ON PRIME SEQUENCES OVER AN IDEAL

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1. Introduction. In [10], Rees introduced the concepts of an asymptotic sequence, an asymptotic sequence over an ideal, and a prime sequence over an ideal. In [4] it was shown that most of the basic results known to hold for R-sequences have a valid analogue for asymptotic sequences, and it was noted there that, conversely, sometimes a new result on R-sequences can be found by proving the R-sequence version of a new result on asymptotic sequences. The main purpose of this paper is to illustrate this converse phenomenon and thus to derive some new information on R-sequences. Actually, in the present case, the results are the "prime sequence over an ideal" version of "asymptotic sequence over an ideal" results.

Specifically, in [6, (5.6.1) and (5.7)] it is shown that $b_1, \ldots, b_s$ are an asymptotic sequence over an ideal I in a local ring R if and only if $b_1, \ldots, b_s, u$ are an asymptotic sequence in the Rees ring $\mathcal{R} = \mathcal{R}(R, I)$, and this holds if and only if $b_1, \ldots, b_s, u$ are an asymptotic sequence in $\mathcal{R}_{\mathfrak{M}}$, where $\mathfrak{M}$ is the maximal homogeneous ideal in $\mathcal{R}$. Then, among other things, it is shown that: the I-forms of $b_1, \ldots, b_s$ are an asymptotic sequence in the form ring $\mathfrak{F}(R, I)$ [6, (5.10)]; the images of $b_1, \ldots, b_s$ in $R[I/b_1]$ are an asymptotic sequence [6, (7.4)]; each permutation of $b_1, \ldots, b_s$ is an asymptotic sequence over I [6, (6.2)]; $b_1, \ldots, b_s$ are an asymptotic sequence in R [6, (6.4)]; and, any two maximal asymptotic sequences over I have the same length [2]. The prime sequence over an ideal version of each of these results is proved in §2, and certain additional results are also proved, such as: the form ring result just mentioned actually characterizes a prime sequence over an ideal and

$$(b_1^{e_1}, \ldots, b_s^{e_s})R \cap (I+B_s)^n = \sum_{i=1}^{s} b_i^{e_i}(I+B_s)^{n-e_i}$$

for all positive integers e_i and for all $n \geqslant 0$, where $B_s = (b_1, \ldots, b_s)R$.

Since most of the results mentioned in the preceding paragraph are rather natural to consider for any type of sequence of elements, the R-sequence versions probably would have been found without knowing that the asymptotic sequence versions are true. But even so, the close analogy between the two versions of the results does nicely illustrate the converse phenomenon mentioned above.

Prime sequences over an ideal seem to have some interesting and useful properties. Hopefully the results in this paper will be useful in any future research on such elements.

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2. Main results. In this section we first give the relevant definitions and then prove (2.2) which gives three characterizations of a prime sequence over an ideal in a local ring. Then quite a few corollaries of (2.2) are given.

We begin with the definitions.

(2.1) DEFINITION. Let I be an ideal in a Noetherian ring R and let $b_1, \ldots, b_s$ in R. Then:

(2.1.1) $b_1, \ldots, b_s$ are a *prime sequence over* I in case $(I, b_1, \ldots, b_s)R \neq R$ and $(I, b_1, \ldots, b_i)^n R : b_{i+1}R = (I, b_1, \ldots, b_i)^n R$ for $i = 0, 1, \ldots, s-1$ and for all $n \geqslant 1$.

(2.1.2) $b_1, \ldots, b_s$ are an *asymptotic sequence over* I in case $(I, b_1, \ldots, b_s)R \neq R$ and $((I, b_1, \ldots, b_i)^n R)_a : b_{i+1}R = ((I, b_1, \ldots, b_i)^n R)_a$ for $i = 0, 1, \ldots, s-1$ and for all $n \geqslant 1$, where J_a is the integral closure of the ideal J in R.

(2.1.3) $\mathcal{R}(R, I) = R[tI, u]$ (t an indeterminate and $u = 1/t$) is the *Rees ring of* R *with respect to* I.

