# ON THE AUTOMORPHISM GROUP OF HYPERELLIPTIC KLEIN SURFACES

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**1.** In [6] *hyperelliptic Klein surfaces* (HKS) were characterized by means of *non-Euclidean crystallographic groups* (NEC groups). An HKS is a Klein surface with non-empty boundary $X$ that admits an automorphism $\phi$ of order two such that the quotient $X/\langle\phi\rangle$ has *algebraic genus* zero. Then, the automorphism group of $X$ has order $2N$.

Given an HKS $X$ with algebraic genus $p \geq 2$ and $k$ boundary components, and given $|\text{Aut}\, X| = 2N$ ($N$ odd), we obtain in this paper the possible values for $N$. Moreover, for each one of these values we prove that there exists an HKS $X$, with the above conditions, such that $|\text{Aut}\, X| = 2N$.

In the particular cases $p=2$ or $p=3$, the list of the automorphism groups is given in [5] and [7].

**2.** A Klein surface (see [1]) may be expressed as $D/\Gamma$ where $D$ is the hyperbolic plane and $\Gamma$ is a certain NEC group [12]. NEC groups were introduced by Wilkie [15]. Macbeath [10] associated to each NEC group $\Gamma$ a *signature* that has the form

$$(g; \pm; [m_1, \dots, m_r], \{(n_{11}, \dots, n_{1s_1}) \cdots (n_{k1}, \dots, n_{ks_k})\}),$$

and determined the algebraic structure of $\Gamma$. In this signature the numbers are integers and $g \geq 0$, $m_i \geq 2$, $n_{ij} \geq 2$. The number $g$ is the topological genus of the group (and that of $D/\Gamma$). The sign determines the orientability of $D/\Gamma$. The numbers $m_i$ are the *proper periods* and the brackets $(n_{i1}, \dots, n_{is_i})$ are the *period-cycles.* The number $k$ of period-cycles is equal to the number of boundary components of $D/\Gamma$. Numbers $n_{ij}$ are the *periods of the period-cycle* $(n_{i1}, \dots, n_{is_i})$.

The *canonical* presentation of $\Gamma$ is as follows.

*Generators*

(i) $x_i$ $\quad i=1, \dots, r$

(ii) $e_i$ $\quad i=1, \dots, k$

(iii) $c_{ij}$ $\quad i=1, \dots, k;\ j=0, \dots, s_i$

(iv) $a_i, b_i$ $\quad i=1, \dots, g$ (if sign '+')

$\quad d_i$ $\quad i=1, \dots, g$ (if sign '−')

*Relations*

(i) $x_i^{m_i} = 1$ $\qquad i=1, \dots, r$

(ii) $c_{i,j-1}^2 = c_{i,j}^2 = (c_{i,j-1}c_{i,j})^{n_{ij}} = 1$ $\qquad i=1, \dots, k;\ j=1, \dots, s_i$

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(iii) $e_i^{-1}c_{io}e_ic_{is_i}=1$ $\quad i=1,\dots,k$

(iv) $x_1\cdots x_re_1\cdots e_ka_1b_1a_1^{-1}b_1^{-1}\cdots a_gb_ga_g^{-1}b_g^{-1}=1$ (if sign '+')

$x_1\cdots x_re_1\cdots e_kd_1^2\cdots d_g^2=1$ (if sign '−').

Every NEC group $\Gamma$ has associated a *fundamental region* whose area is called the *area of the group* (see [14]) and it is denoted by $|\Gamma|$. If $\Gamma^*$ is a subgroup of $\Gamma$ with finite index $n$ then $|\Gamma^*|=n|\Gamma|$. Given an NEC group $\Gamma$, the subgroup of index 2 of the orientation-preserving elements is called the *canonical Fuchsian subgroup* of $\Gamma$ and is denoted by $\Gamma^+$.

