On the Tensor Products of Simple JC-Algebras

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Introduction

The purpose of this article is to examine the structure of the JC-tensor product of two simple JC-algebras. Unlike the C*-algebra case, the JC-tensor product of two simple JC-algebras is not necessarily simple.

Let A be a JC-algebra and Φ_A the canonical involutory *-anti-automorphism of $C^*(A)$, the universal enveloping C*-algebra of A. We may suppose that $A \subseteq C^*(A)$, so that Φ_A restricts to the identity on A. The real C*subalgebra of $C^*(A)$, $R^*(A) = \{x \in C^*(A) : \Phi_A(x) = x^*\}$, satisfies

$$R^*(A) \cap iR^*(A) = 0 \quad \text{and} \quad C^*(A) = R^*(A) \oplus iR^*(A).$$

Let A be a JC-algebra contained in $\mathbb{C}_{s.a.}$, the self-adjoint part of the C*-algebra $\mathbb{C}$. Then A is said to be *reversible* in $\mathbb{C}$ if $a_1 \cdots a_n + a_n \cdots a_1$ lies in A whenever $a_1, \dots, a_n$ are in A. A is said to be *universally reversible* if it is reversible in $C^*(A)$ [3]. The reader is referred to [1; 4; 7; 8] for a detailed account of the theory of JC-algebras. The relevant background for the theory on tensor products of C*-algebras can be found in [2; 6; 9; 10].

DEFINITION. Let A and B be a pair of JC-algebras. A and B are canonically embedded in their respective universal enveloping C*-algebras $C^*(A)$ and $C^*(B)$. Let λ be a C*-norm on $C^*(A) \otimes C^*(B)$. Then the *JC-tensor product of A and B with respect to* λ is the completion $JC(A \otimes_\lambda B)$ of the real Jordan algebra $J(A \otimes B)$ generated by $A \otimes B$ in $C^*(A) \otimes_\lambda C^*(B)$.

The reader is referred to [5] for the properties of the JC-tensor product of two JC-algebras.

THEOREM. *Let A and B be* JC-*algebras. Then*

$$C^*(JC(A \otimes_\lambda B)) = C^*(A) \otimes_\lambda C^*(B),$$

where λ *is the minimum or the maximum* C*-*norm.*

LEMMA. *Given* JC-*algebras A and B and a* C*-*norm* λ *on* $C^*(A) \otimes C^*(B)$, $JC(A \otimes_\lambda B)$ *is universally reversible unless one of A and B has a scalar rep-*

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resentation and the other has a representation onto a spin factor V_n, where $n \geq 4$.

For $n \geq 2$ let V_n be the $(n+1)$-dimensional spin factor. When $n < \infty$, recall that the table for the eight spin factors given by [1, p. 385] is

n	$R^*(V_n)$		
2	$M_2(\mathbf{R})$	6	$M_4(\mathbf{H})$
3	$M_2(\mathbf{C})$	7	$M_8(\mathbf{C})$
4	$M_2(\mathbf{H})$	8	$M_{16}(\mathbf{R})$
5	$M_2(\mathbf{H}) \oplus M_2(\mathbf{H})$	9	$M_{16}(\mathbf{R}) \oplus M_{16}(\mathbf{R})$

This table is repeated modulo 8 according to the formula

$$R^*(V_{n+8}) \cong R^*(V_n) \otimes M_{16}(\mathbf{R}).$$

Therefore, $R^*(V_n)$ are all simple real C*-algebras except those of the form $R^*(V_{4n+1})$, in which case $R^*(V_{4n+1}) \cong R^*(V_{4n}) \oplus R^*(V_{4n})$.

The Main Result

Takesaki [9, Cor., p. 117] (see also [2, Cor. 3]) proves that the minimum tensor product of two simple C*-algebras is simple. In contrast to the theory of tensor products of C*-algebras, the tensor product of two simple JC-algebras is not simple, in general. The most significant result of this article is the table given in Theorem 8 in which we give a complete structure theory for $JC(A \otimes_\lambda B)$ when A and B are simple JC-algebras.

