THE VARIATION OF HOLOMORPHIC FUNCTIONS ON TANGENTIAL BOUNDARY CURVES

Walter Rudin

To George Piranian, on the occasion of his retirement

In this paper we study the total variation of holomorphic functions belonging to certain Hilbert spaces D_β on certain families of curves Γ in the open unit disc $\mathbf{U}$. In other words, we study the rectifiability of the image curves $f\circ\Gamma$.

For $0\le\beta\le 1$, D_β *denotes the space of all* $f(z)=\Sigma_0^\infty a_n z^n$ *whose coefficients satisfy*

$$\sum_{n=1}^{\infty} n^{2(1-\beta)}|a_n|^2<\infty. \tag{1}$$

Thus $D_1=H^2$, $D_{1/2}$ is the Dirichlet space characterized by $\int_U |f'|^2<\infty$, and $f\in D_0$ if and only if $f'\in H^2$.

When $\beta>0$, then $f\in D_\beta$ if and only if

$$f(z)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{F(\theta)\,d\theta}{(1-e^{-i\theta}z)^\beta} \tag{2}$$

for some $F\in L^2(-\pi,\pi)$. This follows easily from the binomial expansion

$$(1-e^{-i\theta}z)^{-\beta}=\sum_{n=0}^{\infty} b_n z^n e^{-in\theta}$$

in which $b_n=\Gamma(n+\beta)/\Gamma(\beta)\Gamma(n+1)\sim n^{\beta-1}$.

Formula (2) represents f as a certain potential of $F\in L^2$ (although the kernel is not positive). The existence of limits of such potentials, within certain tangential approach regions, was investigated in [2]; the approach curves that we are about to define are essentially the boundaries of some of these approach regions. The present topic is thus closely related to [2], and I wish to thank Alex Nagel and Joel Shapiro for several relevant discussions.

For $1\le\gamma<\infty$ *and* $c>0$, *the approach curves* $\Gamma_{\gamma,c}$ *are defined by*

$$\Gamma_{\gamma,c}(r)=r\exp\{ic(1-r)^{1/\gamma}\}\quad(0\le r<1). \tag{3}$$

Writing $\Gamma_{\gamma,c}(r)=re^{i\theta}$, (3) becomes

$$1-r=(\theta/c)^\gamma \tag{4}$$

so that γ is the *order of contact* between $\Gamma_{\gamma,c}$ and the unit circle $\mathbf{T}$ at $z=1$. In particular, these curves are tangential when $\gamma>1$.

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The total variation of f on $e^{it}\Gamma_{\gamma,c}$ (a rotated copy of $\Gamma_{\gamma,c}$), i.e., the length of the curve $f(e^{it}\Gamma_{\gamma,c})$, will be denoted by $V(f,\gamma,c;t)$. Thus, for holomorphic f,

$$V(f,\gamma,c;t)=\int_0^1 |f'(e^{it}\Gamma_{\gamma,c}(r))|\,|\Gamma'_{\gamma,c}(r)|\,dr. \tag{5}$$

Our results will be stated in terms of a *maximal variation* MV, defined to be

$$\mathrm{MV}(f,\gamma,c_0;t)=\sup_{0<c\le c_0} V(f,\gamma,c;t) \tag{6}$$

and will involve the classical capacities of order α, $0\le\alpha<1$.

Recall that a set $E\subset T$ has "α-capacity 0" provided that $\mu(E)=0$ for every positive Borel measure μ on $\mathbf{T}$ whose "energy integral"

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} H_\alpha(\theta-\varphi)\,d\mu(\theta)\,d\mu(\varphi) \tag{7}$$

is finite, where

$$H_\alpha(\theta)=\begin{cases} |\sin(\theta/2)|^{-\alpha} & \text{if } 0<\alpha<1 \\ -\log|\sin(\theta/2)| & \text{if } \alpha=0. \end{cases} \tag{8}$$

This is not the usual definition of "α-capacity 0" but it is equivalent to it (see [1, Chap. III]) and is precisely what will be useful in the present context. Note that every set of α-capacity 0 has Lebesgue measure 0.

The following theorems show how the product $\beta\gamma$ governs the size of the "exceptional set" where the total variation of an $f\in D_\beta$ can be infinite on curves $\Gamma_{\gamma,c}$. This size varies from empty (when $\beta\gamma<1/2$) to all of $\mathbf{T}$ (when $\beta\gamma\ge 1$).

THEOREM 1. *If* $\beta\gamma<\frac{1}{2}$, $f\in D_\beta$, *and* $c_0<\infty$, *then* $\mathrm{MV}(f,\gamma,c_0;t)$ *is a bounded function of* t *on* $[-\pi,\pi]$.