Let $X$ be a Klein surface with algebraic genus $p$. Then $X=D/\Gamma$, where $\Gamma$ is an NEC group with the following signature [12]:

$$(g;\pm;[-],\{(-)\overset{k}{\cdots}(-)\}),\tag{2.1}$$

where $p=\alpha g+k-1$, $\alpha=2$ (resp. $\alpha=1$) if $D/\Gamma$ is orientable (resp. non-orientable).

May [11] proved that if $H$ is a group of automorphisms of $X=D/\Gamma$ with algebraic genus $p\geq 2$ then $H$ may be expressed as $\Gamma^*/\Gamma$, $\Gamma^*$ being another NEC group. The full group of automorphisms of $X$ is $N_{\mathcal{G}}(\Gamma)/\Gamma$, where $N_{\mathcal{G}}(\Gamma)$ is the normalizer of $\Gamma$ in the group $\mathcal{G}$ of isometries of $D$.

**3.** From now on, we suppose that $p\geq 2$. In [6] the following characterization of HKS was obtained.

THEOREM 3.1. *Let $\Gamma$ be an NEC group with signature as in* (2.1). *Then $D/\Gamma$ is a hyperelliptic surface if and only if there exists a unique NEC group $\Gamma_1$ with $|\Gamma_1:\Gamma|=2$ and with signature*

(i) $(0;+;[-],\{(2,\overset{2k}{\dots},2)\})$ (*if* $g=0$);

(ii) $(0;+;[2,\overset{2g+k}{\dots},2],\{(-)\})$ (*if* $g\neq 0$ *and* $\Gamma$ *has sign* '+');

(iii) $(0;+;[2,\overset{g}{\dots},2],\{(2,\overset{2k}{\dots},2)\})$ (*if* $\Gamma$ *has sign* '−').

Let $\Gamma_1/\Gamma=\langle I_d,\phi\rangle$. It was proved in [6] that $\phi$, the automorphism of hyperellipticity, is a central element of $\operatorname{Aut}X$.

COROLLARY 3.2. *Let $X=D/\Gamma$ be an HKS with boundary of algebraic genus $p\geq 2$ such that* $|\operatorname{Aut}X|=2N$ ($N$ odd); *then* $\operatorname{Aut}X\simeq Z_{2N}$.

*Proof.* As $X=D/\Gamma$ is an HKS, by Theorem 3.1 there exists a $\Gamma_1$ such that $\Gamma\lhd\Gamma_1$ and the quotient space $D/\Gamma_1$ is the disc or the sphere. If $\operatorname{Aut}X=\Gamma^*/\Gamma$ then $\Gamma_1\lhd\Gamma^*$, and so $\Gamma^*/\Gamma_1$ is cyclic as it has odd order. Put $\Gamma_1/\Gamma=C$ ($\simeq Z_2$) and $\Gamma^*/\Gamma_1=H$. Then $\operatorname{Aut}X/C=H\simeq Z_N$; thus $\operatorname{Aut}X=C\cup Ct\cup Ct^2\cup\cdots\cup Ct^{N-1}$. If $\phi$ generates $C$ then $\phi t$ has order $2N$ (as $N$ is odd), and so $\operatorname{Aut}X\simeq Z_{2N}$. □

Let $G=\operatorname{Aut}X$ the group of automorphisms of $X=D/\Gamma$ then there exists an NEC group $\Gamma^*$ such that $G\simeq\Gamma^*/\Gamma$. Thus we must study the possible groups $\Gamma^*$ such that there is an epimorphism $\theta_1$ from $\Gamma$ onto $G/\langle\phi\rangle$ with kernel $\Gamma_1$ and $\Gamma\lhd\Gamma_1\lhd\Gamma^*$. According to the possible signatures for $\Gamma_1$, the three following cases can occur:

(a) $D/\Gamma$ is a compact planar Klein surface,
(b) $D/\Gamma$ is orientable with topological genus $g \neq 0$ with one or two components in its boundary,
(c) $D/\Gamma$ is non-orientable.

