ON THE STRONG SUMMABILITY OF DOUBLE ORTHOGONAL SERIES

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1. Introduction. Let (X, F, μ) be an arbitrary positive measure space and let $\{\varphi_{ik}(x): i, k=1,2,\dots\}$ be an orthonormal system on X. We consider the double orthogonal series

$$\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} a_{ik} \varphi_{ik}(x), \tag{1.1}$$

where $\{a_{ik}: i, k=1,2,\dots\}$ is a sequence of coefficients for which

$$\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} a_{ik}^2 < \infty. \tag{1.2}$$

Then, by the well-known Riesz–Fischer theorem, there exists a function $f(x) \in L^2 = L^2(X, F, \mu)$ such that the rectangular partial sums

$$s_{mn}(x) = \sum_{i=1}^{m} \sum_{k=1}^{n} a_{ik} \varphi_{ik}(x) \quad (m, n=1,2,\dots)$$

of series (1.1) converge to $f(x)$ in the L^2-metric:

$$\int [s_{mn}(x) - f(x)]^2 \, d\mu(x) \to 0 \quad \text{as } \min(m,n) \to \infty.$$

Here and in the sequel, the integrals are taken over the whole space X.

It is a basic fact that condition (1.2) itself does not ensure the pointwise convergence of $s_{mn}(x)$ to $f(x)$. The extension of the famous Rademacher–Menšov theorem proved by a number of authors (see e.g. [1], [7], etc.) reads as follows.

THEOREM A. *If*

$$\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} a_{ik}^2 [\log(i+1)]^2 [\log(k+1)]^2 < \infty,$$

then

$$s_{mn}(x) \to f(x) \quad \textit{a.e. as } \min(m,n) \to \infty$$

and there exists a function $F(x) \in L^2$ *such that*

$$\sup_{m,n \geq 1} |s_{mn}(x)| \leq F(x) \quad \textit{a.e.}$$

In this paper all logarithms are to the base 2.

The next theorem (see e.g. [8]) gives information on the order of magnitude of $s_{mn}(x)$ in the more general setting of (1.2).

THEOREM B. *Under condition* (1.2),

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$$\frac{s_{mn}(x)}{\log(m+1)\log(n+1)} \to 0 \quad a.e.\ as\ \max(m,n)\to\infty \tag{1.3}$$

and there exists a function $F(x)\in L^2$ *such that*

$$\sup_{m,n\geqslant 1} \frac{|s_{mn}(x)|}{\log(m+1)\log(n+1)} \leqslant F(x) \quad a.e.$$

We note that both theorems are exact in the sense that $\log t$ cannot be replaced in them by any sequence $\rho(t)$ tending to ∞ slower than $\log t$ as $t\to\infty$ (cf. [11]).

The convergence properties improve when the first arithmetic means

$$\sigma_{mn}(x)=\frac{1}{mn}\sum_{i=1}^{m}\sum_{k=1}^{n} s_{ik}(x)$$
$$=\sum_{i=1}^{m}\sum_{k=1}^{n}\left(1-\frac{i-1}{m}\right)\left(1-\frac{k-1}{n}\right)a_{ik}\varphi_{ik}(x) \quad (m,n=1,2,\ldots)$$

are considered instead of the rectangular partial sums $s_{mn}(x)$. The following extension of the summation theorem of Menšov and Kaczmarz was proved in [9]. We mention that it was stated earlier in [5] and [4], but the proofs are not complete there.

THEOREM C. *If*

$$\sum_{i=1}^{\infty}\sum_{k=1}^{\infty} a_{ik}^2[\log\log(i+3)]^2[\log\log(k+3)]^2<\infty, \tag{1.4}$$

then

$$\sigma_{mn}(x)\to f(x) \quad a.e.\ as\ \min(m,n)\to\infty \tag{1.5}$$

and there exists a function $F(x)\in L^2$ *such that*

$$\sup_{m,n\geqslant 1}|\sigma_{mn}(x)|\leqslant F(x) \quad a.e.$$

The order of magnitude of $\sigma_{mn}(x)$, under condition (1.2), is also better in general than that of $s_{mn}(x)$. The following theorem was proved in [10].

THEOREM D. *Under condition* (1.2),

$$\frac{\sigma_{mn}(x)}{\log\log(m+3)\log\log(n+3)}\to 0 \quad a.e.\ as\ \max(m,n)\to\infty$$

and there exists a function $F(x)\in L^2$ *such that*

$$\sup_{m,n\geqslant 1}\frac{|\sigma_{mn}(x)|}{\log\log(m+3)\log\log(n+3)}\leqslant F(x) \quad a.e.$$

It was pointed out in [5] and [10] that Theorems C and D are the best possible in the same sense as Theorems A and B are.

2. Main results. Relation (1.5) can be rewritten into the form

$$\sigma_{MN}(x)-f(x)=\frac{1}{MN}\sum_{m=1}^{M}\sum_{n=1}^{N}[s_{mn}(x)-f(x)]\to 0 \quad \text{a.e. as } \min(M,N)\to\infty.$$

The following theorem reveals that the average of $s_{mn}(x)-f(x)$ is small, not because of the cancellation of positive and negative terms, but because the indices (m,n) for which $|s_{mn}(x)-f(x)|$ is large are fairly sparse.

THEOREM 1. *Under condition* (1.4),

$$\frac{1}{MN}\sum_{m=1}^{M}\sum_{n=1}^{N}[s_{mn}(x)-f(x)]^2\to 0 \quad \textit{a.e. as } \min(M,N)\to\infty \tag{2.1}$$

and there exists a function $F(x)\in L^2$ such that

$$\sup_{M,N\geqslant 1}\left\{\frac{1}{MN}\sum_{m=1}^{M}\sum_{n=1}^{N}[s_{mn}(x)-f(x)]^2\right\}^{1/2}\leqslant F(x) \quad \textit{a.e.}$$

This theorem can be considered as an extension of a theorem of Borgen [3] from single orthogonal series to double ones.

