On Tensor Stable Operator Ideals

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A quasi-Banach ideal $[\mathfrak{A}, A]$ of operators is said to be tensor stable with respect to a tensor norm α if, for all $S \in \mathfrak{A}(E, G)$ and $T \in \mathfrak{A}(F, H)$,

$$S \tilde{\otimes}_\alpha T \in \mathfrak{A}(E \tilde{\otimes}_\alpha F, G \tilde{\otimes}_\alpha H);$$

in this case there is an $a \geq 1$ such that

$$A(S \tilde{\otimes}_\alpha T) \leq a A(S) A(T)$$

(in all concrete cases the constant a is obtained free of charge within the proof of tensor stability). Recently, Pietsch [15] has successfully exploited the tensor stability of certain Banach ideals to improve constants in the eigenvalue estimates of operators in them. In [2] tensor stability techniques were very useful to improve constants in the ideal norm estimates of certain operators. The earliest examples of tensor stable ideals with respect to ϵ are those of compact operators and of absolutely p-summing operators; these results are due to Vala [18] and Holub [6], respectively.

In Section 1 we prove that every nonproper ideal $[\mathfrak{A}, A]$ (i.e., there is at least one infinite-dimensional Banach space E such that the identity operator on E is in $\mathfrak{A}$) with

$$\sup_n A(\mathrm{id}_{\ell_\infty^n}) = \infty \quad (\text{resp., } \sup_n A(\mathrm{id}_{\ell_1^n}) = \infty)$$

cannot be tensor stable with respect to an injective (resp., a projective) tensor norm. (Recall that ϵ is injective and π is projective.) In Section 2 we completely identify which of the ideals $\mathfrak{L}_{p,q}$ of (p,q)-factorable operators as well as their injective, surjective, and minimal hulls are ϵ- or π-tensor stable. This is done by using some preliminary lemmas which are broadly of the type of permanence properties. The last section contains some applications.

Each section includes the relevant definitions; for notions and notations not explicitly defined here, we refer the reader to Pietsch [14].

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0. Preliminaries

First we recall the basic notion of Grothendieck's metric theory of tensor products (see, e.g., [3], [4], [5], [12]). A mapping which assigns to each pair (E,F) of Banach spaces a reasonable norm $\alpha(\cdot;E,F)$, $\epsilon\le\alpha\le\pi$, on the algebraic tensor product $E\otimes F$ is called a *tensor norm* if it satisfies the metric mapping property; namely, for $S\in\mathcal{L}(E,G)$ and $T\in\mathcal{L}(F,H)$,

$$\|S\otimes T: E\otimes_\alpha F\to G\otimes_\alpha H\|\le\|S\|\|T\|$$

(note that this definition differs from the original one given in [5]). The Banach space $E\tilde{\otimes}_\alpha F$ is obtained by completing $E\otimes_\alpha F$, and the continuous extension $S\tilde{\otimes}_\alpha T\in\mathcal{L}(E\tilde{\otimes}_\alpha F, G\tilde{\otimes}_\alpha H)$ of $S\otimes T$ is called the *α-tensor product* of the operators S and T. The most important examples are the injective tensor norm ϵ, which is the smallest, and the projective tensor norm π, which is the largest. For other basic examples we refer to [3], [4], or [12].

A quasi-Banach ideal $[\mathfrak{A},A]$ of operators is said to be *tensor stable* with respect to the tensor norm α if, for all $S\in\mathfrak{A}(E,G)$ and $T\in\mathfrak{A}(F,H)$, $S\tilde{\otimes}_\alpha T\in\mathfrak{A}(E\tilde{\otimes}_\alpha F, G\tilde{\otimes}_\alpha H)$. Then it is also easy to show that there exists a constant $a\ge1$ so that, for all S,T as above,

$$A(S\tilde{\otimes}_\alpha T)\le aA(S)A(T).$$

In this case we say $[\mathfrak{A},A]$ is α-tensor stable with constant a; if $a=1$ we merely say $[\mathfrak{A},A]$ is α-tensor stable.

The metric mapping property of tensor norms implies that $[\mathcal{L},\|\cdot\|]$ is α-tensor stable for each tensor norm α. The same holds also for the ideal $[\mathfrak{N},N]$ of nuclear operators and the ideal $[\mathfrak{I},I]$ of integral operators. The first result on the tensor products of compact operators goes back to Vala [18], who proved that $[\mathfrak{K},\|\cdot\|]$ is ϵ-tensor stable; the same holds for π, but it seems to be unknown whether this is valid for an arbitrary tensor norm. Holub ([6], [7]) proved the important result that the ideal $[\mathfrak{P}_p,P_p]$ of all absolutely p-summing operators is ϵ-tensor stable (but is not tensor stable with respect to π). Moreover, it can be easily seen that the ideal $[\mathcal{L}_2,L_2]$ of all Hilbertian operators is not tensor stable with respect to ϵ or π, since $\ell_2\tilde{\otimes}_\epsilon\ell_2$ and $\ell_2\tilde{\otimes}_\pi\ell_2$ are not reflexive. We finally remark that Pietsch [15] and König [9] have studied the tensor stability of quasi-Banach ideals defined by s-numbers.

Recall that a tensor norm α is said to be *injective* if, for all Banach spaces E and F and all subspaces G of E and H of F, the canonical embedding

$$J_G^E\otimes J_H^F: G\otimes_\alpha H\to E\otimes_\alpha F$$

is a metric injection; it is called *projective* if, for E,F,G,H as above, the canonical surjection

$$Q_G^E\otimes Q_H^F: E\otimes_\alpha F\to (E/G)\otimes_\alpha(F/H)$$

is a metric surjection; standard arguments show that ϵ is injective and π is projective.

If α is a tensor norm then the *dual tensor norm* α' for pairs (M,N) of finite-dimensional Banach spaces is defined by

$$\alpha'(z;M,N)=\sup\{|\langle z,u\rangle|: u\in M'\otimes N', \alpha(u;M',N')\le 1\},$$

and for pairs (E,F) of arbitrary Banach spaces by

$$\alpha'(z;E,F)=\inf \alpha'(z;M,N),$$

where the infimum is taken over all finite-dimensional subspaces M of E and N of F with $z\in M\otimes N$. It can be easily seen that $\epsilon'=\pi$ and $\pi'=\epsilon$. Moreover, the dual of an injective (resp., projective) tensor norm is projective (resp., injective); see for example [3].