THEOREM 3.3. *Let $X = D/\Gamma$ be an HKS of algebraic genus $p \geq 2$, with $k$ boundary components such that* $|\mathrm{Aut}\, X| = 2N$ ($N$ odd). *Then:*

(a) *If $X$ is planar then $N$ divides $p+1$ and $N \neq p+1$. Furthermore, for each $p \geq 2$ and each $N$ satisfying these conditions there exists a planar HKS $X$ of algebraic genus $p$ with $|\mathrm{Aut}\, X| = 2N$.*

(b) *If $X$ is orientable and $k = 1$ or 2, then $N$ divides $p+1$ or $N$ divides $p$ ($N \neq p+1$, $N \neq p$). Moreover, for every $p \geq 2$ and each $N$ satisfying the preceding conditions there exists an orientable HKS $X$ with $k = 1$ or 2 and algebraic genus $p$ with $|\mathrm{Aut}\, X| = 2N$.*

(c) *If $X$ is non-orientable then $N$ divides $g$ or $g-1$ and $N$ divides $k$ (where $g = p-k+1$ is the topological genus of $X$). Besides, for every $p \geq 2$ and each $N$ satisfying these conditions there exists a non-orientable HKS $X$ of algebraic genus $p$ with $|\mathrm{Aut}\, X| = 2N$.*

*Proof.* (a) As $X = D/\Gamma$ is a compact planar Klein surface of algebraic genus $p$, by (2.1) the signature of $\Gamma$ is

$$(0; +; [-], \{(-)\overset{p+1}{\cdots}(-)\}).$$

Since $X$ is hyperelliptic, by Theorem 3.1(i) there exists an NEC group $\Gamma_1$ with signature

$$(0; +; [-], \{(2, \overset{2(p+1)}{\ldots\ldots}, 2)\}), \tag{3.1}$$

with $\Gamma \triangleleft \Gamma_1 \triangleleft \Gamma^*$ and $|\Gamma_1 : \Gamma| = 2$. By Corollary 3.2, $\Gamma^*/\Gamma_1 \simeq Z_N$; then the signature of $\Gamma^*$ is, by [3],

$$(g^*; +; [m_1, \ldots, m_r], \{(n_1, \ldots, n_s)\}).$$

Let $e, c_0, \ldots, c_s$ be the generators associated to the period-cycle (see §2). The minimal number $t$ such that $e^t \in \Gamma_1$ is $t = N$ by [3]. Therefore, the period-cycle of $\Gamma_1$ must be $(n_1, \ldots, n_s, \overset{N}{\ldots}, n_1, \ldots, n_s)$. From (3.1) we may see that $n_i = 2$ for $i = 1, \ldots, s = 2(p+1)/N$. Since $|\Gamma_1|/|\Gamma^*| = N$, we have

$$N\left(2g^* - 1 + \sum_{i=1}^{r}\left(1 - \frac{1}{m_i}\right) + \frac{p+1}{2N}\right) = \frac{p+1}{2} + 1,$$

and thus

$$2g^*N + N\left(\sum_{i=1}^{r}\left(1 - \frac{1}{m_i}\right)\right) = N - 1.$$

We may deduce $g^* = 0$, $r = 1$, and $m_1 = N$. Hence, the signature of $\Gamma^*$ is

$$(0; +; [N], \{(2, \overset{2(p+1)/N}{\ldots\ldots\ldots}, 2)\}),$$

and so $N$ must divide $p+1$. If $N$ were equal to $p+1$ the signature of $\Gamma^*$ would be $(0; +; [N], \{(2,2)\})$, and then by [4, (2.4)] $X$ would have a dihedral automorphism group of order $4N$. Hence $N \neq p+1$.