Applying the Cauchy inequality, from (2.1) it follows that

$$\frac{1}{MN}\sum_{m=1}^{M}\sum_{n=1}^{N}|s_{mn}(x)-f(x)|\to 0 \quad \text{a.e. as } \min(M,N)\to\infty.$$

In particular, Theorem C is a corollary of Theorem 1.

In the more general case of (1.2) we can only estimate the order of magnitude of the average

$$\delta_{MN}(x)=\left\{\frac{1}{MN}\sum_{m=1}^{M}\sum_{n=1}^{N}s_{mn}^2(x)\right\}^{1/2} \quad (M,N=1,2,\dots).$$

THEOREM 2. *Under condition* (1.2),

$$\frac{\delta_{MN}(x)}{\log\log(M+3)\log\log(N+3)}\to 0 \quad \textit{a.e. as } \max(M,N)\to\infty \tag{2.2}$$

and there exists a function $F(x)\in L^2$ such that

$$\sup_{M,N\geqslant 1}\frac{\delta_{MN}(x)}{\log\log(M+3)\log\log(N+3)}\leqslant F(x) \quad \textit{a.e.}$$

Estimate (2.2) is very unexpected in comparison with that which follows from (1.3).

Theorem 1 will be obtained as the consequence of Theorem C and the following.

THEOREM 3. *Under condition* (1.4),

$$\epsilon_{MN}(x)=\left\{\frac{1}{MN}\sum_{m=1}^{M}\sum_{n=1}^{N}[s_{mn}(x)-\sigma_{mn}(x)]^2\right\}^{1/2}\to 0 \quad \textit{a.e.} \tag{2.3}$$

$$\textit{as } \min(M,N)\to\infty$$

and there exists a function $F(x)\in L^2$ such that

$$\sup_{M,N\geqslant 1} \epsilon_{MN}(x) \leqslant F(x) \quad a.e.$$

Theorem 2 will be directly proved. However, Theorem 2 could be also derived from Theorem D and the following.

THEOREM 4. *Under condition* (1.2),

$$\frac{\epsilon_{MN}(x)}{\log\log(M+3)\log\log(N+3)} \to 0 \quad a.e. \text{ as } \max(M,N) \to \infty$$

and there exists a function $F(x) \in L^2$ *such that*

$$\sup_{M,N\geqslant 1} \frac{\epsilon_{MN}(x)}{\log\log(M+3)\log\log(N+3)} \leqslant F(x) \quad a.e.$$

On the other hand, Theorem 4 immediately follows from Theorem D and Theorem 2.

Before proving Theorems 2 and 3 we make an agreement in notation for the sake of brevity. Given a sequence $\{f_p(x)\} \subset L^2$ of functions and a sequence $\{\lambda(p)\}$ of positive numbers, we write

$$f_p(x) = o_x\{\lambda(p)\} \quad \text{a.e.}$$

if

$$f_p(x)/\lambda(p) \to 0 \quad \text{a.e. as } p \to \infty,$$

and there exists a function $F(x) \in L^2$ such that

$$\sup_p |f_p(x)|/\lambda(p) \leqslant F(x) \quad \text{a.e.}$$

Here p runs over either $0, 1, \ldots$ or $1, 2, \ldots$. The same convention is made for the case of a double sequence $\{f_{pq}(x)\} \subset L^2$ of functions and a double sequence $\{\lambda(p,q)\}$ of positive numbers:

$$f_{pq}(x) = o_x\{\lambda(p,q)\} \quad \text{a.e. as } \max(p,q) \to \infty \quad (\text{or as } \min(p,q) \to \infty)$$

means that

$$f_{pq}(x)/\lambda(p,q) \to 0 \quad \text{a.e. as } \max(p,q) \to \infty \quad (\text{or as } \min(p,q) \to \infty),$$

and there exists a function $F(x) \in L^2$ such that

$$\sup_{p,q} |f_{pq}(x)|/\lambda(p,q) \leqslant F(x) \quad \text{a.e.}$$

3. Proof of Theorem 2. We begin with a few simple observations.

(i) Let $2^{p-1} < M \leqslant 2^p$ and $2^{q-1} < N \leqslant 2^q$ with some $p, q \geqslant 0$. Then clearly

$$\delta_{MN}(x) \leqslant 2\delta_{2^p,2^q}(x).$$

Also, it is enough to prove that

$$\delta_{2^p,2^q}(x) = o_x\{\log(p+2)\log(q+2)\} \quad \text{a.e. as } \max(p,q) \to \infty.$$

(ii) On the basis of the representation

$$[\delta_{2^p,2^q}(x)]^2=\sum_{r=0}^{p}\sum_{t=0}^{q}2^{r+t-p-q}\frac{1}{2^r2^t}\sum_{m=2^{r-1}+1}^{2^r}\sum_{n=2^{t-1}+1}^{2^t}s_{mn}^2(x),$$

with the agreement that for $r=0$ we take $2^{r-1}+1$ to equal 1 and similarly for $t=0$ we take $2^{t-1}+1$ to equal 1, we can make one more reduction: If we prove that

$$\left\{\frac{1}{2^p2^q}\sum_{m=2^{p-1}+1}^{2^p}\sum_{n=2^{q-1}+1}^{2^q}s_{mn}^2(x)\right\}^{1/2}$$
$$=o_x\{\log(p+2)\log(q+2)\}\quad \text{a.e. as } \max(p,q)\to\infty,$$

then we are done.

