TRANSITIVE GROUP ACTIONS AND RICCI CURVATURE PROPERTIES

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1. Introduction. When considering left invariant metrics on Lie groups, Milnor [11] discusses the problem of determining the real Lie groups admitting left invariant metrics and satisfying certain curvature restrictions. For example, it is shown in [11] that if for some metric the Ricci curvature is nonnegative then the group must be unimodular, and if the Ricci curvature is positive then the group must be compact and semi-simple.

We generalize in Proposition 2 and Corollary 2 the above results for homogeneous M. On the other hand, we prove in Theorem 1 a splitting result related to transitive subgroups of $I(M)$, where M is a homogeneous manifold of nonnegative Ricci curvature, and derive as a corollary that if the subgroup is solvable then the manifold must be flat; if the Ricci curvature is positive there exists a compact and semi-simple Lie group acting transitively by isometries.

Related to negatively Ricci curved homogeneous manifolds, it is not yet known in general whether any $M=G/H$ does admit an invariant metric with $\mathrm{Ric}<0$, even in the simply transitive case. In Theorem 2 we find a strong restriction on the transitive unimodular Lie subgroups of the full isometry group: they are semi-simple (noncompact). In particular, from results of Gordon [9], the full isometry group is semi-simple.

2. Ricci curvature of Riemannian submersions. Let $P\xrightarrow{\pi}M$ be a Riemannian submersion; that is, for x in M there is an orthogonal splitting $T_xP=V_x\oplus H_x$ into vertical and horizontal subspaces, and $H_x\xrightarrow{\pi_*}T_{\pi(x)}M$ is an isometry for all x.

Let $\mathcal{H}$ and $\mathcal{V}$ denote (respectively) the orthogonal projections onto the horizontal and vertical subspaces of T_xP at any point, and set (see [13])

$$A_EF=\mathcal{V}\nabla_{\mathcal{H}E}F+\mathcal{H}\nabla_{\mathcal{V}F}E$$

for E and F vector fields on P.

The following lemma is a consequence of O'Neill's identities for the curvature of a Riemannian submersion. See also Chapter 9 of [4].

LEMMA 1. *Let $P\xrightarrow{\pi}M$ be a Riemannian submersion with totally geodesic fibers, and let X and Y be horizontal unit vectors at x in P. Then*

$$\mathrm{Ric}(\pi_*X,\pi_*Y)=\mathrm{Ric}(X,Y)+2\sum_i\langle A_{H_i}X,A_{H_i}Y\rangle,$$

where $\{H_i\}$ is an orthonormal local basis of horizontal vector fields.

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In order to apply the previous lemma to the case of homogeneous Riemannian manifolds, we first need to recall some known facts.

Let M be a homogeneous Riemannian manifold and let G be a connected Lie group acting transitively by isometries on M. Then M is diffeomorphic to G/H, where H is the isotropy subgroup at a point in M. If the action of G on M is effective (this can always be assumed) then $\overline{\mathrm{Ad}(H)}$ is compact. Thus G/H is reductive; that is, there exists a decomposition $\underline{g} = \underline{h} \oplus \underline{m}$ where $\underline{g}$ is the Lie algebra of G, $\underline{h}$ is the Lie algebra of H, and $\underline{m}$ is an $\mathrm{Ad}(H)$-invariant complement of $\underline{h}$. Fix one such decomposition and give to $\underline{m}$ an $\mathrm{Ad}(H)$-invariant inner product such that π_* is orthogonal when restricted to $\underline{m}$. Extend this inner product to $\underline{g}$ by requiring it to be $\mathrm{Ad}(H)$-invariant and also that the above reductive decomposition be orthogonal.

LEMMA 2. *Let M be a homogeneous Riemannian manifold and let G be any connected Lie group acting transitively and effectively by isometries on M. Then, if G is endowed with a left-invariant metric as above, $G \xrightarrow{\pi} G/H$ is a Riemannian submersion with totally geodesic fibers.*

Proof. Let V, W, and X be left-invariant vector fields; then

$$\langle \nabla_V W, X \rangle = \tfrac{1}{2}\{\langle [V,W], X\rangle - \langle [W,X], V\rangle + \langle [X,V], W\rangle\}.$$

Thus the last assertion follows by taking V, W to be vertical and X to be horizontal. The lemma is now clear. □

In the next proposition we apply Lemma 1 to a submersion as in Lemma 2.