Now let $N$ be an odd number, $N \neq p+1$ and $N$ dividing $p+1$. Let $\Gamma^*$ be an NEC group with signature

$$\sigma^*: (0; +; [N], \{(2, \overset{2(p+1)/N}{\ldots\ldots}, 2)\}). \tag{3.2}$$

Let $\theta$ be the epimorphism from $\Gamma^*$ onto $Z_{2N}$ defined by:

$$\begin{aligned} \theta(x) &= \bar{2}, \\ \theta(e) &= \overline{2N-2}, \\ \theta(c_0) = \theta(c_i) &= \bar{N} \quad \text{for } i \text{ even}, \\ \theta(c_i) &= \bar{0} \quad \text{for } i \text{ odd}, \end{aligned}$$

where $x, e, c_i$ are the generators of the group $\Gamma$. From [2], [8], and [9] we may deduce that the signature of $\ker\theta$ is

$$(0; +; [-], \{(-) \overset{2(p+1)}{\ldots\ldots} (-)\}).$$

Let $\Gamma_1 = \theta^{-1}(\langle N \rangle)$. As $x^N \in \Gamma_1$, $e^N \in \Gamma_1$, and $c_i \in \Gamma_1$ for $i = 1, \ldots, 2(p+1)/N$, the signature of $\Gamma_1$ is, by [3],

$$(0; +; [-], \{(2, \overset{2(p+1)}{\ldots\ldots}, 2)\}),$$

with $|\Gamma_1 : \ker\theta| = 2$. By Theorem 3.1, $X = D/\ker\theta$ is a hyperelliptic compact planar Klein surface with algebraic genus $p$.

Now, we need to prove that there exists a maximal NEC group $\Gamma^*$ with signature $\sigma^*$ since the group $\operatorname{Aut} X$ is $N_{\mathfrak{G}}(\ker\theta)/\ker\theta$, and if $\Gamma^*$ is maximal then $N_{\mathfrak{G}}(\ker\theta) \simeq \Gamma^*$; thus $\operatorname{Aut} X \simeq \Gamma^*/\ker\theta$ $(\simeq Z_{2N})$. From (3.2) we obtain

$$\sigma^{*+} = (0; \pm; [N, N, 2, \overset{2(p+1)/N}{\ldots\ldots}, 2], \{-\}),$$

with $2(p+1)/N \geq 3$. By [13], $\sigma^{*+}$ has associated a maximal Fuchsian group, and therefore $\sigma^*$ does also.

(b) By hypothesis upon $X = D/\Gamma$ and (2.1), the signature of $\Gamma$ is

$$(g; +; [-], \{(-)^k\}),$$

where $k = 1$ or 2. By Theorem 3.1(ii) there exists an NEC group $\Gamma_1$ with signature

$$(0; +; [2, \overset{2g+k}{\ldots}, 2], \{(-)\}), \tag{3.3}$$

with $\Gamma \triangleleft \Gamma_1 \triangleleft \Gamma^*$ and $[\Gamma_1 : \Gamma| = 2$. Since $\Gamma^*/\Gamma_1 \simeq Z_N$, by [3] the signature of $\Gamma^*$ is

$$(g^*; +; [m_1, \ldots, m_r], \{(-)\}).$$

Let $x_i$ be the elliptic generators of $\Gamma^*$ and let $s_i$ be the minimal number such that $x_i^{s_i} \in \Gamma_1$. We have seen in (3.3) that all the numbers are 2 by [3], so

$$s_i = m_i/2 \quad \text{or} \quad s_i = m_i. \tag{3.4}$$

We may suppose that for $i=1,\dots,h$ the $s_i$ are of the first type and that for $i=h+1,\dots,r$ the $s_i$ are of the second. From [2], the number of proper periods of $\Gamma_1$ is

$$2g+k=\sum_{i=1}^{h}\frac{N}{s_i}. \tag{3.5}$$

The relation between the areas of $\Gamma_1$ and $\Gamma^*$ gives us

$$N\left(2g^*-1+\sum_{i=1}^{r}\left(1-\frac{1}{m_i}\right)\right)=\frac{2g+k}{2}-1,$$

thus

$$N-1=2g^*N+N\left(\sum_{i=1}^{r}\left(1-\frac{1}{m_i}\right)\right)-\frac{2g+k}{2}.$$