(iii) In the following we neglect the cases $p=0$ or $q=0$, only because of the simplicity in notation, and instead we deal with the quantity

$$\delta^{(1)}_{2^p,2^q}(x)=\left\{\frac{1}{2^{p-1}2^{q-1}}\sum_{m=2^{p-1}+1}^{2^p}\sum_{n=2^{q-1}+1}^{2^q}s_{mn}^2(x)\right\}^{1/2}\quad (p,q=1,2,\dots).$$

By the triangle inequality,

$$\begin{aligned}\delta^{(1)}_{2^p,2^q}(x)&\leqslant|s_{2^{p-1},2^{q-1}}(x)|\\&+\left\{\frac{1}{2^{p-1}}\sum_{m=2^{p-1}+1}^{2^p}[s_{m,2^{q-1}}(x)-s_{2^{p-1},2^{q-1}}(x)]^2\right\}^{1/2}\\&+\left\{\frac{1}{2^{q-1}}\sum_{n=2^{q-1}+1}^{2^q}[s_{2^{p-1},n}(x)-s_{2^{p-1},2^{q-1}}(x)]^2\right\}^{1/2}\\&+\left\{\frac{1}{2^{p-1}2^{q-1}}\right.\\&\left.\times\sum_{m=2^{p-1}+1}^{2^p}\sum_{n=2^{q-1}+1}^{2^q}[s_{mn}(x)-s_{m,2^{q-1}}(x)-s_{2^{p-1},n}(x)+s_{2^{p-1},2^{q-1}}(x)]^2\right\}^{1/2}.\end{aligned}$$

Now, Theorem 2 will be completely proved if we take into consideration the following Lemmas 1, 3, 4 and 5.

LEMMA 1. *Under condition* (1.2),

(3.1) $\quad s_{2^{p-1},2^{q-1}}(x)=o_x\{\log(p+1)\log(q+1)\}$ *a.e. as* $\max(p,q)\to\infty$.

Proof. This is an easy consequence of Theorem B. To this effect, we set

$$A_{rt}=\left\{\sum_{i=2^{r-1}+1}^{2^r}\sum_{k=2^{t-1}+1}^{2^t}a_{ik}^2\right\}^{1/2}\quad (r,t=0,1,\dots)$$

and

$$\Phi_{rt}(x)=\begin{cases}\dfrac{1}{A_{rt}}\displaystyle\sum_{i=2^{r-1}+1}^{2^r}\sum_{k=2^{t-1}+1}^{2^t}a_{ik}\varphi_{ik}(x) & \text{if } A_{rt}\neq 0,\\ \varphi_{2^r,2^t}(x) & \text{if } A_{rt}=0,\end{cases}$$

where we agree again that for $r=0$ we take $2^{r-1}+1$ to equal 1, etc.

It is obvious that $\{\Phi_{rt}(x): r, t=0, 1, \dots\}$ is an ONS and by (1.2)

$$\sum_{r=0}^{\infty} \sum_{t=0}^{\infty} A_{rt}^2 < \infty.$$

Thus we can apply Theorem B which results

$$S_{p-1,q-1}(x) = o_x\{\log(p+1)\log(q+1)\} \quad \text{a.e. as } \max(p,q)\to\infty, \tag{3.2}$$

where

$$S_{p-1,q-1}(x) = \sum_{r=0}^{p-1} \sum_{t=0}^{q-1} A_{rt}\Phi_{rt}(x) = s_{2^{p-1},2^{q-1}}(x).$$

That is, (3.2) is equivalent to (3.1) to be proved.

LEMMA 2. *Under condition* (1.2),

$$\delta_{2^p,n}^{(2)}(x) = \left\{\frac{1}{2^{p-1}} \sum_{m=2^{p-1}+1}^{2^p} \left[\sum_{i=2^{p-1}+1}^{m} \sum_{k=1}^{n} a_{ik}\varphi_{ik}(x)\right]^2\right\}^{1/2} \tag{3.3}$$

$$= o_x\{\log(n+1)\} \quad \textit{a.e. as } \max(p,n)\to\infty.$$

Proof. (i) First we prove (3.3) for the special case $n=2^q$:

$$\delta_{2^p,2^q}^{(2)}(x) = o_x\{q\} \quad \text{a.e. as } \max(p,q)\to\infty. \tag{3.4}$$

Indeed, by the Cauchy inequality

$$[\delta_{2^p,2^q}^{(2)}(x)]^2 \leqslant \frac{q+1}{2^{p-1}} \sum_{m=2^{p-1}+1}^{2^p} \sum_{t=0}^{q} \left[\sum_{i=2^{p-1}+1}^{m} \sum_{k=2^{t-1}+1}^{2^t} a_{ik}\varphi_{ik}(x)\right]^2.$$

This inequality motivates the following definition:

$$F_2^2(x) = \sum_{p=1}^{\infty} \sum_{t=0}^{\infty} \frac{1}{2^{p-1}} \sum_{m=2^{p-1}+1}^{2^p} \left[\sum_{i=2^{p-1}+1}^{m} \sum_{k=2^{t-1}+1}^{2^t} a_{ik}\varphi_{ik}(x)\right]^2.$$

If we prove that $F_2(x)\in L^2$, then Levi's theorem implies (3.4). Now, the termwise integration gives that

$$\int F_2^2(x)\,d\mu(x) = \sum_{p=1}^{\infty} \sum_{t=0}^{\infty} \frac{1}{2^{p-1}} \sum_{m=2^{p-1}+1}^{2^p} \sum_{i=2^{p-1}+1}^{m} \sum_{k=2^{t-1}+1}^{2^t} a_{ik}^2$$

$$= \sum_{p=1}^{\infty} \sum_{t=0}^{\infty} \frac{1}{2^{p-1}} \sum_{i=2^{p-1}+1}^{2^p} \sum_{k=2^{t-1}+1}^{2^t} (2^p-i+1)a_{ik}^2$$

$$\leqslant \sum_{p=1}^{\infty} \sum_{t=0}^{\infty} \sum_{i=2^{p-1}+1}^{2^p} \sum_{k=2^{t-1}+1}^{2^t} a_{ik}^2 = \sum_{i=2}^{\infty} \sum_{k=1}^{\infty} a_{ik}^2 < \infty.$$