1. Nonproper Tensor Stable Ideals

A quasi-Banach ideal $[\mathfrak{A},A]$ is called *nonproper* if Space($\mathfrak{A}$) contains an infinite-dimensional Banach space (a Banach space E belongs to Space($\mathfrak{A}$) if the identity on E is in $\mathfrak{A}$). In this section we prove some general results from which it follows that every nonproper maximal Banach ideal which is ϵ-tensor stable (resp., π-tensor stable) contains $\mathfrak{L}_\infty$ (resp., $\mathfrak{L}_1$). The proofs are based on essentially finite-dimensional arguments and therefore also are relevant in providing a partial answer to a question of Pelczyński ([13, p. 136]).

A Banach space E is said to be *finitely representable* in F, written briefly as $E \overset{\text{f.r.}}{\hookrightarrow} F$, if for each $\epsilon>0$ and each finite-dimensional subspace M of E there is a finite-dimensional subspace N of F and a bijective operator $T\in\mathfrak{L}(M,N)$ such that $\|T\|\,\|T^{-1}\|\le 1+\epsilon$. Every Banach space is finitely representable in ℓ_∞. Dvoretzky's theorem states that ℓ_2 is finitely representable in every infinite-dimensional Banach space.

If $\mathcal{P}$ is a property defined for Banach spaces (i.e., if $\mathcal{P}$ is a subclass of the class of all Banach spaces) then a Banach space E has property "super $\mathcal{P}$" if every Banach space finitely representable in E also has $\mathcal{P}$. A property $\mathcal{P}$ is called a *super property* if $\mathcal{P}=$ super $\mathcal{P}$. For more information on this important notion we refer to [16].

We start with the following.

1.1. PROPOSITION. *Let α be an injective tensor norm and $\mathcal{P}$ a property of Banach spaces which is nontrivial (i.e., there is at least one Banach space without $\mathcal{P}$). Then the α-tensor product $E\tilde{\otimes}_\alpha F$ of two infinite-dimensional Banach spaces E and F cannot have property* super $\mathcal{P}$.

Proof. Let E and F be two infinite-dimensional Banach spaces. We prove that

$$\ell_\infty \overset{\text{f.r.}}{\hookrightarrow} E\tilde{\otimes}_\alpha F,$$

since then $E\tilde{\otimes}_\alpha F$ cannot have super $\mathcal{P}$ (otherwise every Banach space would have $\mathcal{P}$).

We will need Grothendieck's fundamental theorem of the metric theory of tensor products (see, e.g., [4] or [5]). There is a universal constant $K_G>0$ such that, for arbitrary Banach spaces G and H,

$$w_2(\cdot;G,H) \le /\pi\backslash(\cdot;G,H) \le K_G w_2(\cdot;G,H);$$

here $/\pi\backslash$ denotes the largest injective tensor norm and w_2 the tensor norm defined by

$$w_2(z;G,H) := \inf\left[\sup_{\|x'\|\le 1}\left(\sum_{i=1}^n |\langle x_i,x'\rangle|^2\right)^{1/2} \sup_{\|y'\|\le 1}\left(\sum_{i=1}^n |\langle y_i,y'\rangle|^2\right)^{1/2}\right],$$

where the infimum is taken over all finite representations $z=\sum_{i=1}^n x_i\otimes y_i$.

First we prove that $E\tilde{\otimes}_\alpha F$ contains all spaces ℓ_∞^n $(1+\epsilon)K_G$-uniformly; that is, for all n and ϵ there is a finite-dimensional subspace G of $E\tilde{\otimes}_\alpha F$ and a bijection $R:\ell_\infty^n \to G$ such that $\|R\|\|R^{-1}\| \le (1+\epsilon)K_G$. This then completes the proof, since by a result of James [8] a Banach space contains all ℓ_∞^n λ-uniformly for all $\lambda>1$ if it does this for some $\lambda>1$, and by the fact that ℓ_∞ is an $\mathcal{L}_{\infty,\lambda}$-space for all $\lambda>1$. Fix $n\in\mathbf{N}$ and $\epsilon>0$. Define the metric injection

$$\mathfrak{J}:\ell_\infty^n \to \ell_2^n\otimes_\epsilon \ell_2^n$$

$$\xi \to \sum_{i=1}^n \xi_i e_i\otimes e_i.$$

By Dvoretzky's theorem we can find a finite-dimensional subspace M of E and a bijection $S:\ell_2^n\to M$ such that $\|S\|\le 1$ and $\|S^{-1}\|\le 1+\epsilon$, and similarly a finite-dimensional subspace N of F and a bijection $T:\ell_2^n\to N$ with $\|T\|\le 1$ and $\|T^{-1}\|\le 1+\epsilon$. Now consider the subspace $G:=(S\tilde{\otimes}_\alpha T)\mathfrak{J}(\ell_\infty^n)$ of $E\tilde{\otimes}_\alpha F$ and the bijection

$$R:\ell_\infty^n \to G$$

$$\xi \to (S\tilde{\otimes}_\alpha T)\mathfrak{J}(\xi).$$

Because α is injective, $M\tilde{\otimes}_\alpha N$ is a subspace of $E\tilde{\otimes}_\alpha F$ and hence $\|R^{-1}\|\le (1+\epsilon)^2$. Moreover, by an easy calculation we have

$$w_2(\cdot;\ell_2^n,\ell_2^n) = \epsilon(\cdot;\ell_2^n,\ell_2^n),$$

so that by the injectivity of α and Grothendieck's theorem

$$\alpha(\cdot;\ell_2^n,\ell_2^n) \le /\pi\backslash(\cdot;\ell_2^n,\ell_2^n) \le K_G\epsilon(\cdot;\ell_2^n,\ell_2^n).$$

This proves $\|R\|\le K_G$. □

REMARK. Without the injectivity of α the above result need not hold; for example, the g_2-tensor product of two Hilbert spaces has the super property of being a Hilbert space (for the definition of g_2 see [3] or [4]); also, the above proposition does not hold for $\alpha=\pi$ since, for example, by [17] the Schatten–von Neumann class $\mathfrak{A}(\ell_2)=\ell_2\tilde{\otimes}_\pi \ell_2$ has the super property cotype 2.

In order to apply the preceding result to our investigation of tensor stable Banach ideals we need the following lemma.