PROPOSITION 1. *Let M and G, with a left-invariant metric, be as in Lemma 2. Then the Ricci curvatures on M are given by*

$$\begin{aligned}\mathrm{Ric}(\pi_* X, \pi_* X) = {} & -\frac{1}{2}\operatorname{tr}(\mathrm{ad}_X)^2 - \frac{1}{2}\sum_i \|[X, H_i]_{\underline{m}}\| \\ & - \operatorname{tr} \mathrm{ad}_{\nabla_X X} + \frac{1}{2}\sum_{i<j} \langle [H_i, H_j], X\rangle^2,\end{aligned}$$

where $\underline{m}$ is any $\mathrm{Ad}(H)$-invariant complement of $\underline{h}$, $\{H_i\}$ is an orthonormal basis of $\underline{m}$, and X is a unit vector in $\underline{m}$.

Proof. Let X and H_i be left-invariant vector fields whose values at e coincide with a unit vector in $\underline{m}$ and with vectors of an orthonormal basis of $\underline{m}$, respectively.

Lemma 1, together with the fact that $A_X Y = \frac{1}{2}\mathcal{V}[X, Y]$ for X, Y horizontal vector fields, implies

$$\mathrm{Ric}(\pi_* X, \pi_* X) = \mathrm{Ric}(X, X) + \frac{1}{2}\sum_i \|\mathcal{V}[X, H_i]\|^2.$$

On the other hand, for a Lie group with a left-invariant metric, the Ricci curvatures can be expressed as follows (see [7]):

$$\operatorname{Ric}(X,X) = -\frac{1}{2}\operatorname{tr}(\operatorname{ad}_X)^2 - \frac{1}{2}\sum_j \|[X,V_j]\|^2 - \frac{1}{2}\sum_i \|[X,H_i]\|^2$$
$$-\operatorname{tr}\operatorname{ad}_{\nabla_X X} + \frac{1}{2}\sum_{j,i}\langle [V_j,H_i],X\rangle^2 + \frac{1}{2}\sum_{j<k}\langle [H_j,H_k],X\rangle^2,$$

where $\{V_j\}$ is an orthonormal basis of $\underline{h}$. Since ad_V is skew-symmetric, V in $\underline{h}$, the second and fifth term in the expression above cancel out and the identity thus follows. □

3. Invariant metrics with Ric ≥ 0. We recall that a connected Lie group G is unimodular if $\operatorname{tr}\operatorname{ad}_X = 0$ for X in $\underline{g}$.

PROPOSITION 2 (cf. [11]). *Let M be a homogeneous Riemannian manifold of nonnegative Ricci curvature. Then every connected Lie group acting transitively and effectively by isometries on M must be unimodular.*

Proof. Let G be such a Lie group and let $\underline{g} = \underline{h} \oplus \underline{m}$ be a reductive decomposition of $\underline{g}$, the Lie algebra of G. If $\underline{u} = \{X \in \underline{g} : \operatorname{tr}\operatorname{ad}_X = 0\}$ then clearly $\underline{h} \subset \underline{u}$. Assume $\underline{u} \neq \underline{g}$ and let X in $\underline{m}$ be such that $\langle X, \underline{u}\rangle = 0$. In particular X is orthogonal to $[\underline{g},\underline{g}]$, the conmutator subalgebra.

Let $\{V_i\}$ and $\{H_j\}$ denote (respectively) orthonormal basis of $\underline{h}$ and $\underline{m}$. Next we analyze the different terms appearing in the identity of Proposition 1:

$$-\frac{1}{2}\operatorname{tr}\operatorname{ad}_X = -\frac{1}{2}\sum_{j,k}\langle [X,H_j],H_k\rangle\langle [X,H_k],H_j\rangle;$$

since $\langle [X,V_i],H_j\rangle = \langle X,[V_i,H_j]\rangle = 0$,

$$-\frac{1}{2}\sum_j \|\mathcal{H}[X,H_j]\|^2 = -\frac{1}{2}\sum_{j,k}\langle [X,H_j],H_k\rangle^2,$$

$$\operatorname{tr}\operatorname{ad}_{\nabla_X X} = 0, \quad \text{and} \quad -\frac{1}{2}\sum_{j<k}\langle [H_j,H_k],X\rangle^2 = 0.$$