(ii) Let $2^{q-1}<n\leqslant 2^q$ $(q\geqslant 1)$. Then

$$\delta_{2^p,n}^{(2)}(x) \leqslant \delta_{2^p,2^{q-1}}^{(2)}(x) + \left\{\frac{1}{2^{p-1}} \sum_{m=2^{p-1}+1}^{2^p} \left[\sum_{i=2^{p-1}+1}^{m} \sum_{k=2^{q-1}+1}^{n} a_{ik}\varphi_{ik}(x)\right]^2\right\}^{1/2},$$

whence

$$\max_{2^{q-1}<n\leqslant 2^q} \delta^{(2)}_{2^p,n}(x) \leqslant \delta^{(2)}_{2^p,2^{q-1}}(x) + M^{(2)}_{pq}(x), \tag{3.5}$$

where

$$M^{(2)}_{pq}(x) = \left\{ \frac{1}{2^{p-1}} \sum_{m=2^{p-1}+1}^{2^p} \max_{2^{q-1}<n\leqslant 2^q} \left[\sum_{i=2^{p-1}+1}^{m} \sum_{k=2^{q-1}+1}^{n} a_{ik}\varphi_{ik}(x) \right]^2 \right\}^{1/2}.$$

Applying the Rademacher–Menšov inequality (see e.g. [2, p. 79] or [6, Theorem 3]) separately for each fixed m:

$$\int [M^{(2)}_{pq}(x)]^2 d\mu(x) \leqslant \frac{1}{2^{p-1}} \sum_{m=2^{p-1}+1}^{2^p} (\log 2^q)^2 \sum_{i=2^{p-1}+1}^{m} \sum_{k=2^{q-1}+1}^{2^q} a^2_{ik}$$
$$\leqslant q^2 \sum_{i=2^{p-1}+1}^{2^p} \sum_{k=2^{q-1}+1}^{2^q} a^2_{ik}.$$

Consequently,

$$\sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \frac{1}{q^2} \int [M^{(2)}_{pq}(x)]^2 d\mu(x) \leqslant \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \sum_{i=2^{p-1}+1}^{2^p} \sum_{k=2^{q-1}+1}^{2^q} a^2_{ik} = \sum_{i=2}^{\infty} \sum_{k=2}^{\infty} a^2_{ik} < \infty.$$

Hence Levi's theorem implies that

$$M^{(2)}_{pq}(x) = o_x\{q\} \quad \text{a.e. as } \max(p,q) \to \infty. \tag{3.6}$$

Combining (3.4), (3.5) and (3.6) we obtain (3.3).

LEMMA 3. *Under condition* (1.2),

$$\begin{aligned} \delta^{(2)}_{2^p,2^{q-1}}(x) &= \left\{ \frac{1}{2^{p-1}} \sum_{m=2^{p-1}+1}^{2^p} [s_{m,2^{q-1}}(x) - s_{2^{p-1},2^{q-1}}(x)]^2 \right\}^{1/2} \\ &= o_x\{\log(q+1)\} \quad \textit{a.e. as } \max(p,q) \to \infty. \end{aligned} \tag{3.7}$$

Proof. We set

$$A_{ir} = \left\{ \sum_{k=2^{r-1}+1}^{2^r} a^2_{ik} \right\}^{1/2} \quad (i=1,2,\ldots;\ r=0,1,\ldots) \tag{3.8}$$

and

$$\Phi_{ir}(x) = \begin{cases} \dfrac{1}{A_{ir}} \displaystyle\sum_{k=2^{r-1}+1}^{2^r} a_{ik}\varphi_{ik}(x) & \text{if } A_{ir} \neq 0, \\ \varphi_{i,2^r}(x) & \text{if } A_{ir} = 0. \end{cases} \tag{3.9}$$

Obviously, $\{\Phi_{ir}(x) : i=1,2,\ldots;\ r=0,1,\ldots\}$ is an ONS and by (1.2)

$$\sum_{i=1}^{\infty} \sum_{r=0}^{\infty} A^2_{ir} < \infty.$$

Thus we can apply Lemma 2 and obtain

$$\Delta^{(2)}_{2^p,q-1}(x) = o_x\{\log(q+1)\} \quad \text{a.e. as } \max(p,q) \to \infty, \tag{3.10}$$

where

$$\Delta^{(2)}_{2^p,q-1}(x)=\left\{\frac{1}{2^{p-1}}\sum_{m=2^{p-1}+1}^{2^p}\left[\sum_{i=2^{p-1}+1}^{m}\sum_{r=0}^{q}A_{ir}\Phi_{ir}(x)\right]^2\right\}^{1/2}=\delta^{(2)}_{2^p,2^{q-1}}(x).$$

By this (3.10) is equivalent to (3.7) to be proved.

In the same way we can prove the symmetric counterpart of Lemma 3 expressed in the following.

LEMMA 4. *Under condition* (1.2),

$$\left\{\frac{1}{2^{q-1}}\sum_{n=2^{q-1}+1}^{2^q}[s_{2^{p-1},n}(x)-s_{2^{p-1},2^{q-1}}(x)]^2\right\}^{1/2}$$
$$=o_x\{\log(p+1)\}\quad a.e.\ as\ \max(p,q)\to\infty.$$

Finally, we still need the following.