1.2. LEMMA. *Let* $[\mathfrak{A}, A]$ *be a maximal quasi-Banach ideal of operators. Then the following are equivalent:*

(i) $\mathrm{Space}(\mathfrak{A}^{\mathrm{inj}})$ *is a nontrivial super property*;
(ii) $\ell_\infty \notin \mathrm{Space}(\mathfrak{A})$.

Proof. Assume (i) holds and that $\ell_\infty \in \mathrm{Space}(\mathfrak{A})$. Then the maximality of $\mathfrak{A}$ implies that $\mathfrak{L}_\infty \subseteq \mathfrak{A}$ and hence $\mathfrak{L} = \mathfrak{L}_\infty^{\mathrm{inj}} = \mathfrak{A}^{\mathrm{inj}}$, which is a contradiction.

Conversely, assume $\ell_\infty \notin \mathrm{Space}(\mathfrak{A})$. Let $F \in \mathrm{Space}(\mathfrak{A}^{\mathrm{inj}})$ and $E \overset{\text{f.r.}}{\hookrightarrow} F$. Let M be a finite-dimensional subspace of E. Then there is a finite-dimensional subspace N of F and a bijection $T \in \mathfrak{L}(M,N)$ with $\|T\|\|T^{-1}\| \leq 1+\epsilon$. Hence

$$\begin{aligned} A^{\mathrm{inj}}(J_M^E) &= A^{\mathrm{inj}}(\mathrm{id}_M) = A^{\mathrm{inj}}(T^{-1}\,\mathrm{id}_N\, T) \\ &\leq (1+\epsilon)A^{\mathrm{inj}}(\mathrm{id}_N) \leq (1+\epsilon)A^{\mathrm{inj}}(\mathrm{id}_F). \end{aligned}$$

This implies $E \in \mathrm{Space}(\mathfrak{A}^{\mathrm{inj}})$ since $[\mathfrak{A}^{\mathrm{inj}}, A^{\mathrm{inj}}]$ is maximal. It only remains to check that $\mathfrak{A}^{\mathrm{inj}} \neq \mathfrak{L}$, but this is true since ℓ_∞, as a complemented subspace of $\ell_\infty^{\mathrm{inj}}$, cannot be contained in $\mathrm{Space}(\mathfrak{A}^{\mathrm{inj}})$. □

We are now ready to make some general statements on tensor stability of Banach operator ideals.

1.3. THEOREM. *Let* $[\mathfrak{A}, A]$ *be a quasi-Banach ideal of operators and* α *a tensor norm. If either*

(i) α *is injective and* $\sup A(\mathrm{id}_{\ell_\infty^n}) = \infty$

or

(ii) α *is projective and* $\sup A(\mathrm{id}_{\ell_1^n}) = \infty$,

then $E\tilde{\otimes}_\alpha F \notin \mathrm{Space}(\mathfrak{A})$ *for all infinite-dimensional* E *and* $F \in \mathrm{Space}(\mathfrak{A})$. *In particular, every nonproper Banach ideal* $[\mathfrak{A}, A]$ *cannot be* α*-tensor stable in each of the above two cases.*

Proof. (i) Since $\ell_\infty \notin \mathrm{Space}(\mathfrak{A}^{\max})$, we have from the preceding lemma that the property $\mathrm{Space}(\mathfrak{A}^{\max\,\mathrm{inj}})$ is a nontrivial super property. Now the conclusion on $E\tilde{\otimes}_\alpha F$ follows from 1.1, since $\mathrm{Space}(\mathfrak{A}) \subseteq \mathrm{Space}(\mathfrak{A}^{\max\,\mathrm{inj}})$.

(ii) If α is projective and $\ell_1 \notin \mathrm{Space}(\mathfrak{A}^{\max})$, then α' is injective and

$$\ell_\infty \notin \mathrm{Space}(\mathfrak{A}^{\max\,\mathrm{dual}}) = \mathrm{Space}(\mathfrak{A}^{\mathrm{dual}\,\max}).$$

Note that an operator T belongs to a maximal ideal $\mathfrak{B}$ if and only if this holds for its bi-adjoint. Assume that there are infinite-dimensional Banach spaces E and F with $E\tilde{\otimes}_\alpha F \in \mathrm{Space}(\mathfrak{A}) \subseteq \mathrm{Space}(\mathfrak{A}^{\max})$. Since α is projective, the canonical embedding $E'\tilde{\otimes}_{\alpha'} F' \hookrightarrow (E\tilde{\otimes}_\alpha F)'$ is a metric injection (see, e.g., [3, 3.4 and 9.1]), and $(E\tilde{\otimes}_\alpha F)' \in \mathrm{Space}(\mathfrak{A}^{\mathrm{dual}\,\max})$, we have $E'\tilde{\otimes}_{\alpha'} F' \in \mathrm{Space}(\mathfrak{A}^{\mathrm{dual}\,\max\,\mathrm{inj}})$. Again, by 1.1 and 1.2 this is a contradiction. □

As an immediate consequence we have the following.

1.4. COROLLARY. *Let* $[\mathfrak{A}, A]$ *be maximal and nonproper. If* $[\mathfrak{A}, A]$ *is tensor stable with respect to an injective* (*resp., a projective*) *tensor norm* α, *then* $\mathfrak{L}_\infty \subseteq \mathfrak{A}$ (*resp.,* $\mathfrak{L}_1 \subseteq \mathfrak{A}$). *In particular, if* α *is an injective* (*resp., a projective*) *tensor norm then* $[\mathfrak{L}, \|\cdot\|]$ *is the only* α*-tensor stable ideal which is injective* (*resp., surjective*), *maximal, and nonproper.*

This last remark follows from $\mathfrak{L}_\infty^{\text{inj}} = \mathfrak{L}$ and $\mathfrak{L}_1^{\text{sur}} = \mathfrak{L}$.

1.5. COROLLARY. *For a maximal quasi-Banach ideal* $[\mathfrak{A}, A]$ *the following are equivalent:*

(1) $\ell_\infty \notin \text{Space}(\mathfrak{A})$ (*resp.*, $\ell_1 \notin \text{Space}(\mathfrak{A})$);

(2) $E \tilde{\otimes}_\epsilon F \notin \text{Space}(\mathfrak{A})$ (*resp.*, $E \tilde{\otimes}_\pi F \notin \text{Space}(\mathfrak{A})$) *for all infinite-dimensional* $E, F \in \text{Space}(\mathfrak{A})$.