In order to examine the structure of $JC(A \otimes_\lambda B)$ when A and B are simple JC-algebras, we shall need an elementary but important result (Lemma 1) from the theory of real C*-algebras. First, recall that if $\mathbb{B}$ is a real C*-algebra then $\mathbb{B}$ can be *regarded* as a complex C*-algebra if there is a complex C*-algebra $\mathbb{C}$ and a real C*-algebra isomorphism $\pi: \mathbb{C} \xrightarrow{\cong} \mathbb{B}$. The complex identity j acts on $\mathbb{B}$ as follows:

$$jr = \pi(i\pi^{-1}(r)) \quad \text{for } r \text{ in } \mathbb{B}.$$

LEMMA 1. *Let A be a* JC-*algebra (not necessarily unital). Then the following are equivalent:*

(i) $R^*(A)$ *can be realized as a complex* C*-*algebra.*

(ii) *There exists a norm closed ideal I of* $C^*(A)$ *such that*

$$C^*(A) = I \oplus \Phi_A(I).$$

In that case $R^*(A) \cong I$.

Proof. (i) ⇒ (ii). Assume (i), and let j be the complex identity acting on $R^*(A)$. Put $I = \{jx + ix : x \in R^*(A)\}$. Then I is a norm closed ideal of $C^*(A) = R^*(A) \oplus iR^*(A)$.

It is easily seen that $\Phi_A(I)=\{-jx+ix: x\in R^*(A)\}$, $I\cap\Phi_A(I)=0$, and if $x+iy\in R^*(A)\oplus iR^*(A)$ when x and $y\in R^*(A)$ then

$$x+iy=(ja+ia)+(-jb+ib)\in I\oplus\Phi_A(I),$$

where $a=(-jx+y)/2$ and $b=(jx+y)/2\in R^*(A)$. Hence $C^*(A)=I\oplus\Phi_A(I)$.

To prove the final statement we define $\pi: I\to R^*(A)$ by $\pi(x)=x+\Phi_A(x^*)$. It is not difficult to see that π is an injective real C*-algebra homomorphism, and if $a\in R^*(A)\subseteq I\oplus\Phi_A(I)$ then $a=x+\Phi_A(y)$ for some x and y in I. Thus $y=x^*$, that is, $a=x+\Phi_A(x^*)$, and hence π is surjective. This proves that π is a real C*-algebra isomorphism of I onto $R^*(A)$.

Let x be in I. Then $x=ja+ia$ for some a in $R^*(A)$, $\pi(x)=2ja$, and $\pi(ix)=-2a$. Thus $j\pi(x)=-2a$ and so $\pi(ix)=j\pi(x)$. Hence π is also a complex linear map.

(ii) $\Rightarrow$ (i). Suppose that (ii) holds, and note that

$$R^*(A)=\{x+\Phi_A(x^*): x\in I\}.$$

Then the map $\pi: I\to R^*(A)$ defined by $\pi(x)=x+\Phi_A(x^*)$ is a real C*-algebra isomorphism, and (i) follows. □

Recall that if J is a norm closed Jordan ideal of a JC-algebra A, then $R^*(J)$ is a norm closed idea of $R^*(A)$ such that $R^*(J)\cap A=J$, and if I is a norm closed ideal of $R^*(A)$ then $I\oplus iI$ is a norm closed ideal of $C^*(A)$. We may therefore immediately deduce the next lemma.

LEMMA 2. *Let A be a* JC-*algebra. If* $C^*(A)$ *is simple then* $R^*(A)$ *and A are simple.*

REMARK. Let A be a simple JC-algebra. If A has a 1-dimensional representation then $A\cong\mathbf{R}$ by the simplicity of A; hence, for each JC-algebra B, $\mathrm{JC}(A\otimes_{\min}B)\cong\mathbf{R}\otimes_{\mathbf{R}}B\cong B$.