LEMMA 5. *Under condition* (1.2),

$$\delta^{(5)}_{2^p,2^q}(x)=\left\{\frac{1}{2^{p-1}2^{q-1}}\right.$$
$$\left.\times\sum_{m=2^{p-1}+1}^{2^p}\sum_{n=2^{q-1}+1}^{2^q}[s_{mn}(x)-s_{m,2^{q-1}}(x)-s_{2^{p-1},n}(x)+s_{2^{p-1},2^{q-1}}(x)]^2\right\}^{1/2}$$
$$=o_x\{1\}\quad a.e.\ as\ \max(p,q)\to\infty.$$

Proof. Considering

$$s_{mn}(x)-s_{m,2^{q-1}}(x)-s_{2^{p-1},n}(x)+s_{2^{p-1},2^{q-1}}(x)=\sum_{i=2^{p-1}+1}^{2^p}\sum_{k=2^{q-1}+1}^{2^q}a_{ik}\varphi_{ik}(x),$$

we can easily calculate that

$$\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\int[\delta^{(5)}_{2^p,2^q}(x)]^2\,d\mu(x)$$
$$=\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\frac{1}{2^{p-1}2^{q-1}}\sum_{m=2^{p-1}+1}^{2^p}\sum_{n=2^{q-1}+1}^{2^q}\sum_{i=2^{p-1}+1}^{m}\sum_{k=2^{q-1}+1}^{n}a^2_{ik}$$
$$\leqslant\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{i=2^{p-1}+1}^{2^p}\sum_{k=2^{q-1}+1}^{2^q}a^2_{ik}=\sum_{i=2}^{\infty}\sum_{k=2}^{\infty}a^2_{ik}<\infty.$$

It only remains to apply Levi's theorem in order to get the statement in Lemma 5.

4. Proof of Theorem 3. This is similar to the proof of Theorem 2.

(i) Let $2^{p-1}<M\leqslant 2^r$ and $2^{q-1}<N\leqslant 2^q$ $(p,q\geqslant 0)$. Then $\epsilon_{MN}(x)\leqslant 2\epsilon_{2^p,2^q}(x)$. Thus, it is sufficient to show that

$$\epsilon_{2^p,2^q}(x)=o_x\{1\}\quad \text{a.e. as }\min(p,q)\to\infty.$$

(ii) Taking into account the representation

$$[\epsilon_{2^p,2^q}(x)]^2=\sum_{r=0}^{p}\sum_{t=0}^{q}2^{r+t-p-q}\frac{1}{2^r2^t}\sum_{m=2^{r-1}+1}^{2^r}\sum_{n=2^{t-1}+1}^{2^t}[s_{mn}(x)-\sigma_{mn}(x)]^2,$$

if we prove that

$$\epsilon^{(1)}_{2^p,2^q}(x)=\left\{\frac{1}{2^{p-1}2^{q-1}}\sum_{m=2^{p-1}+1}^{2^p}\sum_{n=2^{q-1}+1}^{2^q}[s_{mn}(x)-\sigma_{mn}(x)]^2\right\}^{1/2}$$
$$=o_x\{1\}\quad \text{a.e. as } \min(p,q)\to\infty\quad (p,q\geqslant 1),$$

then (2.3) will also be proved.

(iii) By the triangle inequality,

$$\epsilon^{(1)}_{2^p,2^q}(x)\leqslant|s_{2^{p-1},2^{q-1}}(x)-\sigma_{2^{p-1},2^{q-1}}(x)|$$
$$+\left\{\frac{1}{2^{p-1}}\sum_{m=2^{p-1}+1}^{2^p}[s_{m,2^{q-1}}(x)-\sigma_{m,2^{q-1}}(x)]^2\right\}^{1/2}$$
$$+\left\{\frac{1}{2^{q-1}}\sum_{n=2^{q-1}+1}^{2^q}[s_{2^{p-1},n}(x)-\sigma_{2^{p-1},n}(x)]^2\right\}^{1/2}$$
$$+\left\{\frac{1}{2^{p-1}2^{q-1}}\sum_{m=2^{p-1}+1}^{2^p}\sum_{n=2^{q-1}+1}^{2^q}[s_{mn}(x)-s_{m,2^{q-1}}(x)-s_{2^{p-1},n}(x)+s_{2^{p-1},2^{q-1}}(x)]^2\right\}^{1/2}$$
$$+\left\{\frac{1}{2^{p-1}2^{q-1}}\sum_{m=2^{p-1}+1}^{2^p}\sum_{n=2^{q-1}+1}^{2^q}[\sigma_{mn}(x)-\sigma_{m,2^{q-1}}(x)-\sigma_{2^{p-1},n}(x)+\sigma_{2^{p-1},2^{q-1}}(x)]^2\right\}^{1/2}.$$

The fourth term on the right-hand side is already estimated in Lemma 5. We still have to estimate the other four terms. This will be done in the subsequent Lemmas 6, 11, 12 and 13.

LEMMA 6. *Under condition* (1.4),

$$\sum_{i=1}^{2^{p-1}}\sum_{k=1}^{2^{q-1}}\frac{k-1}{2^{q-1}}a_{ik}\varphi_{ik}(x)=o_x\{1\}\quad \textit{a.e. as } q\to\infty \tag{4.1}$$

uniformly in p, and

$$s_{2^{p-1},2^{q-1}}(x)-\sigma_{2^{p-1},2^{q-1}}(x)=o_x\{1\}\quad \textit{a.e. as } \min(p,q)\to\infty.$$

This lemma is due to Csernyák [4]. (Although (4.1) is proved there in the case when $\min(p,q)\to\infty$.)

LEMMA 7. *Under the condition*

$$\sum_{i=1}^{\infty}\sum_{k=1}^{\infty}a_{ik}^2[\log(k+1)]^2<\infty,$$

we have

$$\epsilon^{(7)}_{2^p,n}(x)=\left\{\frac{1}{2^{p-1}}\sum_{m=2^{p-1}+1}^{2^p}\left[\sum_{i=1}^{m}\sum_{k=1}^{n}\frac{i-1}{m}a_{ik}\varphi_{ik}(x)\right]^2\right\}^{1/2}=o_x\{1\}\quad \textit{a.e. as } p\to\infty$$

uniformly in n.