Proof. Obviously the theorem shows that (1) implies (2). Conversely, if $\ell_\infty \in \text{Space}(\mathfrak{A})$ then $\mathfrak{L}_\infty \subseteq \mathfrak{A}$, and hence in particular

$$\ell_\infty \tilde{\otimes}_\epsilon \ell_\infty \in \text{Space}(\mathfrak{A}),$$

but this contradicts (2). The argument for π is analogous. □

2. Tensor Stability of the Ideals $\mathfrak{L}_{p,q}$ and $\mathfrak{K}_{p,q}$

We shall in this section completely identify which of the Banach ideals $[\mathfrak{L}_{p,q}, L_{p,q}]$ and $[\mathfrak{K}_{p,q}, K_{p,q}]$, as well as their injective and surjective hulls, are ϵ- or π-tensor stable. The ideal $\mathfrak{L}_{p,q}$ of (p,q)-factorable operators is related to the ideal $\mathfrak{K}_{p,q}$ of (p,q)-compact operators by $\mathfrak{L}_{p,q}^{\min} = \mathfrak{K}_{p,q}$ and $\mathfrak{K}_{p,q}^{\max} = \mathfrak{L}_{p,q}$. Moreover, $\mathfrak{I}_p := \mathfrak{L}_{p,1}$ is the class of all p-integral operators, $\mathfrak{L}_p := \mathfrak{L}_{p,p'}$ the class of all p-factorable operators, $\mathfrak{N}_p := \mathfrak{K}_{p,1}$ the class of all p-nuclear operators, and $\mathfrak{K}_p := \mathfrak{K}_{p,p'}$ the class of all p-compact operators. We recall that $\mathfrak{L}_{p,q}^{\text{dual}} = \mathfrak{L}_{q,p}$ and $\mathfrak{I}_p^{\text{inj}} = \mathfrak{P}_p$. For definitions and various properties and relationships with other ideals, the reader is referred to Pietsch [14], especially Chapters 18 and 19.

2.1. THEOREM. *Let* $1 \le p, q \le \infty$ *and* $1/p + 1/q \ge 1$. *The following tables characterize which of the Banach ideals on the first column of each of the tables are* ϵ- *or* π-*tensor stable, respectively.*

(1)

	ϵ	π
$\mathfrak{L}_{p,q}$	$\mathfrak{I}_p$	$\mathfrak{I}_p^{\text{dual}}$
$\mathfrak{L}_{p,q}^{\text{inj}}$	$\mathfrak{P}_p$	$\mathfrak{L}$
$\mathfrak{L}_{p,q}^{\text{sur}}$	$\mathfrak{L}$	$\mathfrak{P}_p^{\text{dual}}$

(2)

	ϵ	π
$\mathfrak{K}_{p,q}$	$\mathfrak{N}_p$	$\mathfrak{K}_{1,p}$
$\mathfrak{K}_{p,q}^{\text{inj}}$	$\mathfrak{N}_p^{\text{inj}}$	$\mathfrak{K}$
$\mathfrak{K}_{p,q}^{\text{sur}}$	$\mathfrak{K}$	$\mathfrak{K}_{1,p}^{\text{sur}}$

[For example, $\mathfrak{L}_{p,1} = \mathfrak{I}_p$ is ϵ-tensor stable (with constant 1) and $\mathfrak{L}_{p,q}$, for $q \ne 1$, is not tensor stable with respect to ϵ.]

Before giving a proof of this statement, we shall collect together some elementary basic results on how far tensor stability of an ideal with respect to a tensor norm is inherited by the ideals $\mathfrak{A}^{\text{sur}}$, $\mathfrak{A}^{\text{inj}}$, $\mathfrak{A}^{\max}$, $\mathfrak{A}^{\min}$, and $\mathfrak{A}^{\text{dual}}$.

2.2. LEMMA. *Let* $[\mathfrak{A}, A]$ *be a quasi-Banach ideal which is* α-*tensor stable with constant* a. *Then so are*

(1) $[\mathfrak{A}^{\text{inj}}, A^{\text{inj}}]$, *if* α *is injective*;

(2) $[\mathfrak{A}^{\text{sur}}, A^{\text{sur}}]$, *if* α *is projective*.

Proof. (2) Let $S \in \mathfrak{A}^{sur}(E,G)$ and $T \in \mathfrak{A}^{sur}(F,H)$. Then

$$(S\tilde{\otimes}_\alpha T)(Q_E\tilde{\otimes}_\alpha Q_F) = SQ_E\tilde{\otimes}_\alpha TQ_F \in \mathfrak{A}(E^{sur}\tilde{\otimes}_\alpha F^{sur}, G\tilde{\otimes}_\alpha H)$$

and $A((S\tilde{\otimes}_\alpha T)(Q_E\tilde{\otimes}_\alpha Q_F)) \le aA(SQ_E)A(TQ_F)$. Since $(E\tilde{\otimes}_\alpha F)^{sur}$ has the metric lifting property and $Q_E\tilde{\otimes}_\alpha Q_F : E^{sur}\tilde{\otimes}_\alpha F^{sur} \to E\tilde{\otimes}_\alpha F$ is a metric surjection, there is $R \in \mathfrak{L}((E\tilde{\otimes}_\alpha F)^{sur}, E^{sur}\tilde{\otimes}_\alpha F^{sur})$ such that $\|R\| \le 1+\epsilon$ and $Q_{E\tilde{\otimes}_\alpha F} = (Q_E\tilde{\otimes}_\alpha Q_F)R$. Hence $S\tilde{\otimes}_\alpha T \in \mathfrak{A}^{sur}(E\tilde{\otimes}_\alpha F, G\tilde{\otimes}_\alpha H)$ and

$$\begin{aligned} A^{sur}(S\tilde{\otimes}_\alpha T) &= A((S\tilde{\otimes}_\alpha T)Q_{E\tilde{\otimes}_\alpha F}) \\ &\le (1+\epsilon)aA^{sur}(S)A^{sur}(T). \end{aligned}$$

The proof of (1) is similar. □

REMARK. By Holub [6], the ideal $[\mathfrak{N}, N]$ of nuclear operators is π-tensor stable while its injective hull (all quasi-nuclear operators) is not. As mentioned earlier, $[\mathfrak{I}, I]$ and hence also $[\mathfrak{I}^{inj}, I^{inj}]$ are ϵ-tensor stable; since $\mathfrak{L}_2 = (\mathfrak{I}^{inj})^{sur}$ (by Grothendieck's theorem every operator from a Banach space $\ell_1(\Gamma)$ into a Hilbert space is absolutely summing [14, 22.4.4] and hence $\mathfrak{L}_2 = \mathfrak{L}_2^{sur} \subseteq \mathfrak{P}^{sur} = (\mathfrak{I}^{inj})^{sur} \subseteq \mathfrak{L}_2$), we see that the surjective hull of an ϵ-tensor stable Banach ideal need not be tensor stable with respect to ϵ.