THEOREM 2. *If* $\frac{1}{2}\le\beta\gamma<1$, $f\in D_\beta$, $c_0<\infty$, *and* $\alpha=2\beta\gamma-1$, *then the set of all* e^{it} *where* $\mathrm{MV}(f,\gamma,c_0;t)=\infty$ *has* α*-capacity* 0.

THEOREM 3. *If* $\beta\gamma=1$ *then there exists an* $f\in D_\beta$ *that has* $V(f,\gamma,c;t)=\infty$ *for every* $c>0$ *and every* $t\in[-\pi,\pi]$.

For example, when f is in the Dirichlet space $D_{1/2}$, the exceptional set has logarithmic capacity 0 when $\gamma=1$ (in which case the approach curves are essentially rectilinear) as was proved by Beurling and Zygmund ([1, p. 49], [3, p. 344]); its possible size increases as γ approaches 2, but still has Lebesgue measure 0 for $\gamma<2$. Theorem 3 shows that the situation changes drastically when $\gamma=2$; some $f\in D_{1/2}$ has infinite variation on every parabolic approach curve.

The proofs of Theorems 1 and 2 will use the following estimate of the total variation of the elementary functions G_β given by

$$G_\beta(z)=(1-z)^{-\beta}\quad (|z|<1,\ 0<\beta<1). \tag{9}$$

LEMMA. *There exists* $A=A(\beta,\gamma,c_0)<\infty$ *so that*

$$\mathrm{MV}(G_\beta,\gamma,c_0;t)\le A|t|^{-\beta\gamma}\quad (-\pi\le t\le\pi). \tag{10}$$

Proof: Write G for G_β, Γ for $\Gamma_{\gamma,c}$. The left side of (10) is the supremum, as c ranges over $(0, c_0)$, of

$$\int_0^1 |G'(e^{it}\Gamma(r))|\,|\Gamma'(r)|\,dr. \tag{11}$$

Since $G'(z) = \beta(1-z)^{-\beta-1}$, the inequality

$$\frac{1-r+|x|}{|1-re^{ix}|} \le 1 + \frac{|x|}{|1-re^{ix}|} \le 1+\pi,$$

valid for $0 \le r < 1$, $-\pi \le x \le \pi$, shows that

$$|G'(e^{it}\Gamma(r))| \le A\{1-r+||t|-c(1-r)^{1/\gamma}|\}^{-\beta-1}$$

where $A = \beta(1+\pi)^{\beta+1}$. Since (3) gives

$$|\Gamma'(r)| \le 1 + \frac{c}{\gamma}(1-r)^{-1+1/\gamma},$$

the change of variables $1-r = x^\gamma$ shows that (11) is not larger than the sum of

$$\mathrm{I} = A\int_0^1 \frac{\gamma x^{\gamma-1}\,dx}{\{x^\gamma + ||t|-cx|\}^{1+\beta}} \quad \text{and} \quad \mathrm{II} = A\int_0^1 \frac{c\,dx}{\{x^\gamma+||t|-cx|\}^{1+\beta}}.$$

It will be enough to obtain upper bounds for I and II for $0 < t < 1/2$.

To estimate I, put $\delta = 1/2c_0$. Then $t - cx > t/2$ if $0 < x < \delta t$ and $0 < c < c_0$, so that

$$\int_0^{\delta t} < \left(\frac{2}{t}\right)^{1+\beta} (t\delta)^\gamma \le 2^{1+\beta}\delta^\gamma t^{-\beta\gamma}$$

because $\gamma - 1 - \beta = (\gamma-1)(\beta+1) - \beta\gamma \ge -\beta\gamma$, and

$$\int_{\delta t}^1 < \gamma\int_{\delta t}^1 x^{\gamma-1-\gamma-\beta\gamma}\,dx < \frac{1}{\beta}(\delta t)^{-\beta\gamma}.$$

To estimate II, put $cx = ts$. This shows that

$$\mathrm{II} < A\int_0^\infty \frac{t\,ds}{\{(ts/c)^\gamma + t|1-s|\}^{1+\beta}}.$$

When $\gamma = 1$, it follows that $\mathrm{II} < A(c_0, \beta)t^{-\beta}$, for $0 < c \le c_0$. When $\gamma > 1$, put $\epsilon = \frac{1}{2}t^{\gamma-1}$, and split the last integral into two parts:

$$\int_{|1-s|>\epsilon} < 2t^{-\beta}\int_\epsilon^\infty \frac{d\sigma}{\sigma^{1+\beta}} = \frac{2^{1+\beta}}{\beta} t^{-\beta\gamma}$$

and

$$\begin{aligned}\int_{|1-s|\le\epsilon} &< c^{\gamma(1+\beta)} t^{1-\gamma-\beta\gamma}\int_{1-\epsilon}^{1+\epsilon} s^{-\gamma(1+\beta)}\,ds \\ &< c^{\gamma(1+\beta)}\cdot t^{1-\gamma-\beta\gamma}\cdot 2\epsilon\cdot(1-\epsilon)^{-\gamma(1+\beta)} \\ &< (2c)^{\gamma(1+\beta)} t^{-\beta\gamma}.\end{aligned}$$