Henceforth, whenever A is a simple JC-algebra, it is assumed that A has no 1-dimensional representations. Thus, if A and B are simple JC-algebras then $\mathrm{JC}(A\otimes_{\min}B)$ is universally reversible, and so

$$\mathrm{JC}(A\otimes_{\min}B)\cong(R^*(A)\otimes_{\min}R^*(B))_{\mathrm{s.a.}}.$$

Let A be a simple universally reversible JC-algebra. If $C^*(A)$ is not simple then it contains a nonzero norm closed two-sided ideal J such that $J\cap A=0$. Then, by [8, Lemma 3.1], $J_{\mathrm{s.a.}}$ is (Jordan) isomorphic to a Jordan ideal of A. But the simplicity of A implies that $A\cong J_{\mathrm{s.a.}}$.

For later reference, note that we have established the following lemma.

LEMMA 3. *Let A be a simple universally reversible* JC-*algebra. Then A is not isomorphic to the self-adjoint part of a* C*-*algebra if and only if* $C^*(A)$ *is simple.*

LEMMA 4. *Let A be a simple universally reversible* JC*-algebra. Then the following are equivalent:*

(i) *A is isomorphic to the self-adjoint part of a* C*-*algebra*;
(ii) *there exists a simple norm closed ideal I of* $\mathrm{C}^*(A)$ *such that* $\mathrm{C}^*(A)=I\oplus\Phi_A(I)$;
(iii) $\mathrm{R}^*(A)$ *is simple and complex.*

Proof. (i) ⇒ (ii). Assume that $A=\mathbb{C}_{\mathrm{s.a.}}$ for some complex C*-algebra $\mathbb{C}$. Then $\mathrm{C}^*(A)=\mathbb{C}\oplus\mathbb{C}^{\circ}$ by [4, 7.4.15], where the canonical *-anti-automorphism of $\mathrm{C}^*(A)$ is defined by $\Phi_A(x\oplus y^{\circ})=y\oplus x^{\circ}$ for x and y in $\mathbb{C}$. It follows that $\mathrm{C}^*(A)=\mathbb{C}\oplus\Phi_A(\mathbb{C})$. Since A is simple, $\mathbb{C}$ is simple and (ii) follows.

(ii) ⇔ (iii). This is Lemma 1.

(iii) ⇒ (i). The implication is immediate, since A is universally reversible and so $A\cong\mathrm{R}^*(A)_{\mathrm{s.a.}}$. □

REMARK. Recall that the universal enveloping C*-algebra $\mathrm{C}^*(V)$ of an infinite-dimensional spin factor V is a simple C*-algebra and hence $\mathrm{R}^*(V)$ is simple. It is clear that, when $2\le n<\infty$, $\mathrm{R}^*(V_{2n})$ is a simple matrix algebra over $\mathbf{R}$ or $\mathbf{H}$, $\mathrm{R}^*(V_{4n-1})=M_{2^{2n-1}}(\mathbf{C})$ (which is simple and complex), and $\mathrm{R}^*(V_{4n+1})\cong\mathrm{R}^*(V_{4n})\oplus\mathrm{R}^*(V_{4n})$, which is the direct sum of two copies of matrix algebras over $\mathbf{R}$ or $\mathbf{H}$. Hence, if A is a simple JC-algebra we have the following two cases.

(a) *A is not a spin factor.* In this case A is universally reversible and hence $\mathrm{R}^*(A)$ is simple, by Lemmas 2, 3, and 4.
(b) *A is a spin factor.* In this case $\mathrm{R}^*(A)$ is simple except when A is of the form V_{4n+1} for $2\le n<\infty$. It follows that if $\mathrm{R}^*(A)$ is complex then A cannot be of the form V_{4n+1}, and therefore $\mathrm{R}^*(A)$ is simple.