We call a quasi-Banach ideal $[\mathfrak{A}, A]$ *right accessible* if for each $\epsilon > 0$ and each $T \in \mathfrak{L}(M,F)$, where M is a finite-dimensional Banach space and F is an arbitrary Banach space, there is a finite-dimensional subspace N of F and $T_1 \in \mathfrak{L}(M,N)$ such that $T = J_N^F T_1$ and $A(T_1) \le (1+\epsilon)A(T)$. All Banach ideals mentioned in 2.1(1) and 2.1(2) have this property (see [3] or [4]). We remark that no ideal without this property is known.

2.3. LEMMA. *Let $[\mathfrak{A}, A]$ be a Banach ideal and α a tensor norm.*

(1) *If $[\mathfrak{A}, A]$ is α-tensor stable with constant a, then so is $[\mathfrak{A}^{min}, A^{min}]$.*

(2) *Assume that α is injective and that there is $a \ge 1$ such that*

$$A(S\tilde{\otimes}_\alpha T) \le aA(S)A(T)$$

for all operators S, T between finite-dimensional Banach spaces. Then $[\mathfrak{A}^{max}, A^{max}]$ is α-tensor stable with constant a, provided it is right accessible.

Proof. (1) By definition $[\mathfrak{A}^{min}, A^{min}] = [\mathfrak{G}, \|\cdot\|] \circ [\mathfrak{A}, A] \circ [\mathfrak{G}, \|\cdot\|]$, where $[\mathfrak{G}, \|\cdot\|]$ denotes the ideal of all approximable operators. Obviously the product of an α-tensor stable ideal with constant a by an α-tensor stable ideal with constant b is α-tensor stable with constant ab. This implies (1), since an easy argument using the metric mapping property of α shows that $[\mathfrak{G}, \|\cdot\|]$ is α-tensor stable.

(2) Let $S \in \mathfrak{A}^{max}(E,G)$ and $T \in \mathfrak{A}^{max}(F,H)$. In order to prove that $S\tilde{\otimes}_\alpha T \in \mathfrak{A}$, it suffices to check that

$$A((S\tilde{\otimes}_\alpha T)(J_M^E\tilde{\otimes}_\alpha J_N^F)) \le aA^{max}(S)A^{max}(T)$$

for all finite-dimensional subspaces M of E and N of F; this follows by an argument based on the principle of local reflexivity, since the union of all subspaces $M\tilde{\otimes}_\alpha N$ of $E\tilde{\otimes}_\alpha F$ is dense (cf. [12, p. 38]). For M, N as above, by the right accessibility of $[\mathfrak{A}^{\max}, A^{\max}]$ we have M_1, S_1, N_1, T_1 such that $SJ_M^E = J_{M_1}^E S_1$ and $TJ_N^F = J_{N_1}^F T_1$, with

$$A(S_1) \le (1+\epsilon)A^{\max}(S) \quad \text{and} \quad A(T_1) \le (1+\epsilon)A^{\max}(T).$$

Now the desired inequality follows from

$$\begin{aligned} A((S\tilde{\otimes}_\alpha T)(J_M^E \tilde{\otimes}_\alpha J_N^F)) &\le A(J_{M_1}^E \tilde{\otimes}_\alpha J_{N_1}^F)(S_1 \tilde{\otimes}_\alpha T_1)) \\ &\le aA(S_1)A(T_1) \\ &\le a(1+\epsilon)^2 A^{\max}(S)A^{\max}(T). \qquad \square \end{aligned}$$

As an easy consequence we have the following.

2.4. LEMMA. *Let α be a projective tensor norm and $[\mathfrak{A}, A]$ a Banach ideal that is α-tensor stable with constant a. Then $[\mathfrak{A}^{\max\,\mathrm{dual}}, A^{\max\,\mathrm{dual}}]$ is α'-tensor stable with constant a provided it is right accessible.*

Proof. This follows by $[\mathfrak{A}^{\max\,\mathrm{dual}}, A^{\max\,\mathrm{dual}}] = [\mathfrak{A}^{\mathrm{dual}\,\max}, A^{\mathrm{dual}\,\max}]$ and part (2) of the preceding lemma because, for operators S, T between finite-dimensional spaces,

$$\begin{aligned} A^{\mathrm{dual}}(S\tilde{\otimes}_{\alpha'} T) &= A((S\tilde{\otimes}_{\alpha'} T)') \\ &= A(S'\tilde{\otimes}_\alpha T') \le aA^{\mathrm{dual}}(S)A^{\mathrm{dual}}(T). \qquad \square \end{aligned}$$

We are now ready to give the following.

Proof of 2.1 *Theorem.* We start with the table (1).

(A) First, we consider the column under π and prove the positive results for $\mathfrak{L}_{p,q}$.

(A1) In the case $p=1$, $q=1$, Holub [6] has proved that $\mathfrak{L}_{1,1} = \mathfrak{I}$ is π-tensor stable. For $p=1$, $1<q<\infty$ and $S \in \mathfrak{L}_{1,q}(E,G)$, $T \in \mathfrak{L}_{1,q}(F,H)$, we obtain the following commuting diagram, using the "if" part of the factorization theorem for $\mathfrak{L}_{p,q}$ ([14, 19.4.4]):

$$\begin{array}{ccccc} E\tilde{\otimes}_\pi F & \xrightarrow{S\tilde{\otimes}_\pi T} & G\tilde{\otimes}_\pi H & \xhookrightarrow{K_G\tilde{\otimes}_\pi K_H} & G''\tilde{\otimes}_\pi H'' \\ \searrow U\tilde{\otimes}_\pi V & & & & \nearrow X\tilde{\otimes}_\pi Y \\ L_{q'}(\mu)\tilde{\otimes}_\pi L_{q'}(\nu) & & \xrightarrow{I_\mu \tilde{\otimes}_\pi I_\nu} & & L_1(\mu)\tilde{\otimes}_\pi L_1(\nu), \end{array}$$

