Acylindrical Surfaces in 3-Manifolds

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An annulus properly embedded in a 3-manifold is said to be *essential* if it is incompressible and not boundary parallel. An incompressible surface F in a 3-manifold M is *acylindrical* if the closure of its complement contains no essential annuli. Acylindrical surfaces play a special rule in 3-dimensional hyperbolic geometry. For example, it is a consequence of Thurston's work that an acylindrical surface in a compact irreducible 3-manifold has a complement that admits a hyperbolic metric with totally geodesic boundary [T1].

In this paper we will show that if M is a closed hyperbolic manifold and F is an incompressible acylindrical surface that has large genus, then M has large volume. An immediate topological consequence of this is that any 3-manifold contains only finitely many acylindrical surfaces. This extends the result, implicit in the theory of normal surfaces as developed by Haken, that there are only finitely many incompressible surfaces of fixed genus in a closed atoroidal 3-manifold [Ha]. Haken's result is false without the atoroidal assumption, since there are many 3-manifolds that contain infinitely many nonisotopic incompressible surfaces. The simplest of these is the 3-torus, with its many nonisotopic tori. Examples can also be constructed that are hyperbolic 3-manifolds, such as those that fiber over the circle and have noncyclic second homology groups [T2]. These have incompressible surfaces of unbounded genus, though only finitely many of any given genus.

Acylindrical surfaces also arise naturally in the study of finitely generated groups. Recent work of Rips and Sela shows that the structures of arbitrary finitely generated groups have parallels with the structure of 3-manifold fundamental groups. An acylindrical surface in a 3-manifold gives a splitting of the fundamental group of the 3-manifold along a malnormal surface subgroup. Sela has recently generalized the finiteness results obtained here to obtain bounds on the number of malnormal splittings for a much larger class of groups, including freely indecomposable groups with no 2-torsion [Se]. Sela's methods are completely different.

Definitions. In a Riemannian manifold M we let $d(x,y)$ denote the distance between two points x and y and $B(x,r)$, the open ball of radius r

Received July 6, 1994. Revision received March 21, 1995.
This work was carried out while the author was a member of the Institute for Advanced Study in 1990–91. Partially supported by the NSF.
Michigan Math. J. 42 (1995).

around x. A surface is *minimal* if its mean curvature is zero. A minimal surface is *stable* if its second variation of area is ≥ 0. An embedded surface in a 3-manifold M is *incompressible* if its fundamental group injects into the fundamental group of M under the inclusion map. A properly embedded annulus in a 3-manifold with boundary is *essential* if it is incompressible and not boundary parallel. By a *minimal plane* we mean a complete simply connected minimal surface. Hyperbolic 3-space is denoted H^3.

The following result, which follows from a theorem of Schoen [Sc], plays a central role in the estimates we will obtain.

LEMMA 1. *The principal curvatures of all complete stable minimal surfaces in H^3 are uniformly bounded in absolute value.*

Lemma 1 applies more generally in a manifold whose sectional curvatures are uniformly bounded above and below.

COROLLARY 2. *There is a constant $s_0 > 0$ such that a sphere of radius $r < s_0$ in H^3 has principal curvatures larger than that of any stable minimal surface in H^3.*

The next lemma states that two disjoint stable minimal surfaces in H^3 that are close at one point stay close in a large ball around that point.

LEMMA 3. *Let P and Q be two disjoint complete stable minimal surfaces in H^3, and let x be a point on P. Given constants $r > 0$ and $\epsilon > 0$, there is a $\delta > 0$ such that if $d(x, Q) < \delta$ and y is a point on the component of $P \cap B(x, r)$ that contains x, then $d(y, Q) < \epsilon$. Moreover, δ can be chosen so that P is C^1-close to Q at any such y in $B(x, r)$.*

Proof. If the first assertion fails, we can find an $r > 0$ and $\epsilon > 0$, as well as a sequence of stable minimal surfaces P_i, Q_i and points x_i, y_i on the same component of $P_i \cap B(x_i, r)$ such that $\lim_{i \to \infty} d(x_i, Q_i) = 0$ and $d(y_i, Q_i) > \epsilon$.