PROPOSITION 5. *Let A be a simple* JC*-algebra such that* $\mathrm{R}^*(A)$ *is complex. Then, for any simple* JC*-algebra B,*

$$\mathrm{JC}(A\otimes_{\min}B)\cong(I\otimes_{\min}\mathrm{C}^*(B))_{\mathrm{s.a.}},$$

where I is a simple norm closed ideal of $\mathrm{C}^*(A)$ *such that* $\mathrm{C}^*(A)=I\oplus\Phi_A(I)$.

Proof. Note that by the assumption on $\mathrm{R}^*(A)$, A is not isomorphic to a spin factor of the form V_{4n+1}, and hence $\mathrm{R}^*(A)$ is simple. Thus, by Lemma 1, there exists a simple norm closed ideal I of $\mathrm{C}^*(A)$ such that $I\cong\mathrm{R}^*(A)$ and $\mathrm{C}^*(A)=I\oplus\Phi_A(I)$. Hence

$$\begin{aligned}\mathrm{C}^*(\mathrm{JC}(A\otimes_{\min}B))&=\mathrm{C}^*(A)\otimes_{\min}\mathrm{C}^*(B)\\&=(I\otimes_{\min}\mathrm{C}^*(B))\oplus(\Phi_A(I)\otimes_{\min}\mathrm{C}^*(B))\\&=(I\otimes_{\min}\mathrm{C}^*(B))\oplus(\Phi_A\otimes_{\min}\Phi_B)(I\otimes_{\min}\mathrm{C}^*(B)).\end{aligned}$$

It follows that

$$\mathrm{R}^*(\mathrm{JC}(A\otimes_{\min}B))\cong I\otimes_{\min}\mathrm{C}^*(B),$$

by Lemma 4. Since $\mathrm{JC}(A \otimes_{\min} B)$ is universally reversible, $\mathrm{JC}(A \otimes_{\min} B) \cong (I \otimes_{\min} \mathrm{C}^*(B))_{\mathrm{s.a.}}$ and the proof is complete. □

PROPOSITION 6. *Let A and B be simple* JC-*algebras where neither A nor B is of the form V_{4n+1}. Then:*

(i) *if* $\mathrm{R}^*(A)$ (*or* $\mathrm{R}^*(B)$) *is not complex then* $\mathrm{JC}(A \otimes_{\min} B)$ *is simple; and*
(ii) *if* $\mathrm{R}^*(A)$ *and* $\mathrm{R}^*(B)$ *are complex then* $\mathrm{JC}(A \otimes_{\min} B)$ *is a direct sum of two simple* JC-*algebras.*

Proof. (i)(a) Suppose that $\mathrm{R}^*(A)$ and $\mathrm{R}^*(B)$ are not complex. Then A and B are not of the form V_{4n-1}, which implies that $\mathrm{C}^*(A)$ and $\mathrm{C}^*(B)$ are simple (cf. [4, 6.2.2]). Therefore

$$\mathrm{C}^*(\mathrm{JC}(A \otimes_{\min} B)) = \mathrm{C}^*(A) \otimes_{\min} \mathrm{C}^*(B)$$

is simple by [2, Cor. 3] and so $\mathrm{JC}(A \otimes_{\min} B)$ is simple.

(i)(b) Suppose that $\mathrm{R}^*(A)$ is complex and $\mathrm{R}^*(B)$ is not complex. Then, by Proposition 5, $\mathrm{JC}(A \otimes_{\min} B) \cong (I \otimes_{\min} \mathrm{C}^*(B))_{\mathrm{s.a.}}$ where I is a simple norm closed ideal of $\mathrm{C}^*(A)$. Thus $\mathrm{JC}(A \otimes_{\min} B)$ is simple, as before, because $\mathrm{C}^*(B)$ is simple.