where $\|U\|\|X\| \le (1+\epsilon)L_{1,q}(S)$ and $\|V\|\|Y\| \le (1+\epsilon)L_{1,q}(T)$; here the maps I_μ and I_ν are the canonical embeddings. Moreover, it can be seen easily that $I_\mu \tilde{\otimes}_\pi I_\nu$ factorizes in a canonical way through

$$I_{\mu\times\nu}: L_{q'}(\mu\times\nu) \to L_1(\mu\times\nu),$$

and that there is a metric injection $R \in \mathfrak{L}(G''\tilde{\otimes}_\pi H'', (G\tilde{\otimes}_\pi H)'')$ such that

$$K_{G\tilde{\otimes}_\pi H} = R(K_G \tilde{\otimes}_\pi K_H)$$

(because each $\varphi \in (G \tilde{\otimes}_\pi H)'$ has a canonical extension $\hat{\varphi} \in (G \tilde{\otimes}_\pi H'')'$ with $\|\varphi\| = \|\hat{\varphi}\|$). Hence the "only if" part of the above-used factorization theorem shows that $S \tilde{\otimes}_\pi T \in \mathfrak{L}_{1,q}$ and $L_{1,q}(S \tilde{\otimes}_\pi T) \le (1+\epsilon)^2 L_{1,q}(S) L_{1,q}(T)$.

The remaining case $p=1$, $q=\infty$, follows in a manner similar to the one above, by use of the factorization theorem ([14, 19.3.7]).

(A2) Of course, $\mathfrak{L}_\infty^{\mathrm{inj}} = \mathfrak{L}$ is π-tensor stable.

(A3) Since $\mathfrak{I}_p^{\mathrm{dual\,sur}} = \mathfrak{I}_p^{\mathrm{inj\,dual}} = \mathfrak{P}_p^{\mathrm{dual}}$, the positive results for $\mathfrak{L}_{p,q}^{\mathrm{sur}}$ follow from 2.2(2) and the result proved in (A1).

(B) Next, we observe that the positive results concerning the tensor stability of $\mathfrak{L}_{p,q}$ with respect to ϵ follow by 2.4 and the results of (A), since

$$\mathfrak{L}_{p,q}^{\mathrm{dual}} = \mathfrak{L}_{q,p}, \qquad \mathfrak{L}_{p,q}^{\mathrm{inj\,dual}} = \mathfrak{L}_{q,p}^{\mathrm{sur}}, \qquad \mathfrak{L}_{p,q}^{\mathrm{sur\,dual}} = \mathfrak{L}_{q,p}^{\mathrm{inj}}.$$

(C) Now we prove the negative results for the column under ϵ.

(C1) The negative results on the tensor stability of $\mathfrak{L}_{p,q}$ with respect to ϵ follow by 2.2(1) and (C2).

(C2) For $1 < p, q < \infty$ the factorization theorem mentioned in (A1) shows that $\mathfrak{L}_{p,q}^{\mathrm{inj}}$ is contained in the ideal $\mathfrak{W}$ of all weakly compact operators. Moreover, by [14, 19.4.4], $\ell_2 \in \mathrm{Space}(\mathfrak{L}_{p,q})$. Since $\ell_2 \tilde{\otimes}_\epsilon \ell_2$ is not reflexive, none of the ideals $\mathfrak{L}_{p,q}^{\mathrm{inj}}$ $(1 < p, q < \infty)$ can be tensor stable with respect to ϵ. Let $p=1$ and $1 < q < \infty$. Then

$$Q_{\ell_2} \in \mathfrak{P}_1(\ell_2^{\mathrm{sur}}, \ell_2) \subseteq \mathfrak{I}_2(\ell_2^{\mathrm{sur}}, \ell_2) \subseteq \mathfrak{I}_q(\ell_2^{\mathrm{sur}}, \ell_2)$$

(see [14, 22.4.4 and 22.4.2]) and this proves

$$\mathrm{id}_{\ell_2} \in \mathfrak{I}_q^{\mathrm{sur}} = \mathfrak{L}_{1,q}^{\mathrm{inj\,dual}},$$

hence $\mathrm{id}_{\ell_2} \in \mathfrak{L}_{1,q}^{\mathrm{inj}}$. Again $\mathfrak{L}_{1,q}^{\mathrm{inj}}$ cannot be tensor stable with respect to ϵ, since $\mathfrak{L}_{1,q}^{\mathrm{inj}} \subseteq \mathfrak{W}$. The remaining case $\mathfrak{L}_{1,\infty}^{\mathrm{inj}} = \mathfrak{L}_1^{\mathrm{inj}}$ follows, for example, by 1.4 Corollary.

(C3) Assume that $\mathfrak{L}_{p,q}^{\mathrm{sur}}$ is tensor stable with respect to ϵ. Then, by 2.2(1), $\mathfrak{L}_{p,q}^{\mathrm{sur\,inj}}$ is tensor stable with respect to ϵ. Since

$$\mathfrak{L}_2 = (\mathfrak{I}^{\mathrm{sur}})^{\mathrm{inj}} \subseteq (\mathfrak{L}_{p,q}^{\mathrm{sur}})^{\mathrm{inj}} \quad \text{for all } p, q$$

(see the remark after 2.2) and the latter ideal is contained in $\mathfrak{W}$ if $1 < p < \infty$ or $1 < q < \infty$, it remains to check the case $(p,q) = (\infty, 1)$ (use the argument of (C2)). In view of 2.4 we show that $\mathfrak{L}_1^{\mathrm{inj}}$ is not tensor stable with respect to π (note that $\mathfrak{L}_1^{\mathrm{inj}} = \mathfrak{L}_\infty^{\mathrm{sur\,dual}}$). But this, for example, follows from [11] where it is proved that $\ell_2 \tilde{\otimes}_\pi \ell_2$ cannot be a subspace of some Banach space $L_1(\mu)$.

(D) The negative results concerning the column under π follow, as in (B), by 2.4 and the results shown in (C). Hence the proof of table (1) is complete. Next we consider the table for $\mathfrak{K}_{p,q}$ and first establish the positive results.

(E) By 2.3(1) Lemma, $\mathfrak{N}_p = \mathfrak{I}_p^{\min}$ is ϵ-tensor stable and $\mathfrak{K}_{1,p} = \mathfrak{I}_p^{\mathrm{dual\,min}}$ π-tensor stable; it then follows from 2.2 that $\mathfrak{N}_p^{\mathrm{inj}}$ is ϵ-tensor stable and $\mathfrak{K}_{1,p}^{\mathrm{sur}}$ is π-tensor stable. As mentioned, $\mathfrak{K}$ is both ϵ- and π-tensor stable.