From (3.4) and (3.5),

$$N-1=2g^*N+N\sum_{i=1}^{r}\left(1-\frac{1}{s_i}\right),$$

hence $g^*=0$ and $s_i=1$ for each $i$ with the exception that one of them is $N$. Therefore, the signature of $\Gamma^*$ must be

$$(0;+;[2,\overset{(2g+k)/N}{\dots\dots},2,N],\{(\text{—})\})$$

or

$$(0;+;[2,\overset{(2g+k-1)/N}{\dots\dots},2,2N],\{(\text{—})\}),$$

and thus $N$ divides $p+1$ in the first case and $N$ divides $p$ in the second. If $N$ were $p+1$ (resp. $p$), the signature of $\Gamma^*$ would be one of the following signatures:

$$(0;+;[2,N],\{(\text{—})\})\quad\text{or}\quad(0;+;[2,2N],\{(\text{—})\}),$$

and by [4] the group $Z_{2N}\ltimes Z_2$ would be a group of automorphisms of the surface $X$, contradicting the hypothesis that the order is $2N$.

Conversely, let $N$ be an odd number that divides $p+1$ and $N\neq p+1$. Let $\Gamma^*$ be an NEC group with signature

$$(0;+;[2,\overset{(p+1)/N}{\dots\dots},2,N],\{(\text{—})\})$$

and let $\theta$ be the epimorphism from $\Gamma^*$ onto $Z_{2N}$ defined by:

$$\theta(x_i)=\bar{N},\quad i=1,\dots,(p+1)/N;$$

$$\theta(x_{(p+1)/N+1})=\bar{u},\quad (\bar{u}\text{ is an element of order }N);$$

$$\theta(e)=-(x_1,\dots,x_{(p+1)/N+1});$$

$$\theta(c)=\bar{0}.$$

From [2], [8], and [9], the signature of $\ker\theta$ is $(g;+;[\text{—}],\{(\text{—})^k\})$.

Let $\Gamma_1=\theta^{-1}(\langle N\rangle)$. As $x_i\in\Gamma_1$ $(i=1,\dots,(p+1)/N)$, $x^N_{(p+1)/N+1}\in\Gamma_1$, $e^N\in\Gamma_1$, and $c_i\in\Gamma_1$, by [3] the signature of $\Gamma_1$ is

$$(0; +; [2,\overset{2g+k}{\ldots},2], \{(\text{—})\})$$

and $|\Gamma_1 : \ker\theta| = 2$. By Theorem 3.1, $X = D/\ker\theta$ is an orientable HKS with one boundary component. The canonical Fuchsian subgroup $\Gamma^{*+}$ has the signature

$$(p; +; [2,\overset{2(p+1)/N}{\ldots\ldots},2,N,N], \{\text{—}\}),$$

where $2(p+1)/N \geq 3$. By means of a similar argument to one of part (a), we may conclude that $Z_{2N}$ is the automorphism group of $X$.

If $N$ divides $p$ and $N \neq p$, let $\theta$ be the epimorphism from $\Gamma^*$ onto $Z_{2N}$ defined by:

$$\theta(x_i) = \bar{N}, \quad i = 1, \ldots, p/N = (2g+k-1)/N;$$

$$\theta(x_{p/N+1}) = \bar{u}, \quad (\bar{u} \text{ an element of order } 2N);$$

$$\theta(e) = -\theta(x_1, \ldots, x_{p/N+1});$$

$$\theta(c) = \bar{0}.$$

Then $X = D/\ker\theta$ is an orientable HKS with two boundary components whose automorphism group is $Z_{2N}$.

(c) By (2.1), the signature of $\Gamma$ is $(g; -; [\text{—}], \{(\text{—})^k\})$.