These inequalities prove (10). □

Proof of Theorems 1 *and* 2. Since $D_0 \subset D_\beta$ if $\beta > 0$, we may assume $\beta > 0$ in Theorem 1. Every $f \in D_\beta$ is then related to an $F \in L^2$ as in (2). If $0 < c \leq c_0$, it follows that

$$V(f, \gamma, c; t) \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |F(\theta)| \, \mathrm{MV}(G_\beta, \gamma, c_0; t-\theta) \, d\theta. \tag{12}$$

By (8) and the lemma, (12) implies

$$\mathrm{MV}(f, \gamma, c_0; t) \leq A \int_{-\pi}^{\pi} |F(\theta)| H_{\beta\gamma}(t-\theta) \, d\theta \tag{13}$$

for some $A = A(\beta, \gamma, c_0) < \infty$. When $\beta\gamma < \frac{1}{2}$ then $H_{\beta\gamma} \in L^2$, and the Schwarz inequality shows that the integral in (13) is bounded. This proves Theorem 1.

To prove Theorem 2, it will suffice to show that

$$\int_{-\pi}^{\pi} \mathrm{MV}(f, \gamma, c_0; t) \, d\mu(t) < \infty \tag{14}$$

for every μ for which (7) is finite when $\alpha = 2\beta\gamma - 1$. By (13) and the Schwarz inequality,

$$\int_{-\pi}^{\pi} \mathrm{MV}(f, \gamma, c_0; t) \, d\mu(t) \leq A \int_{-\pi}^{\pi} |F(\theta)| \, d\theta \int_{-\pi}^{\pi} H_{\beta\gamma}(t-\theta) \, d\mu(t)$$

$$\leq A \|F\|_2^2 \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} d\mu(s) \, d\mu(t) \int_{-\pi}^{\pi} H_{\beta\gamma}(t-\theta) H_{\beta\gamma}(\theta - s) \, d\theta.$$

This proves (14), because the innermost integral, the convolution of $H_{\beta\gamma}$ with $H_{\beta\gamma}$, is less than a constant times $H_{2\beta\gamma-1} = H_\alpha$. This follows most easily from the fact that the Fourier coefficients of H_α are $\approx |n|^{\alpha-1}$ [4, p. 186].

REMARK. The preceding proof shows that

$$\int_{-\pi}^{\pi} \mathrm{MV}(f, \gamma, c_0; t) \, dt < \infty \tag{15}$$

if $f \in D_\beta$ and $\beta\gamma < 1$, since (7) is finite with Lebesgue measure in place of μ, for all $\alpha < 1$.

Proof of Theorem 3. We assume now that $\beta\gamma = 1$. If $\{n_k\}$ is an increasing sequence of positive integers, put

$$a_k = k^{-1} n_k^{\beta-1}, \quad f(z) = \sum_{k=1}^{\infty} a_k z^{n_k}.$$

Then $f \in D_\beta$, and

$$zf'(z) = \sum_{k=1}^{\infty} a_k n_k z^{n_k} = \sum_{k=1}^{\infty} k^{-1} n_k^{\beta} z^{n_k}.$$

If $\{n_k\}$ increases sufficiently rapidly, familiar estimates show that the last power series is dominated in the annulus

$$\Omega_k = \left\{ z : 1 - \frac{1}{n_k} < |z| < 1 - \frac{1}{2n_k} \right\}$$

by its kth term; to be more precise, one can choose $\{n_k\}$ so that

$$|f'(z)| > \frac{a_k n_k}{10} = \frac{n_k^\beta}{10k} \quad (z \in \Omega_k). \tag{16}$$

The length of $\Gamma_{\gamma,c} \cap \Omega_k$ exceeds

$$A \cdot \left(\frac{1}{n_k} \right)^{1/\gamma} = A n_k^{-\beta} \tag{17}$$

since $\beta\gamma = 1$, where $A > 0$ depends on γ and c.

It follows from (16) and (17) that the part of $e^{it}\Gamma_{\gamma,c}$ in Ω_k contributes more than $A/(10k)$ to $V(f, \gamma, c; t)$. Since $\sum 1/k = \infty$, we have $V(f, \gamma, c; t) = \infty$.

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Department of Mathematics
University of Wisconsin-Madison
Madison, Wisconsin 53706