(F) Finally, we shall prove the negative results. The relationships

$$\mathfrak{K}_{p,q}^{\max} = \mathfrak{L}_{p,q}, \qquad \mathfrak{K}_{p,q}^{\mathrm{inj\,max}} = \mathfrak{L}_{p,q}^{\mathrm{inj}}, \qquad \mathfrak{K}_{p,q}^{\mathrm{sur\,max}} = \mathfrak{L}_{p,q}^{\mathrm{sur}}$$

and 2.3(2) yield the negative results for the column under ϵ for $\mathcal{K}_{p,q}$ from those of $\mathcal{L}_{p,q}$ et cetera. Since

$$\mathcal{K}_{p,q}^{\text{max dual}} = \mathcal{L}_{q,p}, \qquad \mathcal{K}_{p,q}^{\text{inj max dual}} = \mathcal{L}_{q,p}^{\text{sur}}, \qquad \mathcal{K}_{p,q}^{\text{sur max dual}} = \mathcal{L}_{q,p}^{\text{inj}},$$

the negative results for the column under π follow, by 2.4, from the negative results for the column under ϵ for $\mathcal{L}_{p,q}$ et cetera. This completes the proof. □

We remark that 2.1 includes several already-mentioned results of [6], [7], and [18].

3. Applications

We start with the following lemma.

3.1. LEMMA. *Let $[\mathfrak{A}, A]$ be an α-tensor stable Banach ideal with constant a. Then, for all $S \in \mathfrak{A}^*(E,G)$ and $T \in \mathfrak{A}^*(F,H)$ with $S \tilde{\otimes}_\alpha T \in \mathfrak{A}^*$:*

$$A^*(S)A^*(T) \le aA^*(S \tilde{\otimes}_\alpha T),$$

where $[\mathfrak{A}^*, A^*]$ denotes the adjoint ideal of $[\mathfrak{A}, A]$.

Proof. Let S, T be as above. For subspaces M_1 of G, M_2 of H, N_1 of E, and N_2 of F (all of finite co-dimension), and for all operators $L_1 \in \mathcal{L}(G/M_1, N_1)$, $L_2 \in \mathcal{L}(H/M_2, N_2)$ with $A(L_1) \le 1$, $A(L_2) \le 1$, one gets

$$\begin{aligned} A^*(S \tilde{\otimes}_\alpha T) &\ge \frac{1}{a} |\operatorname{trace}((J_{N_1}^E \tilde{\otimes}_\alpha J_{N_2}^F)(L_1 \tilde{\otimes}_\alpha L_2)(Q_{M_1}^G \tilde{\otimes}_\alpha Q_{M_2}^H)(S \tilde{\otimes}_\alpha T))| \\ &\ge \frac{1}{a} |\operatorname{trace}(J_{N_1}^E L_1 Q_{M_1}^G S \tilde{\otimes}_\alpha J_{N_2}^F L_2 Q_{M_2}^H T)| \\ &= \frac{1}{a} |\operatorname{trace}(J_{N_1}^E L_1 Q_{M_1}^G S)| \, |\operatorname{trace}(J_{N_2}^F L_2 Q_{M_2}^H T)|, \end{aligned}$$

and therefore

$$A^*(S \tilde{\otimes}_\alpha T) \ge \frac{1}{a} A^*(S) A^*(T).$$ □

3.2. COROLLARY. *Let $[\mathfrak{A}, A]$ be an α-tensor stable maximal Banach ideal such that its adjoint is α-tensor stable. Then, for all $S, T \in \mathfrak{A}$,*

$$A(S \tilde{\otimes}_\alpha T) = A(S)A(T).$$

For a proof apply the lemma to $[\mathfrak{A}^*, A^*]$ and recall that $[\mathfrak{A}^{**}, A^{**}] = [\mathfrak{A}, A]$.

By [14, 19.2.13],

$$[\mathcal{P}_r, P_r] = [\mathcal{I}_{r'}^*, I_{r'}^*], \qquad [\mathcal{I}_r, I_r] = [\mathcal{P}_{r'}^*, P_{r'}^*]$$

for all $1 \le r \le \infty$. Hence it follows from 2.1 that

$$P_r(S \tilde{\otimes}_\epsilon T) = P_r(S)P_r(T) \quad \text{for all } S, T \in \mathcal{P}_r;$$

$$I_r(S \tilde{\otimes}_\epsilon T) = I_r(S)I_r(T) \quad \text{for all } S, T \in \mathcal{I}_r.$$

If $P(E)$ denotes the projection constant of a finite-dimensional Banach space E then, for finite-dimensional E and F, the equation $P(E\tilde{\otimes}_\epsilon F) = P(E)P(F)$ is a well-known fact of the geometry of Banach spaces (see, e.g., [13, p. 135]). Since $P(E)$ indeed equals $I_\infty(\mathrm{id}_E)$ (cf. [14, 28.2.5]), the equality $I_r(S\tilde{\otimes}_\epsilon T) = I_r(S)I_r(T)$ established above for $S, T \in \mathfrak{I}_r$ can be considered as an extension of the equation on projection constants.

A second application will be to extend a result of Vala [18] to the ideal $\mathcal{P}_p$. Vala proved that, for two compact operators $S \in \mathcal{K}(E,F)$ and $T \in \mathcal{K}(G,H)$, the operator

$$\begin{aligned}\mathrm{Hom}(S,T)\colon \mathcal{L}(F,G) &\to \mathcal{L}(E,H)\\ U &\to TUS\end{aligned}$$

is again compact. For absolutely p-summing operators we have an analogous result.

3.3. PROPOSITION. *Let* $S \in \mathcal{P}_p^{\mathrm{dual}}(E,F)$ *and* $T \in \mathcal{P}_p(G,H)$. *Then*

$\mathrm{Hom}(S,T) \in \mathcal{P}_p(\mathcal{L}(F,G), \mathcal{L}(E,H))$ *and* $P_p(\mathrm{Hom}(S,T)) = P_p^{\mathrm{dual}}(S)P_p(T)$.