A standard argument shows that a subsequence of the surfaces $\{P_i\}$, $\{Q_i\}$ converges in H^3 to stable minimal surfaces P_0 and Q_0. We present this argument briefly. After translating and rotating by a hyperbolic isometry, we can assume that x_i is a fixed point x in H^3 and that the tangent planes of P_i at x_i are constant. Lemma 1 implies that the principal curvatures of a complete stable minimal surface in H^3 are bounded in absolute value. Thus there is a coordinate neighborhood V_x of x with coordinates (u^1, u^2, u^3) in which x is the origin and each P_i is tangent at x to $u^3 = 0$. Each P_i in some neighborhood of x is a union of graphs over the u^1–u^2 plane. We consider only the component of $\{P_i \cap V_x\}$ which meets x and one component of $\{Q_i \cap V_x\}$ with the property that $\lim_{i \to \infty} d(x_i, Q_i) = 0$ for this component. The uniform curvature bounds imply that the second derivatives of these graphs are uniformly bounded in V_x, and that there is a neighborhood of x in the u^1-u^2 plane over which all P_i are graphs tangent to the u^1-u^2 plane with uniformly bounded second derivatives. This implies that the graphs and their first derivatives are uniformly bounded over a smaller neighborhood of x. The surfaces Q_i satisfy the same curvature bounds and, for i large enough, are also graphs with uniformly bounded first derivatives over a neighborhood of x in the u^1-u^2 plane. Ascoli's theorem then implies the existence of convergent subsequences of $\{P_i \cup Q_i\}$ in a neighborhood U_x of x that converge in U_x to stable minimal surfaces P_0 and Q_0. By repeating this argument in a collection of balls covering H^3 and taking a diagonal subsequence, we can arrange for $\{P_i \cup Q_i\}$ to converge in all of H^3 to stable minimal surfaces P_0 and Q_0 (these surfaces may not be properly embedded). P_0 and Q_0 intersect at x without crossing, and the maximum principle for minimal surfaces then implies that P_0 and Q_0 have the same image everywhere in H^3.

All the points y_i lie in a compact ball of radius r around x, so by passing to a subsequence we can assume that the points y_i are converging to a limit point y on P. Since $d(y_i, Q_i) \geq \epsilon$ for each i, it follows that $d(y, Q) \geq \epsilon$, a contradiction since $d(y, Q) = 0$. Thus δ sufficiently small implies that $d(y, Q) < \epsilon$ at any point y on the component of $P \cap B(x, r)$ that contains x, proving the first assertion of the lemma. If the tangent plane of P_i at y_i is uniformly bounded away from the tangent planes of Q_i as $i \to \infty$, then Lemma 1 implies that P_i and Q_i must intersect near y_i for large i, proving the second assertion. □

The next lemma shows that two lifts P and Q of an acylindrical surface cannot remain close on too large a set.

LEMMA 4. *Let F be a closed, acylindrical, incompressible surface in a hyperbolic* 3*-manifold M, with $\epsilon > 0$ a constant. Then there is an $r > 0$ such that if P and Q are two distinct lifts to H^3 of F and $x \in P$, then for any $y \in P$ with $d(x, y) > r$, $d(y, Q) \geq \epsilon$.*

Proof. If the lemma is false, then there is a sequence of points $\{y_i\}$ in P with $d(x, y_i) \to \infty$ and $d(y_i, Q) < \epsilon$. We will use these points to construct an essential annulus in the complement of F. Since F is compact, a subsequence of $\{y_i\}$ has the property that $\pi(y_i)$ converges to a point y in F, where $\pi: P \to F$ is the covering projection. We can assume, by passing to a subsequence and moving the points y_i slightly, that $\pi(y_i) = y$ for each i. Let a_i denote the shortest geodesic arc in H^3 which meets P at y_i and whose other endpoint lies in Q. Then the lengths of the arcs a_i are uniformly bounded by ϵ and their projections $\pi(a_i)$ converge to an arc a in M with one endpoint at y and the other endpoint also on F. By perturbing a_i slightly, we can assume that $\pi(a_i)$ is a fixed arc. For each pair of arcs a_i, a_j we can form a disk in H^3 bounded by a_i, a_j and arcs on P and Q connecting the endpoints of a_i and a_j.

Projecting to M, we get an annulus in M, not necessarily embedded in or missing F, whose boundary lies on F. This annulus is not homotopic into F since its boundary components lift up to distinct components of the preimage of F. The annulus theorem [Ja] implies the existence of an essential embedded annulus in the closure of $(M-F)$, contradicting the assumption that F is acylindrical. □

We now describe the local picture in H^3 when two stable minimal planes lie close together.