Proof. (i) First we treat the case $n=2^q$. Using the Cauchy inequality,

$$[\epsilon^{(7)}_{2^p,2^q}(x)]^2$$
$$\leqslant\frac{1}{2^{p-1}}\sum_{m=2^{p-1}+1}^{2^p}\sum_{t=0}^{q}(t+1)^2\left[\sum_{i=1}^{m}\sum_{k=2^{t-1}+1}^{2^t}\frac{i-1}{m}a_{ik}\varphi_{ik}(x)\right]^2\sum_{t=0}^{q}\frac{1}{(t+1)^2}.$$

This is the reason why we set

$$F_7^2(x)=\sum_{p=1}^{\infty}\sum_{t=0}^{\infty}\frac{(t+1)^2}{2^{p-1}}\sum_{m=2^{p-1}+1}^{2^p}\left[\sum_{i=1}^{m}\sum_{k=2^{t-1}+1}^{2^t}\frac{i-1}{m}a_{ik}\varphi_{ik}(x)\right]^2.$$

The termwise integration gives

$$\int F_7^2(x)\,d\mu(x)=\sum_{p=1}^{\infty}\sum_{t=0}^{\infty}\frac{(t+1)^2}{2^{p-1}}\sum_{m=2^{p-1}+1}^{2^p}\sum_{i=1}^{m}\sum_{k=2^{t-1}+1}^{2^t}\frac{(i-1)^2}{m^2}a_{ik}^2$$
$$\leqslant\sum_{p=1}^{\infty}\sum_{t=0}^{\infty}(t+1)^2\sum_{i=2}^{2^p}\sum_{k=2^{t-1}+1}^{2^t}\frac{(i-1)^2}{2^{2(p-1)}}a_{ik}^2$$
$$\leqslant\sum_{p=1}^{\infty}\sum_{k=1}^{\infty}\sum_{i=2}^{2^p}\frac{(i-1)^2}{2^{2(p-1)}}(\log 4k)^2a_{ik}^2$$
$$=\sum_{i=2}^{\infty}\sum_{k=1}^{\infty}(i-1)^2(\log 4k)^2a_{ik}^2\sum_{p:2^p\geqslant i}\frac{1}{2^{2(p-1)}}$$
$$\leqslant 16\sum_{i=2}^{\infty}\sum_{k=1}^{\infty}a_{ik}^2(\log 4k)^2<\infty.$$

Hence Levi's theorem implies

$$\epsilon^{(7)}_{2^p,2^q}(x)=o_x\{1\}\quad\text{a.e. as } p\to\infty \tag{4.2}$$

uniformly in q.

(ii) Let $2^{q-1}<n\leqslant 2^q$ $(q\geqslant 1)$. Then

$$\epsilon^{(7)}_{2^p,n}(x)\leqslant\epsilon^{(7)}_{2^p,2^{q-1}}(x)+\left\{\frac{1}{2^{p-1}}\sum_{m=2^{p-1}+1}^{2^p}\left[\sum_{i=1}^{m}\sum_{k=2^{q-1}+1}^{n}\frac{i-1}{m}a_{ik}\varphi_{ik}(x)\right]^2\right\}^{1/2}.$$

Consequently,

$$\max_{2^{q-1}<n\leqslant 2^q}\epsilon^{(7)}_{2^p,n}(x)\leqslant\epsilon^{(7)}_{2^p,2^{q-1}}(x)+M^{(7)}_{pq}(x), \tag{4.3}$$

where

$$M^{(7)}_{pq}(x)=\left\{\frac{1}{2^{p-1}}\sum_{m=2^{p-1}+1}^{2^p}\max_{2^{q-1}<n\leqslant 2^q}\left[\sum_{i=1}^{m}\sum_{k=2^{q-1}+1}^{n}\frac{i-1}{m}a_{ik}\varphi_{ik}(x)\right]^2\right\}^{1/2}.$$

Applying again the Rademacher-Menšov inequality (as in the proof of Lemma 2) separately for each fixed m:

$$\int [M_{pq}^{(7)}(x)]^2 d\mu(x) \leqslant \frac{1}{2^{p-1}} \sum_{m=2^{p-1}+1}^{2^p} (\log 2^q)^2 \sum_{i=1}^{m} \sum_{k=2^{q-1}+1}^{2^q} \frac{(i-1)^2}{m^2} a_{ik}^2$$
$$\leqslant (\log 2^q)^2 \sum_{i=1}^{2^p} \sum_{k=2^{q-1}+1}^{2^q} \frac{(i-1)^2}{2^{2(p-1)}} a_{ik}^2$$
$$\leqslant \sum_{i=2}^{2^p} \sum_{k=2^{q-1}+1}^{2^q} \frac{(i-1)^2}{2^{2(p-1)}} (\log 2k)^2 a_{ik}^2.$$

Forming the termwise integrated series, we can easily see that

$$\sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \int [M_{pq}^{(7)}(x)]^2 d\mu(x) \leqslant \sum_{p=1}^{\infty} \sum_{k=2}^{\infty} \sum_{i=2}^{2^p} \frac{(i-1)^2}{2^{2(p-1)}} (\log 2k)^2 a_{ik}^2$$
$$= \sum_{i=2}^{\infty} \sum_{k=2}^{\infty} (i-1)^2 (\log 2k)^2 a_{ik}^2 \sum_{p:\, 2^p \geqslant i} \frac{1}{2^{2(p-1)}}$$
$$\leqslant 16 \sum_{i=2}^{\infty} \sum_{k=2}^{\infty} a_{ik}^2 (\log 2k)^2 < \infty.$$

Using Levi's theorem we find that

$$M_{pq}^{(7)}(x) = o_x\{1\} \quad \text{a.e. as } \max(p,q) \to \infty. \tag{4.4}$$

Putting (4.2)-(4.4) together, we can conclude the statement of Lemma 7.