Proof. Since $[\mathcal{P}_p^{\mathrm{dual\,dual}}, P_p^{\mathrm{dual\,dual}}] = [\mathcal{P}_p, P_p]$, we have

$$T' \in \mathcal{P}_p^{\mathrm{dual}} \text{ and } P_p^{\mathrm{dual}}(T') = P_p(T).$$

Hence, by 3.2 and the π-tensor stability of $\mathcal{P}_p^{\mathrm{dual}}$ and its adjoint (see 2.1),

$$P_p^{\mathrm{dual}}(S\tilde{\otimes}_\pi T') = P_p^{\mathrm{dual}}(S)P_p(T).$$

Now the diagram

$$\begin{array}{ccc}(F\tilde{\otimes}_\pi G')' & \xrightarrow{(S\tilde{\otimes}_\pi T')'} & (E\tilde{\otimes}_\pi H')' \\ \| & & \| \\ \mathcal{L}(F,G'') & \xrightarrow{\mathrm{Hom}(T,S'')} & \mathcal{L}(E,H'')\end{array}$$

commutes, so that $\mathrm{Hom}(T,S'') \in \mathcal{P}_p(\mathcal{L}(F,G''), \mathcal{L}(E,H''))$ and

$$P_p(\mathrm{Hom}(T,S'')) = P_p((S\tilde{\otimes}_\pi T')') = P_p^{\mathrm{dual}}(S)P_p(T).$$

This completes the proof since $\mathcal{L}(F,G)$ (resp., $\mathcal{L}(E,H)$) is an isometric subspace of $\mathcal{L}(F,G'')$ (resp., $\mathcal{L}(E,H'')$) and $[\mathcal{P}_p, P_p]$ is injective. □

Another application we shall present is to provide an alternative proof of two known inequalities on ideal norms. We shall again make use of the two equations following 3.2. In [2], tensor stability techniques have been profitably employed to improve ideal norm estimates of certain operators. The following lemma is a variation of a basic lemma used in the above-mentioned investigation; we refer the reader to [2, 2.1.1] for a proof.

3.4. LEMMA. *Let* $[\mathfrak{A}, A]$, $[\mathfrak{B}, B]$ *be two quasi-Banach ideals and* α *a tensor norm. Assume that for each* $\epsilon > 0$ *there exists a constant* $c(\epsilon) \geq 1$ *such that, for all finite rank operators* S,

$$A(S) \le c(\epsilon)(\operatorname{rank} S)^{\lambda+\epsilon} B(S),$$

where $\lambda \ge 0$ is some constant. Moreover, let $a, b \ge 1$ be constants such that $A(S)^2 \le aA(S \tilde{\otimes}_\alpha S)$ and $B(S \tilde{\otimes}_\alpha S) \le bB(S)^2$ for all finite rank operators S. Then, for all such S,

$$A(S) \le ab(\operatorname{rank} S)^{\lambda} B(S).$$

Now we can re-prove the following known facts (see [1] and [10]).

3.5. PROPOSITION. *Let E be an n-dimensional space. Then, for each $S \in \mathcal{L}(E,F)$,*

$$P_2(S) \le n^{1/2-1/r} P_r(S), \quad 2 \le r \le \infty;$$

$$I_r(S) \le n^{1/2-1/r'} I_2(S), \quad 1 \le r \le 2.$$

From [2, 1.2.3] one has, for $2 \le r \le \infty$,

$$\sup\{P_2(S\,\mathrm{id}_E) \mid S \in \mathcal{P}_r(F,G), P_r(S) \le 1\} \le 24(1+\log n) n^{1/2-1/r}$$

and dually, for $1 \le r \le 2$,

$$\sup\{I_r(\mathrm{id}_E S) \mid S \in \mathcal{I}_2(G,F), I_2(S) \le 1\} \le 24(1+\log n) n^{1/2-1/r'}.$$

Since, for finite rank operators S,

$$P_r(S \tilde{\otimes}_\epsilon S) = P_r(S)^2, \quad 1 \le r \le \infty,$$

and

$$I_r(S \tilde{\otimes}_\epsilon S) = I_r(S)^2, 1 \le r \le \infty,$$

the desired results in Proposition 3.5 follow from the preceding lemma. □

References

1. B. Carl, *Inequalities between absolutely (p,q)-summing norms,* Studia Math. 69 (1980), 143–148.
2. B. Carl and A. Defant, *Tensor products, entropy estimates and Grothendieck type inequalities of operators in L_p-spaces,* preprint.
3. A. Defant and K. Floret, *Aspects of the metric theory of tensor products and operator ideals,* Notas Mat. 8 (1988), to appear.
4. J. E. Gilbert and T. Leih, *Factorization, tensor products and bilinear forms in Banach space theory.* Notes in Banach Spaces, pp. 182–305, Univ. of Texas Press, Austin, 1980.
5. A. Grothendieck, *Resumé de la Théorie metrique des produits tensoriels topologiques,* Bol. Soc. Mat. São Paulo 8 (1956), 1–79.
6. J. R. Holub, *Tensor product mappings,* Math. Ann. 188 (1970), 1–12.
7. ———, *Tensor product mappings, II,* Proc. Amer. Math. Soc. 42 (1974), 437–441.
8. R. C. James, *Uniformly non-square Banach spaces,* Ann. of Math. (2) 80 (1964), 542–550.
9. H. König, *On the tensor stability of s-number ideals,* Math. Ann. 269 (1984), 77–93.

10. D. R. Lewis, *Finite dimensional subspaces of L_p*, Studia Math. 63 (1978), 207–212.
11. ———, *An isomorphic characterization of the Schmidt class,* Compositio Math. 30 (1975), 293–297.
12. H. P. Lotz, *Grothendieck ideals of operators in Banach spaces,* Lecture notes, Univ. Illinois, Urbana, 1973.
13. A. Pełczyński, *Geometry of finite dimensional Banach spaces and operator ideals.* Notes in Banach Spaces, pp. 81–181, Univ. of Texas Press, Austin, 1980.
14. A. Pietsch, *Operator ideals,* North-Holland, Amsterdam, 1980.
15. ———, *Eigenvalues and s-numbers,* Cambridge Univ. Press, Cambridge, 1987.
16. L. Schwartz, *Geometry and probability in Banach spaces,* Lecture Notes in Math., 852, Springer, Berlin, 1981.
17. N. Tomczak-Jaegermann, *The modulii of smoothness and convexity and the Rademacher averages of the trace class operators S_p $(1 \le p < \infty)$*, Studia Math. 50 (1974), 163–182.
18. K. Vala, *On compact sets of compact operators,* Ann. Acad. Sci. Fenn. Ser. A I Math. 351 (1964).

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