Thus

$$\operatorname{Ric}(\pi_* X, \pi_* X) = -\frac{1}{2}\sum_{j,k}\langle \operatorname{ad}_X H_j, H_k\rangle\langle (\operatorname{ad}_X + \operatorname{ad}_X^*)H_k, H_j\rangle.$$

Let T_X denote the transformation $p \circ \operatorname{ad}_X|_{\underline{m}}$, where p is the projection onto the $\underline{m}$ component. Then, the equality above says that

$$\operatorname{Ric}(\pi_* X, \pi_* X) = -\tfrac{1}{4}\operatorname{tr}(T_X + T_X^*)^2 \leq 0.$$

Assume equality holds. Then $-T_X = T_X^*$; that is, $\langle [X,H_i],H_j\rangle = -\langle H_i,[X,H_j]\rangle$ for all i,j. Hence $\operatorname{tr}\operatorname{ad}_X = 0$, showing that either $X = 0$ or $\operatorname{Ric} X < 0$, a contradiction. Thus $\underline{u} = \underline{g}$ and G is unimodular as claimed. □

In the next result we obtain rather precise information on the groups G as in Proposition 2. With $\underline{g}'$ we will denote the derived algebra of $\underline{g}$.

THEOREM 1. *If $M=G/H$ has an invariant metric with nonnegative Ricci curvatures, then there exists an* $\mathrm{Ad}(H)$*-invariant decomposition*

$$\underline{g}=\underline{h}\oplus\underline{a}\oplus\underline{b}$$

such that:

(1) $\underline{h}\oplus\underline{a}=\underline{h}+\underline{g}'$ *and* $\underline{h}\oplus\underline{b}$ *is a subalgebra satisfying* $[\underline{h},\underline{b}]=0$, $[\underline{b},\underline{b}]\subset\underline{h}$;

(2) $\mathrm{Ric}(\pi_*Z,\pi_*Y)=0$ *if Z is in $\underline{a}\oplus\underline{b}$ and Y is in $\underline{b}$*;

(3) *if K is the connected Lie subgroup of G with Lie algebra $\underline{h}\oplus\underline{a}$ then the Ricci curvatures of the induced metric in $K/K\cap H$ coincide with the Ricci curvatures in G/H.*

Proof. Note that since $\underline{g}'$ is an ideal, $\mathrm{Ad}(H)$ preserves $\underline{h}+\underline{g}'$. Let $\underline{n}$ (respectively $\underline{a}$) be the orthogonal complement of $\underline{h}+\underline{g}'$ in $\underline{g}$ (respectively $\underline{h}$ in $\underline{h}+\underline{g}$) with respect to any $\mathrm{Ad}(H)$-invariant inner product in $\underline{g}$. Consider now in $\underline{g}=\underline{h}\oplus\underline{a}\oplus\underline{n}$ an inner product as in Lemma 2.

If $\underline{b}$ is the orthogonal complement of $\underline{a}$ in $\underline{a}\oplus\underline{n}$, since $[\underline{h},\underline{b}]\subset\underline{h}\oplus\underline{a}$ one has that $[\underline{h},\underline{b}]=0$.

Let Y be in $\underline{b}$. Using the same computations as in the proof of Proposition 2 (note that $\langle Y,\underline{g}'\rangle=0$), one obtains

$$\mathrm{Ric}(\pi_*Y,\pi_*Y)=-\tfrac{1}{4}\operatorname{tr}(T_Y+T_Y^*)^2\leq 0;$$

hence T_Y is a skew-symmetric transformation of $\underline{a}\oplus\underline{b}$. Since Y in $\underline{b}$ is arbitrary it follows that $[\underline{b},\underline{b}]\subset\underline{h}$. In fact, if Y_1, Y_2 are in $\underline{b}$ and Z is in $\underline{a}\oplus\underline{b}$ then

$$\langle[Y_1,Y_2],Z\rangle=\langle T_{Y_1}Y_2,Z\rangle=-\langle Y_2,T_{Y_1}Z\rangle=-\langle Y_2,[Y_1,Z]\rangle=0,$$

and the first assertion follows.