LEMMA 8. *Under the condition*

$$\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} a_{ik}^2 [\log\log(k+3)]^2 < \infty, \tag{4.5}$$

we have

$$\epsilon_{2^p,2^{q-1}}^{(7)}(x) = \left\{ \frac{1}{2^{p-1}} \sum_{m=2^{p-1}+1}^{2^p} \left[\sum_{i=1}^{m} \sum_{k=1}^{2^{q-1}} \frac{i-1}{m} a_{ik} \varphi_{ik}(x) \right]^2 \right\}^{1/2} = o_x\{1\} \quad \text{a.e. as } p \to \infty \tag{4.6}$$

uniformly in q.

Proof. We introduce the same coefficients A_{ir} and orthonormal functions $\Phi_{ir}(x)$ as in the proof of Lemma 3 defined by (3.8) and (3.9). On the one hand, by (4.5)

$$\sum_{i=1}^{\infty} \sum_{r=0}^{\infty} A_{ir}^2 [\log(r+2)]^2 < \infty.$$

On the other hand,

$$\mathcal{E}_{2^p,q-1}^{(7)}(x) = \left\{ \frac{1}{2^{p-1}} \sum_{m=2^{p-1}+1}^{2^p} \left[\sum_{i=1}^{m} \sum_{r=0}^{q-1} \frac{i-1}{m} A_{ir} \Phi_{ir}(x) \right]^2 \right\}^{1/2} = \epsilon_{2^p,2^{q-1}}^{(7)}(x).$$

By Lemma 7, we have

$$\mathcal{E}^{(7)}_{2^p,q-1}(x)=o_x\{1\} \quad \text{a.e. as } p\to\infty$$

uniformly in q. This is equivalent to (4.6) to be proved.

LEMMA 9. *Under condition* (1.4),

$$\begin{aligned}\epsilon^{(9)}_{2^p,2^{q-1}}(x)&=\left\{\frac{1}{2^{p-1}}\sum_{m=2^{p-1}+1}^{2^p}\left[\sum_{i=1}^{m}\sum_{k=1}^{2^{q-1}}\frac{k-1}{2^{q-1}}a_{ik}\varphi_{ik}(x)\right]^2\right\}^{1/2}\\ &=o_x\{1\} \quad \textit{a.e. as } q\to\infty\end{aligned}\tag{4.7}$$

uniformly in p.

Proof. We can estimate as follows:

$$\epsilon^{(9)}_{2^p,2^{q-1}}(x)\leqslant\left|\sum_{i=1}^{2^{p-1}}\sum_{k=1}^{2^{q-1}}\frac{k-1}{2^{q-1}}a_{ik}\varphi_{ik}(x)\right|+M^{(9)}_{pq}(x),\tag{4.8}$$

where

$$M^{(9)}_{pq}(x)=\left\{\frac{1}{2^{p-1}}\sum_{m=2^{p-1}+1}^{2^p}\left[\sum_{i=2^{p-1}+1}^{m}\sum_{k=1}^{2^{q-1}}\frac{k-1}{2^{q-1}}a_{ik}\varphi_{ik}(x)\right]^2\right\}^{1/2}.$$

A simple calculation gives

$$\begin{aligned}\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\int[M^{(9)}_{pq}(x)]^2\,d\mu(x)&=\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\frac{1}{2^{p-1}}\sum_{m=2^{p-1}+1}^{2^p}\sum_{i=2^{p-1}+1}^{m}\sum_{k=1}^{2^{q-1}}\frac{(k-1)^2}{2^{2(q-1)}}a_{ik}^2\\ &\leqslant\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{i=2^{p-1}+1}^{2^p}\sum_{k=1}^{2^{q-1}}\frac{(k-1)^2}{2^{2(q-1)}}a_{ik}^2\\ &=\sum_{i=2}^{\infty}\sum_{q=1}^{\infty}\sum_{k=2}^{2^{q-1}}\frac{(k-1)^2}{2^{2(q-1)}}a_{ik}^2\\ &=\sum_{i=2}^{\infty}\sum_{k=2}^{\infty}(k-1)^2a_{ik}^2\sum_{q:2^{q-1}\geqslant k}\frac{1}{2^{2(q-1)}}\\ &\leqslant 4\sum_{i=2}^{\infty}\sum_{k=2}^{\infty}a_{ik}^2<\infty.\end{aligned}$$

Hence Levi's theorem yields

$$M^{(9)}_{pq}(x)=o_x\{1\} \quad \text{a.e. as } \max(p,q)\to\infty.\tag{4.9}$$

Now, the combination of (4.1), (4.8) and (4.9) gives us (4.7).

LEMMA 10. *Under condition* (1.2),

$$\begin{aligned}\epsilon^{(10)}_{2^p,2^{q-1}}(x)&=\left\{\frac{1}{2^{p-1}}\sum_{m=2^{p-1}+1}^{2^p}\left[\sum_{i=1}^{m}\sum_{k=1}^{2^{q-1}}\frac{(i-1)(k-1)}{m2^{q-1}}a_{ik}\varphi_{ik}(x)\right]^2\right\}^{1/2}\\ &=o_x\{1\} \quad \textit{a.e. as } \max(p,q)\to\infty.\end{aligned}\tag{4.10}$$

