 + 46{,}075{,}037{,}026{,}668H^4\lambda(\lambda_2\lambda_3)^4 \\
& - 8{,}228{,}316{,}021{,}120H^2\lambda^2(\lambda_2\lambda_3)^4 - 899{,}891{,}200\lambda^3(\lambda_2\lambda_3)^4 \\
& - 444{,}888{,}559{,}320{,}192H^4(\lambda_2\lambda_3)^5 - 5{,}928{,}306{,}059{,}520H^2\lambda(\lambda_2\lambda_3)^5 \\
& + 13{,}878{,}572{,}032\lambda^2(\lambda_2\lambda_3)^5 + 12{,}466{,}012{,}815{,}360H^2(\lambda_2\lambda_3)^6 \\
& + 21{,}876{,}355{,}072\lambda(\lambda_2\lambda_3)^6 - 124{,}706{,}922{,}496(\lambda_2\lambda_3)^7. \qquad (44)
\end{aligned}
$$

Elimination of $\lambda_2\lambda_3$ between (43) and (44) gives a nontrivial algebraic equation for H with constant coefficients, one that involves the parameter λ. We shall not list this eliminant explicitly because of its length, and since its particular form does not contain the most important information; it can, however, be recovered quickly by applying the standard Mathematica command `Eliminate[ ]` to equations (43) and (44). The simple fact that H satisfies a nontrivial algebraic equation with constant coefficients shows—without having to solve this algebraic equation explicitly and even in the case of a real solution—that H must be a constant, which contradicts our assumption.

In summary, we have proved that the assumption $H \neq C$ for a hypersurface M^3 of $\bar{\mathbb{E}}^4$ satisfying (11) and (12) yields a contradiction. Hence, we have proved the following.

THEOREM. *A hypersurface of $\bar{\mathbb{E}}^4$ satisfying $\Delta \vec{H} = \lambda \vec{H}$ must necessarily have constant mean curvature.*

REMARK. Subsequent to the completion of this paper, it was brought to my attention that the content of the present theorem could be recovered also on the basis of results in [13]. Indeed, [13] completes a characterization of the null 2-type hypersurfaces of $\bar{\mathbb{E}}^4$. A comparison of our method of proof with those in [13] shows that the latter relies on technicalities of the classification of H-surfaces presented in [12]; hence [13] is based on explicit parametrizations and choice of special coordinates. Our proof here is entirely independent of coordinates and is also more self-contained from a structural point of view. It therefore provides better insight into the structure of the hypersurface and, in particular, may open perspectives for generalizations to higher (co)dimensions.

References

[1] B.-Y. Chen, *Total mean curvature and submanifolds of finite type,* World Scientific, Singapore, 1984.
[2] ———, *Null 2-type surfaces in $\mathbb{E}^3$ are circular cylinders,* Kodai Math. J. 11 (1988), 295–299.
[3] ———, *Null 2-type surfaces in Euclidean space,* Proceedings of the symposium in honor of Cheng-Sung Hsu and Kung-Sing Shih: algebra, analysis, and geometry (National Taiwan Univ., 1988), pp. 1–18, World Scientific, Teaneck, NJ.
[4] ———, *Some open problems and conjectures on submanifolds of finite type,* Soochow J. Math. 17 (1991), 169–188.
[5] ———, *Some classification theorems for submanifolds in Minkowski space-time,* Arch. Math. (Basel) 62 (1994), 177–182.
[6] ———, *Submanifolds of Euclidean spaces satisfying $\Delta H = AH$,* Tamkang J. Math. 25 (1994), 71–81.
[7] ———, *Submanifolds in de Sitter space-time satisfying $\Delta H = \lambda H$,* Israel J. Math. 91 (1995), 373–391.
[8] ———, *A report on submanifolds of finite type,* Soochow J. Math. 22 (1996), 117–337.
[9] F. Defever, *Hypersurfaces of $\bar{\mathbb{E}}^4$ with harmonic mean curvature vector,* preprint.
[10] A. Ferrández and P. Lucas, *Null finite type hypersurfaces in space forms,* Kodai Math. J. 14 (1991), 406–419.
[11] O. J. Garay, *A classification of certain 3-dimensional conformally flat Euclidean hypersurfaces,* Pacific J. Math. 162 (1994), 13–25.
[12] Th. Hasanis and Th. Vlachos, *Hypersurfaces in $\mathbb{E}^4$ with harmonic mean curvature vector field,* Math. Nachr. 172 (1995), 145–169.

[13] ———, *Hypersurfaces with constant scalar curvature and constant mean curvature,* Ann. Global Anal. Geom. 13 (1995), 69–77.
[14] T. Takahashi, *Minimal immersions of Riemannian manifolds,* J. Math. Soc. Japan 18 (1966), 380–385.

Zuivere en Toegepaste Differentiaalmeetkunde
Departement Wiskunde
KU Leuven, Celestijnenlaan 200 B
3001 Heverlee (Leuven)
Belgium