In order to verify the second assertion, let Y be in $\underline{b}$ and let Z be in $\underline{a}\oplus\underline{b}$. Then by Lemma 1 we need to compute $\mathrm{Ric}(Z,Y)$ with respect to a left-invariant metric on G, and $A(Z,Y)=2\sum_i\langle A_{X_i}Z,A_{X_i}Y\rangle+2\sum_j\langle A_{Y_j}Z,A_{Y_j}Y\rangle$ where $\{X_i\}$ (respectively $\{Y_j\}$) is an orthonormal basis of $\underline{a}$ (respectively $\underline{b}$).

Since G is unimodular (Proposition 2), the Ricci transformation of a left-invariant metric is given by (see [1]):

$$\mathrm{Ric}(E,F)=\left\langle\left(-\frac{1}{2}\sum_i \mathrm{ad}_{E_i}^*\,\mathrm{ad}_{E_i}+\frac{1}{4}\sum_i \mathrm{ad}_{E_i}\,\mathrm{ad}_{E_i}^*-\frac{1}{2}\hat{B}\right)E,F\right\rangle,$$

where $\{E_i\}$ is an orthonormal basis of $\underline{g}$ and $\langle\hat{B}E,F\rangle=\operatorname{tr}(\mathrm{ad}_E\,\mathrm{ad}_F)$.

Let $\{V_k\}$ be an orthonormal basis of $\underline{h}$. Since $\langle Y,\underline{g}'\rangle=0$, $[\underline{h},\underline{b}]=0$, and ad_V is skew-adjoint for every V in $\underline{h}$, one obtains

$$\mathrm{Ric}(Z,Y)=\left\langle\left(-\frac{1}{2}\sum_i \mathrm{ad}_{X_i}^*\,\mathrm{ad}_{X_i}-\frac{1}{2}\sum_j \mathrm{ad}_{Y_j}^*\,\mathrm{ad}_{Y_j}-\frac{1}{2}\hat{B}\right)Z,Y\right\rangle,$$

with

$$\begin{aligned}\langle\hat{B}Z,Y\rangle&=\sum_k\langle\mathrm{ad}_Y\mathrm{ad}_Z V_k,V_k\rangle+\sum_i\langle\mathrm{ad}_Y\mathrm{ad}_Z X_i,X_i\rangle\\&=\sum_k\langle\mathrm{ad}_{[Y,Z]}V_k,V_k\rangle+\sum_i\langle\mathrm{ad}_Y\mathcal{H}[Z,X_i],X_i\rangle\\&=\sum_i\langle T_Y\mathcal{H}[Z,X_i],X_i\rangle.\end{aligned}$$

On the other hand, $A_E F = \frac{1}{2}\mathcal{V}[E,F]$ for E, F horizontal fields; thus

$$A(Z,Y) = \frac{1}{2}\sum_i \langle \mathcal{V}[X_i, Z], [X_i, Y]\rangle + \frac{1}{2}\sum_j \langle [Y_j, Z], [Y_j, Y]\rangle.$$

Now it is easy to verify that $\mathrm{Ric}(\pi_* Z, \pi_* Y)$ vanishes.

To finish the proof we consider X in $\underline{a}$. Then, with respect to the metric in G/H and according to Proposition 1,

$$\begin{aligned}\mathrm{Ric}(\pi_* X, \pi_* X)\\ &= -\frac{1}{2}\operatorname{tr}\operatorname{ad}_X^2 - \frac{1}{2}\sum_i \|\mathcal{H}[X, X_i]\|^2 - \frac{1}{2}\sum_j \|\mathcal{H}[X, Y_j]\|^2 \\ &\quad + \frac{1}{2}\sum_{i<k}\langle [X_i, X_k], X\rangle^2 + \frac{1}{2}\sum_{i,j}\langle [X_i, Y_j], X\rangle^2 + \frac{1}{2}\sum_{j,l}\langle [Y_j, Y_l], X\rangle^2,\end{aligned}$$

where $\{X_i\}$ (respectively $\{Y_j\}$) is an orthonormal basis of $\underline{a}$ (respectively $\underline{b}$).