Proof. We apply Levi's theorem to prove (4.10), since

$$\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\int[\epsilon^{(10)}_{2^p,2^{q-1}}(x)]^2\,d\mu(x)=\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\frac{1}{2^{p-1}}\sum_{m=2^{p-1}+1}^{2^p}\sum_{i=1}^{m}\sum_{k=1}^{2^{q-1}}\frac{(i-1)^2(k-1)^2}{m^2 2^{2(q-1)}}a_{ik}^2$$

$$\leqslant\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{i=2}^{2^p}\sum_{k=2}^{2^{q-1}}\frac{(i-1)^2(k-1)^2}{2^{2(p-1)}2^{2(q-1)}}a_{ik}^2$$

$$=\sum_{i=2}^{\infty}\sum_{k=2}^{\infty}(i-1)^2(k-1)^2$$

$$\times a_{ik}^2\sum_{p:2^p\geqslant i}\frac{1}{2^{2(p-1)}}\sum_{q:2^{q-1}\geqslant k}\frac{1}{2^{2(q-1)}}$$

$$\leqslant 64\sum_{i=2}^{\infty}\sum_{k=2}^{\infty}a_{ik}^2<\infty.$$

LEMMA 11. *Under condition* (1.4),

$$\epsilon^{(11)}_{2^p,2^{q-1}}(x)=\left\{\frac{1}{2^{p-1}}\sum_{m=2^{p-1}+1}^{2^p}[s_{m,2^{q-1}}(x)-\sigma_{m,2^{q-1}}(x)]^2\right\}^{1/2}$$
$$=o_x\{1\}\quad \textit{a.e. as } \min(p,q)\to\infty.$$

Proof. It is based on the representation

$$s_{m,2^{q-1}}(x)-\sigma_{m,2^{q-1}}(x)=\sum_{i=1}^{m}\sum_{k=1}^{2^{q-1}}\left(\frac{i-1}{m}+\frac{k-1}{2^{q-1}}-\frac{(i-1)(k-1)}{m2^{q-1}}\right)a_{ik}\varphi_{ik}(x).$$

Thus, by the triangle inequality,

$$\epsilon^{(11)}_{2^p,2^{q-1}}(x)\leqslant\epsilon^{(7)}_{2^p,2^{q-1}}(x)+\epsilon^{(9)}_{2^p,2^{q-1}}(x)+\epsilon^{(10)}_{2^p,2^{q-1}}(x).$$

Putting (4.6), (4.7) and (4.10) together, we get the desired statement.

The following counterpart of Lemma 11 can be similarly proved.

LEMMA 12. *Under condition* (1.4),

$$\left\{\frac{1}{2^{q-1}}\sum_{n=2^{q-1}+1}^{2^q}[s_{2^{p-1},n}(x)-\sigma_{2^{p-1},n}(x)]^2\right\}^{1/2}=o_x\{1\}\quad \textit{a.e. as } \min(p,q)\to\infty.$$

Finally, we have to prove the following.

LEMMA 13. *Under condition* (1.2),

$$\epsilon^{(13)}_{2^p,2^q}(x)=\left\{\frac{1}{2^{p-1}2^{q-1}}\sum_{m=2^{p-1}+1}^{2^p}\sum_{n=2^{q-1}+1}^{2^q}[\sigma_{mn}(x)-\sigma_{m,2^{q-1}}(x)-\sigma_{2^{p-1},n}(x)+\sigma_{2^{p-1},2^{q-1}}(x)]^2\right\}^{1/2}$$
$$=o_x\{1\}\quad \textit{a.e. as } \max(p,q)\to\infty.$$

Proof. We begin with the identity

$$\sigma_{mn}(x)-\sigma_{m,2^{q-1}}(x)-\sigma_{2^{p-1},n}(x)+\sigma_{2^{p-1},2^{q-1}}(x)$$
$$=\sum_{i=2^{p-1}+1}^{m}\sum_{k=2^{q-1}+1}^{n}[\sigma_{ik}(x)-\sigma_{i-1,k}(x)-\sigma_{i,k-1}(x)+\sigma_{i-1,k-1}(x)].$$

Hence, by the Cauchy inequality,

$$[\sigma_{mn}(x)-\sigma_{m,2^{q-1}}(x)-\sigma_{2^{p-1},n}(x)+\sigma_{2^{p-1},2^{q-1}}(x)]^2$$
$$\leqslant \sum_{i=2^{p-1}+1}^{m} \sum_{k=2^{q-1}+1}^{n} ik[\sigma_{ik}(x)-\sigma_{i-1,k}(x)-\sigma_{i,k-1}(x)+\sigma_{i-1,k-1}(x)]^2.$$

This implies

$$\epsilon^{(13)}_{2^p,2^q}(x) \leqslant \left\{ \sum_{m=2^{p-1}+1}^{2^p} \sum_{n=2^{q-1}+1}^{2^q} mn[\sigma_{mn}(x)-\sigma_{m-1,n}(x) - \sigma_{m,n-1}(x)+\sigma_{m-1,n-1}(x)]^2 \right\}^{1/2}.$$

We consider the representation

$$\sigma_{mn}(x)-\sigma_{m-1,n}(x)-\sigma_{m,n-1}(x)+\sigma_{m-1,n-1}(x) = \sum_{i=1}^{m} \sum_{k=1}^{n} \frac{(i-1)(k-1)}{m(m-1)n(n-1)} a_{ik}\varphi_{ik}(x),$$

from which we can calculate with ease that

$$\sum_{m=2}^{\infty} \sum_{n=2}^{\infty} mn \int [\sigma_{mn}(x)-\sigma_{m-1,n}(x)-\sigma_{m,n-1}(x)+\sigma_{m-1,n-1}(x)]^2 d\mu(x)$$
$$= \sum_{m=2}^{\infty} \sum_{n=2}^{\infty} \sum_{i=1}^{m} \sum_{k=1}^{n} \frac{(i-1)^2(k-1)^2}{m(n-1)^2 n(n-1)^2} a_{ik}^2$$
$$\leqslant \sum_{i=2}^{\infty} \sum_{k=2}^{\infty} i^2 k^2 a_{ik}^2 \sum_{m=i}^{\infty} \frac{1}{m^3} \sum_{n=k}^{\infty} \frac{1}{n^3} \leqslant 4 \sum_{i=2}^{\infty} \sum_{k=2}^{\infty} a_{ik}^2 < \infty.$$

The application of Levi's theorem completes the proof.

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