LEMMA 5. *Let P, Q be disjoint complete stable minimal planes in H^3, with x a point in P. Given $r > 0$, there is a $\delta_1 > 0$ (depending only on r) such that if $d(x, Q) < \delta_1$ then:*

(1) *the component C_1 of $P \cap B(x, r)$ containing x is homeomorphic to a disk;*
(2) *the component C_2 of $Q \cap B(x, r)$ nearest to x is homeomorphic to a disk; and*
(3) *if a complete stable minimal surface R in H^3, disjoint from P and Q, intersects the region between C_1 and C_2 in $B(x, r)$, then it does so in disks that are parallel in $B(x, r)$ to C_1 and C_2.*

Proof. Since P and Q are minimal, they have nonpositive normal curvatures in H^3. Morse theory implies that the restriction to P and Q of the distance function from any point has no critical points of index 2. Since P and Q are simply connected, each component of the intersection of P and Q with any geodesic ball in H^3 is simply connected. Lemma 1 and Lemma 3 imply that for δ_1 small, Q is C^1-close to P in $B(x, r)$. If R intersects the region between C_1 and C_2 in a component C_3, then P and R also satisfy the conditions of Lemma 3 and the lemma follows. □

The next lemma gives a lower bound for the area of an incompressible surface in a hyperbolic 3-manifold in terms of its genus, and in addition an upper bound for a least-area surface. It works equally well for immersed, π_1-injective surfaces. This lemma was originally observed in an unpublished work of Uhlenbeck [Uh].

LEMMA 6. *The area of any closed incompressible surface of genus g in a hyperbolic 3-manifold is greater than or equal to $2\pi(g-1)$. Furthermore, if the surface is chosen to have least area in its homotopy class, then its area is at most $4\pi(g-1)$. Thus the area of a least-area surface F satisfies*

$$2\pi(g-1) \leq \text{area}(F) \leq 4\pi(g-1).$$

Proof. By [SY] or [SU] there is an immersed minimal surface F that minimizes area among all homotopic surfaces. Let h_{ij} denote the components of the second fundamental form of F, and let R_{ij} denote the components of the sectional curvature of M. Using (as in [SY]) the formula for the second variation of area for a stable minimal surface in a 3-manifold implies that, for any normal variation,

$$\int_F (R_{13}+R_{23}+h_{11}^2+h_{22}^2+2h_{12}^2)f^2\,dv \le \int_F |\nabla f|^2\,dv,$$

where $f\cdot N$ gives a normal variation if N is a unit normal.

Setting $f=1$ and noting that $R_{13}=R_{23}=-1$ in a hyperbolic 3-manifold, we have:

$$\int_F h_{11}^2+h_{22}^2+2h_{12}^2-2\,dv \le 0.$$

Since $h_{11}+h_{22}=0$ for a minimal surface, we have

$$\int_F 2h_{11}^2+2h_{12}^2\,dv \le \int_F 2\,dv = 2\,\mathrm{area}(F).$$

The Gauss-Bonnet theorem and the Gauss formula give

$$\int_F R_{12}+h_{11}h_{22}-h_{12}^2\,dv = 2\pi\chi(F).$$

$R_{12}=-1$ and $h_{11}=-h_{22}$, so for a minimal surface in a hyperbolic manifold:

$$\int_F h_{11}^2+h_{12}^2\,dv = \int_F -1\,dv - 2\pi\chi(F) = -\mathrm{area}(F)-2\pi\chi(F).$$

Hence

$$\begin{aligned}\mathrm{area}(F) &\ge -\mathrm{area}(F)-2\pi\chi(F),\\ &\ge -\pi\chi(F).\end{aligned}$$

Since $\chi(F)=2-2g$, where $g=\mathrm{genus}(F)$,

$$\mathrm{area}(F)\ge 2\pi(g-1).$$

Furthermore, the Gauss-Bonnet theorem implies that for any minimal surface F in a hyperbolic manifold,

$$\int_F -h_{11}^2-h_{12}^2-1\,dv = 2\pi\chi(F),$$

which implies that

$$\mathrm{area}(F) = \int_F -h_{11}^2-h_{12}^2\,dv - 2\pi\chi(F) \le -2\pi\chi(F) = 4\pi(g-1).$$

Thus, any least-area surface of genus g has area between $2\pi(g-1)$ and $4\pi(g-1)$. Any homotopic surface has area no less than the minimizing representative in its homotopy class, and the lemma follows. □

Note that the area of a totally geodesic hyperbolic surface F is $4\pi(g-1)$. Totally geodesic surfaces are always least-area in their homotopy classes, as can be seen by considering the covering space corresponding to their fundamental group. Thus the second inequality is sharp.