Note that: $\langle [Y_j, Y_l], X\rangle = 0$ since $[\underline{b}, \underline{b}] \subset \underline{h}$; that $\operatorname{tr}\operatorname{ad}_X^2 = \operatorname{tr}(\operatorname{ad}_X|_{\underline{h}\oplus\underline{a}})^2$ and T_{Y_j} is skew-symmetric; and that $-\frac{1}{2}\sum_j \|\mathcal{H}[X, Y_j]\|^2 + \frac{1}{2}\sum_{i,j}\langle [X_i, Y_j], X\rangle^2 = 0$. Hence, the last assertion follows. □

COROLLARY 1. *If a connected solvable Lie group G acts transitively and effectively by isometries on a Riemannian manifold M with nonnegative Ricci curvature, then the metric in M must be flat.*

Proof. Assume that the assertion is true if the dimension of G is less than n. If $\underline{h} + \underline{g}' \neq \underline{g}$, Theorem 1 together with the inductive hypothesis imply that M is Ricci flat, and hence flat, by [2].

Otherwise $\underline{h} + \underline{g}' = \underline{g}$. Since we may assume the isotropy subgroup H to be connected it follows that $HG' = G$ (here G' denotes the conmutator subgroup). Thus the nilpotent Lie group G' acts transitively and effectively on M. Now the isotropy subgroup in G' must be discrete, since ad_X nilpotent and skew-symmetric implies that X is central. Hence M is covered by a nilpotent Lie group (with a left-invariant metric). Finally, this group must be abelian since there are no directions of negative Ricci curvature (see [11]). □

In [11] a necessary and sufficient condition for everywhere positive Ricci curvature of a left-invariant metric on a Lie group G is obtained: the group must be compact and semi-simple. For Riemannian homogeneous manifolds the following generalization holds.

COROLLARY 2. *A connected manifold M admits an invariant metric with* $\mathrm{Ric} > 0$ *if and only if there exists a connected, compact, and semi-simple Lie group acting transitively on M.*

Proof. Give to M an invariant metric with $\mathrm{Ric} > 0$. Myer's theorem implies that M, hence $I(M)$, is compact. Since $I(M)$ is compact and G is a connected Lie subgroup of $I(M)$, it follows that G is isomorphic to $A \times K$ where A and K are Lie subgroups of G, A is abelian, and K is compact (see [11] for a proof). Now, as a consequence of Theorem 1, the compact and semi-simple conmutator subgroup

$[K,K]$ acts transitively on M. Because normal metrics (induced by bi-invariant ones) have positive Ricci curvature in any quotient of a compact and semi-simple Lie group, the corollary is clear. □

REMARK. We note that even when G/H is compact, H (hence G) may be noncompact; see in [8] a representation of certain compact symmetric spaces as G/H with H noncompact and G acting effectively. Thus abelian factors can actually occur in a transitive effective group of isometries of M. Also, the previous corollary is known for a compact and simply connected G/H, with no metric assumptions; see [12] and [14].

4. Invariant metrics with Ric < 0. In [15] Wolf showed that if a semi-simple Lie group acts transitively and effectively by isometries on a simply connected manifold M (M with nonpositive sectional curvature and without Euclidean factor) then its center is finite and the isotropy subgroup is a maximal compact subgroup; hence M is a symmetric space. Furthermore, it is known that under these hypotheses ([6]) any connected unimodular Lie group acting transitively by isometries on M must be semi-simple.

We next show that an analogous result holds for negative Ricci curvature. More precisely, we have the following.