(ii) Suppose that $\mathrm{R}^*(A)$ and $\mathrm{R}^*(B)$ are complex. Then, by Proposition 5, there are simple norm closed ideals I and J of $\mathrm{C}^*(A)$ and $\mathrm{C}^*(B)$, respectively, such that $\mathrm{C}^*(B) = J \oplus \Phi_B(J)$ and $\mathrm{JC}(A \otimes_{\min} B) \cong (I \otimes_{\min} \mathrm{C}^*(B))_{\mathrm{s.a.}}$. Hence

$$\begin{aligned}\mathrm{JC}(A \otimes_{\min} B) &\cong (I \otimes_{\min} (J \oplus \Phi_B(J))_{\mathrm{s.a.}} \\ &\cong (I \otimes_{\min} J)_{\mathrm{s.a.}} \otimes (I \otimes_{\min} \Phi_B(J))_{\mathrm{s.a.}}.\end{aligned}$$

Thus the proof is complete, since $I \otimes_{\min} J$ and $I \otimes_{\min} \Phi_B(J)$ are simple C*-algebras. □

If $\mathbb{A}$ is a finite-dimensional C*-algebra and $\mathbb{B}$ is any C*-algebra, then $\mathbb{A} \otimes \mathbb{B}$ is complete relative to its unique C*-norm [6, 11.3.11]. Thus we have the following consequence of Proposition 6.

COROLLARY 7. *Let A be a simple* JC-*algebra that is not of the form V_{4n+1}. Then:*

(i) $\mathrm{JC}(A \otimes_{\min} V_{4n+1})$ *is the direct sum of two simple* JC-*algebras; and*
(ii) $\mathrm{JC}(V_{4n+1} \otimes_{\min} V_{4m+1})$ *is the direct sum of four simple* JC-*algebras.*

Proof. (i) Note that

$$\begin{aligned}\mathrm{JC}(A \otimes_{\min} V_{4n+1}) &\cong (\mathrm{R}^*(A) \otimes \mathrm{R}^*(V_{4n+1}))_{\mathrm{s.a.}} \\ &\cong (\mathrm{R}^*(A) \otimes \mathrm{R}^*(V_{4n}))_{\mathrm{s.a.}} \oplus (\mathrm{R}^*(A) \otimes \mathrm{R}^*(V_{4n}))_{\mathrm{s.a.}} \\ &\cong \mathrm{JC}(A \otimes_{\min} V_{4n}) \oplus \mathrm{JC}(A \otimes_{\min} V_{4n}).\end{aligned}$$

Since $\mathrm{R}^*(V_{4n})$ is not complex, $\mathrm{JC}(A \otimes_{\min} V_{4n})$ is a simple JC-algebra by Proposition 6(i), and (i) is proved.

(ii) It is easy to see that

$$\mathrm{JC}(V_{4n+1}\otimes_{\min}V_{4m+1})\cong\bigoplus_{1}^{4}\mathrm{JC}(V_{4n}\otimes_{\min}V_{4m}).$$

Since $\mathrm{JC}(V_{4n}\otimes_{\min}V_{4m})$ is simple, by Proposition 6(i), (ii) follows. □

In our following main theorem, S^n denotes a direct sum of n simple JC-algebras; V_∞ denotes an infinite-dimensional spin factor; $\cong$ s.a. C* denotes a simple JC-algebra isomorphic to the self-adjoint part of a C*-algebra; and U.R. $\not\cong$ s.a. C* denotes a simple universally reversible JC-algebra not isomorphic to the self-adjoint part of a C*-algebra.

THEOREM 8. *The entries in the following table denote the* JC-*tensor product* $\mathrm{JC}(A\otimes_{\min}B)$ *of a simple* JC-*algebra A in the first column and a simple* JC-*algebra B in the top row.*

	V_{2n}	V_{4n-1}	V_{4n+1}	V_∞	$\cong$ s.a. C*	U.R. $\not\cong$ s.a. C*	**R**
V_{2n}	S	S	S^2	S	S	S	S
V_{4n-1}	S	S^2	S^2	S	S^2	S	S
V_{4n+1}	S^2	S^2	S^4	S^2	S^2	S^2	S
V_∞	S	S	S^2	S	S	S	S
$\cong$ s.a. C*	S	S^2	S^2	S	S^2	S	S
U.R. $\not\cong$ s.a. C*	S	S	S^2	S	S	S	S
R	S	S	S	S	S	S	S

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