We now proceed to relate the volume of a closed hyperbolic 3-manifold to the genus of an incompressible surface embedded in it.

LEMMA 7. *There is an upper bound to the genus of an acylindrical surface in a hyperbolic manifold M. The bound depends only on the volume of M.*

Proof. Let M be a hyperbolic manifold of finite volume. Suppose that F_i is a sequence of closed acylindrical surfaces in M with genus$(F_i) \to \infty$. We can minimize in the homotopy class of each F_i to obtain a sequence of least-area surfaces, which we continue to denote by $\{F_i\}$. These least-area surfaces are embedded [FHS], and by Lemma 6, area$(F_i) \to \infty$ as $i \to \infty$. Since F_i is incompressible and acylindrical, the intersection of F_i with a horocyclic torus T cutting off a cusp of M is a collection of simple curves bounding disks on F_i and on T. A null-homotopic simple closed curve on T bounds a disk on T of area less than area(T). Any least-area disk with boundary on T and area less than area(T) must remain within a fixed neighborhood of T, the radius of this neighborhood being determined by the monotonicity formula for least-area surfaces in hyperbolic space. Thus the geometry of a hyperbolic cusp implies that an area-minimizing sequence of surfaces in the homotopy class of F_i remains in a fixed compact submanifold of M.

Because all of the surfaces $\{F_i\}$ lie in a fixed compact submanifold of M, area(F_i) is unbounded in an open ball U in M around some point x. Let U' be a lift of U to the universal cover H^3 of M. The pre-images of the surfaces $\{F_i\}$ have unbounded area in U' as $i \to \infty$. Let P_i be a lift of F_i to H^3. P_i is a least-area plane [FHS]. The monotonicity formula for least-area surfaces, as in [HS, Lemma 2.3], implies that there is an upper bound to the area of the intersection of P_i with U', so that there is a bound on the number of components of P_i in U', independent of i. Thus for each i we can pick a pair of lifts P_i and Q_i of F_i to H^3 such that $d(P_i \cap U', Q_i \cap U') = 0$. Let p_i be a point in $P_i \cap U'$ such that $\lim_{i\to\infty} d(p_i, Q_i) = 0$ and Q_i is the closest to p_i of all the distinct pre-images of F_i in H^3. Given any constant $\delta_1 > 0$, Lemma 3 implies that there is a sequence of constants r_i with $\lim_{i\to\infty} r_i = \infty$ such that P_i and Q_i are within distance δ_1 on a ball $B(p_i, r_i)$ of radius r_i. After passing to a subsequence, we can assume also that $d(p_i, Q_i)$ is sufficiently small so that Lemma 5 applies to P_i and Q_i on a ball of radius r_i. Lemma 4 implies that there are constants R_i such that, at each point outside a larger ball $B(p_i, R_i)$, the two lifts P_i and Q_i have distance $> \delta_1$. As in Lemma 5, let $C_{i,1}$ denote the component of $P_i \cap B(p_i, R_i)$ containing p_i, and let $C_{i,2}$ denote the component of $Q_i \cap B(p_i, R_i)$ closest to p_i.

Define the set $\Gamma_i \subset P_i$ to be the set of points $\{y \in C_{i,1}: d(y, Q_i) = \delta_1\}$. The set Γ_i lies in the ball of radius R_i in H^3 around p_i and outside the ball of radius r_i. Let $s_1 = \min\{\delta_1/4, s_0\}$, where s_0 is the constant in Corollary 2. For $y \in \Gamma_i$ let $TB(y, s_1)$ denote the ball of radius s_1 in M that is tangent to P_i at y and lies between P_i and Q_i. The ball $TB(y, s_1)$ is disjoint from Q_i because $d(y, Q_i) > \delta_1 > 2s_1$, and it intersects P_i only at y since the curvature of P_i is less than that of $\partial TB(y, s_1)$. Since Q_i is closest to P_i at y, $TB(y, s_1)$ is also disjoint from all other lifts of F.

Let $\{y_{i,j}\}$ be a maximal collection of points on Γ_i such that the balls $TB(y_{i,j}, s_1)$ have centers that are spaced at least $2s_1$ apart and are thus disjoint. The diameters of the sets Γ_i are unbounded as $i \to \infty$, since if they were uniformly bounded there would exist minimal surfaces in H^3 with boundaries inside a fixed diameter ball but with interiors protruding outside that ball; such surfaces would contradict the maximum principle for minimal surfaces. It follows that the number of points $\{y_{i,j}\}$ goes to infinity with i.