THEOREM 2. *If M is a homogeneous Riemannian manifold of negative Ricci curvature then every transitive, connected, and unimodular subgroup G of $I(M)$ must be semi-simple and closed.*

Proof. Let G be a group as in the statement. Let $\underline{u}$ be an ideal of $\underline{g}$, the Lie algebra of G, and let $\underline{h}$ denote the Lie algebra of the isotropy subgroup H. Since $\underline{u}$ is an ideal we obtain $\mathrm{Ad}(H)$ invariant decompositions, $\underline{g}=(\underline{h}+\underline{u})\oplus\underline{n}$ and $\underline{h}+\underline{u}=\underline{h}\oplus\underline{a}$. Give now to G a metric as in Lemma 2. In what follows $\{X_i\}$, $\{Y_j\}$ will denote (respectively) orthonormal bases of $\underline{a}, \underline{b}$, where $\underline{b}$ denotes the orthogonal complement of $\underline{a}$ in $\underline{a}\oplus\underline{n}$.

Assume $\underline{u}$ is abelian and let X be a unit vector in $\underline{a}$. As G is unimodular, it follows from Proposition 1 that

$$\mathrm{Ric}(\pi_*X,\pi_*X) = -\frac{1}{2}\sum_j \|\mathcal{H}[X,Y_j]\|^2 + \frac{1}{2}\sum_{i,j}\langle [X_i,Y_j],X\rangle^2 + \frac{1}{2}\sum_{j<k}\langle [Y_j,Y_k],X\rangle^2.$$

Moreover, if the dimension of $\underline{a}$ is n then

$$\sum_{i=1}^{i=n}\mathrm{Ric}(\pi_*X_i,\pi_*X_i) = \frac{1}{2}\sum_{i=1}^{i=n}\sum_{j<k}\langle [Y_j,Y_k],X_i\rangle^2,$$

showing that if Ric < 0 then $\underline{a}=0$ or equivalently $\underline{u}\subset\underline{h}$. Because the action is effective, it follows that $\underline{g}$ has no abelian ideals; hence it is semi-simple.

To finish the proof, assume that the closure $\bar{G}$ of a semi-simple Lie subgroup G of a Lie group S has a reductive Lie algebra; that is, the radical coincides with the center. Hence $\bar{G}$ is unimodular and (by the first part of the proof) semi-simple. Thus $\bar{G} = G$ as claimed.

For completeness, we include the proof of the fact that $\bar{G}$ is reductive. Note first that G is an invariant subgroup of $\bar{G}$, since if $x = \exp X$ is close to the identity in G and $\lim_{n\to\infty} y_n = y$ then $yxy^{-1} = \exp(\lim_{n\to\infty} \mathrm{Ad}(y_n)X)$.

Let $Z(G)$ (respectively $Z(\bar{G})$) denote the center of G (respectively $\bar{G}$). We show next that the map $G/Z(G) \to \bar{G}/Z(\bar{G})$ is surjective. Since it is clearly injective the assertion will follow.

Let x be in $\bar{G}$ and let $\underline{g}$ and $\bar{\underline{g}}$ denote the respective Lie algebras. Since $\underline{g}$ is an ideal of $\bar{\underline{g}}$, it is preserved by $\mathrm{Ad}(x)$. Since G is semi-simple, $\mathrm{Ad}(G)$ is closed in $GL(\underline{g})$. Thus there exists x_0 in G such that $\mathrm{Ad}(x_0) = \mathrm{Ad}(x)|_{\underline{g}}$ or $x_0^{-1}x$ is in $Z(G)$, as desired. □

REMARK. We recall that G is closed in $I(M)$ if and only if the isotropy subgroup H is compact, and we note that in general it cannot be expected that H be maximal compact. In fact, in [10] and [3] invariant metrics are constructed with $\mathrm{Ric} < 0$ in $SL(n, R)$, $n > 2$, and in $SO(n, 2)/SO(n)$. Berard Bergery [3] also provides examples of noncompact quotients G/H, with G semi-simple, which do not admit invariant metrics of negative Ricci curvature.

Let M be a Riemannian manifold and let $I_0(M)$ denote the connected component of the full isometry group $I(M)$. If G is a connected transitive subgroup of $I(M)$ then $I_0(M) = GL$, where L is a compact subgroup. Moreover, it follows from [9] that if G is reductive then $I_0(M)$ is also reductive. Applying Theorem 2, one easily obtains the following.

COROLLARY 3. *If a homogeneous Riemannian manifold M with* $\mathrm{Ric} < 0$ *admits a unimodular transitive group of isometries, then the full isometry group $I(M)$ is semi-simple.*

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