From Theorem 3.1(iii), there exists an NEC group $\Gamma_1$ with $\Gamma \lhd \Gamma_1 \lhd \Gamma^*$ and $|\Gamma_1 : \Gamma| = 2$ whose signature is

$$(0; +; [2,\overset{g}{\ldots},2], \{(2,\overset{2k}{\ldots},2)\}).$$

Since $\Gamma^*/\Gamma_1 \simeq Z_N$, the signature of $\Gamma^*$ is, by [3],

$$(g^*; \pm; [m_1, \ldots, m_r], \{(n_1, \ldots, n_s)\}),$$

where $m_i = 2s$ for $i = 1, \ldots, h$ and $m_i = s_i$ for $i = 1, \ldots, r$. The relation between areas of $\Gamma_1$ and $\Gamma^*$ yields $g = 0$, $s_i = 1$ for $i = 1, \ldots, r-1$ and $s_r = N$. Therefore, the signature of $\Gamma^*$ is one of the following signatures:

$$(0; +; [2,\overset{g/N}{\ldots},2,N], \{(2,\overset{2k/N}{\ldots},2)\}) \tag{3.6}$$

or

$$(0; +; [2,\overset{(g-1)/N}{\ldots\ldots},2,2N], \{(2,\overset{2k}{\ldots},2)\}). \tag{3.7}$$

In the first case $N$ must divide $g$ and $k$ and hence $p+1$. In the second case $N$ must divide $g-1$ and $k$ and hence $p$.

Now let $N$ be an odd number dividing $k$. Let us also suppose $N$ divides $g$ or $g-1$. Let $\Gamma^*$ be an NEC group with signature (3.6) or (3.7), respectively. In the first case, let $\theta_1$ be the epimorphism from $\Gamma^*$ onto $\mathbf{Z}_{2N}$ defined by:

$$\theta_1(x_i) = \bar{N}, \quad i = 1, \ldots, g/N;$$

$$\theta_1(x_{g/N+1}) = \bar{u}, \quad (\bar{u} \text{ an element of order } N);$$

$$\theta_1(e) = -\theta_1(x_1, \ldots, x_{g/N+1});$$

$$\theta_1(c_0)=\theta_1(c_i)=\bar{N} \quad \text{for } i \text{ even,}$$
$$\theta_1(c_i)=\bar{0} \quad \text{for } i \text{ odd.}$$

In the second case let $\theta_2$ be the epimorphism from $\Gamma^*$ onto $Z_{2N}$:

$$\theta_2(x_i)=\bar{N},$$
$$\theta_2(x_{(g-1)/N+1})=\bar{u},$$
$$\theta_2(e)=-\theta_2(x_1,\dots,x_{(g-1)/N+1}),$$

and $\theta_2(c_i)$ as $\theta_1(c_i)$.

In a manner similar to parts (a) and (b), we may deduce that $X=D/\ker\theta_j$, $j=1$ or 2, is a non-orientable HKS with boundary whose automorphism group is $Z_{2N}$, and so we finish the proof of the Theorem. □

Let $X$ be an HKS with boundary of algebraic genus $p$, with $G$ its group of automorphisms. Let $G^+$ be a maximal subgroup of $G$ with order $N$ ($N$ odd). In these conditions we have the following corollary,

COROLLARY 3.4.
(i) $G^+\simeq Z_N$.
(ii) *$N$ divides $p+1$ if $X$ is planar.*
(iii) *$N$ divides $p+1$ or $N$ divides $p$ if $X$ is orientable with one or two boundary components, respectively.*
(iv) *$N$ divides $g$ and $k$ or $N$ divides $g-1$ and $k$ if $X$ is non-orientable, where $g$ is the topological genus and $k$ the number of boundary components of $X$.*

REMARK. The results on bordered HKS may be expressed in terms of real hyperelliptic algebraic curves (see [1] and [6]), so that we may translate the results of Section 3 obtaining conditions for the hyperelliptic real curves with automorphism group of order $2N$ for $N$ odd.

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