CLAIM 8. *The balls $TB(y_{i,j}, s_1)$ project to disjoint balls in M.*

Proof. If not, two balls $TB(y_{i,1}, s_1)$ and $TB(y_{i,2}, s_1)$, disjoint in H^3, project to overlapping balls in M. Then there is a covering translation τ of H^3 with $\tau \cdot TB(y_{i,1}, s_1) \cap TB(y_{i,2}, s_1) \neq \emptyset$. Since $d(P_i, Q_i) = \delta_1 > 2s_1$ at $y_{i,1}$ and $y_{i,2}$, and since Q_i is the closest plane to P_i at any $y_{i,j}$, τ must preserve both P_i and Q_i, so that $\tau \in \mathrm{stab}(P_i) \cap \mathrm{stab}(Q_i)$. A disk can be constructed in H^3 whose boundary is a geodesic connecting $y_{i,1}$ to $y_{i,2}$ on P_i, two geodesic arcs from P_i to Q_i of length δ_1, and a geodesic arc on Q_i. The two edges of this disk are identified by τ, so that it projects to an essential annulus in M whose boundary lies on F but which cannot be homotoped into F (rel boundary). This contradicts the assumption that F is acylindrical. □

Lemma 7 now follows, since as $i \to \infty$ there is an increasing number of disjoint radius-s_1 balls in M, eventually having volume exceeding the volume of M. □

If one considers only incompressible surfaces of a fixed genus, then it is implicit in the theory of normal surfaces (as developed by Haken) that there are finitely many such surfaces in a closed atoroidal 3-manifold [Ha; Sch]. This result does not assume the acylindrical property. We give a short geometric proof of this result here.

LEMMA 9. *Let M be a finite-volume hyperbolic* 3*-manifold. Then M contains finitely many closed incompressible surfaces of a given genus, up to isotopy.*

Proof. If not, let $\{F_i\}$ be an infinite sequence of nonisotopic closed incompressible surfaces of a fixed genus. If M is compact, then we can isotope each F_i to be area-minimizing in its isotopy class using the results of [FHS]. If M has cusps, an area-minimizing sequence of surfaces in the homotopy class of F_i remains in a fixed compact submanifold of M, as in the proof of Lemma 7.

Now let $x \in M$ be a limit point of the sequence of surfaces. Passing to a subsequence, we can assume that the tangent planes of the $\{F_i\}$ are converging to a tangent plane at x. Since the area and curvatures of $\{F_i\}$ are both uniformly bounded, an application of Ascoli's theorem (as in Lemma 3) shows that there is a convergent subsequence in a neighborhood of x. Repeating this process in a sequence of neighborhoods and taking a diagonal subsequence gives a subsequence that converges smoothly everywhere in M. Surfaces far out in this sequence are C^∞-close and therefore isotopic, a contradiction. □

We can obtain a stronger result, applying to incompressible surfaces of any genus, if we add the acylindrical assumption.

THEOREM 10. *Let M be a compact, orientable 3-manifold. Then M contains finitely many acylindrical incompressible surfaces, up to isotopy.*

Proof. If M is reducible, then any incompressible surface can be isotoped off the reducing spheres, so it suffices to prove the result for each irreducible part. Similarly, if M has boundary we can assume that the boundary is incompressible, as otherwise we can do compressions and any incompressible surface can be isotoped off of the compressing disks. If M is irreducible and non-Haken then it contains no incompressible surfaces. For Haken manifolds with a trivial torus–annulus decomposition, we apply Thurston's geomitrization theorem and consider the cases of the various geometries. If M is hyperbolic, the theorem follows from Lemma 9. If M is Seifert fibered, has the SOL geometry, or is an I-bundle, then incompressible closed surfaces are either vertical or horizontal [Wa] and are never acylindrical. If M has a nontrivial torus–annulus decomposition and an incompressible surface intersects an essential torus or annulus, then it does so in essential curves and is not acylindrical. So the theorem reduces to searching for acylindrical surfaces in finite-volume hyperbolic manifolds, and then follows from Lemmas 7 and 9. □

REMARKS. The results obtained for hyperbolic manifolds can be extended in a straightforward manner from hyperbolic manifolds to compact negatively curved 3-manifolds. D. Gabai has pointed out that Theorem 10 can also be obtained using techniques of branched surface theory, though the result had not previously been observed.

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