It is clear from the definitions that if $b_1, \ldots, b_s$ are a prime (respectively, an asymptotic) sequence over I, then their images in R_S are a prime (respectively, an asymptotic) sequence over I_S for all multiplicatively closed sets S $(0 \notin S)$ in R such that $(I, b_1, \ldots, b_s)R_S \neq R_S$. Also, $\mathcal{R}(R, I)$ is a graded subring of $R[t, u]$ and $u^n \mathcal{R}(R, I) \cap R = I^n$ for all $n \geqslant 1$. These facts will be implicitly used below.

(2.2) is the main result in this paper; it gives three useful characterizations of prime sequences over an ideal in a local ring. Like most of the results in this paper, (2.2) is the R-sequence version of a new result on asymptotic sequences. In each such result, the cited reference is to the known asymptotic sequence result.

(2.2) THEOREM. (Cf. [6, (5.6.1) and (5.7)].) *Let* I *be an ideal in a local ring* (R, M), *let* $b_1, \ldots, b_s$ *in* M, *let* $B_i = (b_1, \ldots, b_i)R$ $(i = 1, \ldots, s)$, *and let* $B_0 = (0)$. *Also let* $\mathcal{R}_i = \mathcal{R}(R, I + B_i)$ $(i = 0, 1, \ldots, s)$ *and let* $\mathfrak{M} = (u, M, tI)\mathcal{R}_0$. *Then the following statements are equivalent:*

(2.2.1) $b_1, \ldots, b_s$ *are a prime sequence over* I.

(2.2.2) $b_1, \ldots, b_s, u$ *are an* $\mathcal{R}_0$*-sequence.*

(2.2.3) $b_1, \ldots, b_s, u$ *are an* $\mathcal{R}_{0\mathfrak{M}}$*-sequence.*

(2.2.4) u, b_{i+1} *are an* $\mathcal{R}_i$*-sequence for* $i = 0, 1, \ldots, s-1$.

Proof. $u^n \mathcal{R}_i \cap R = (I + B_i)^n$ for $i = 0, 1, \ldots, s$ and for all $n \geqslant 1$, and if p is a prime divisor of $u\mathcal{R}_i$ for some i, then $p \cap R$ is a prime divisor of $(I + B_i)^n$ for some $n \geqslant 1$, by [3, (5.1)]. Therefore it follows that (2.2.1) $\Leftrightarrow$ (2.2.4).

Note that $\mathfrak{M}$ is the maximal homogeneous ideal in $\mathcal{R}_0$ and every homogeneous ideal in $\mathcal{R}_0$ is contained in $\mathfrak{M}$. Therefore, since the prime divisors of all the ideals generated by subsets of $u, b_1, \ldots, b_s$ are homogeneous, it follows that $b_1, \ldots, b_s, u$ are an $\mathcal{R}_0$-sequence if and only if they are an $\mathcal{R}_{0\mathfrak{M}}$-sequence, so (2.2.2) $\Leftrightarrow$ (2.2.3).

Now assume that (2.2.4) holds, so u, b_1 are an $\mathcal{R}_0$-sequence. Therefore b_1, u are an $\mathcal{R}_0$-sequence, by (2.2.2) $\Leftrightarrow$ (2.2.3). Thus fix i $(1 \leqslant i < s)$ and assume that $b_1, \ldots, b_i, u$ are an $\mathcal{R}_0$-sequence. Then $\mathcal{R}_i = \mathcal{R}_0[tb_1, \ldots, tb_i]$ and $tb_j = b_j/u$, so every element in $\mathcal{R}_i$ can be written in the form r/u^k for all large k, where $r \in (B_i, u)^k \mathcal{R}_0$. Now $B_i^m \mathcal{R}_0 : u\mathcal{R}_0 = B_i^m \mathcal{R}_0$ for all $m \geqslant 1$, since $b_1, \ldots, b_i, u$ are an $\mathcal{R}_0$-sequence, so

$$(B_i,u)^m \mathfrak{R}_0 : u\mathfrak{R}_0 = (B_i^m, u(B_i,u)^{m-1})\mathfrak{R}_0 : u\mathfrak{R}_0$$
$$= B_i^m \mathfrak{R}_0 : u\mathfrak{R}_0 + (B_i,u)^{m-1}\mathfrak{R}_0 = (B_i,u)^{m-1}\mathfrak{R}_0.$$

Therefore it follows that

$$u^n \mathfrak{R}_i \cap \mathfrak{R}_0 = (B_i,u)^n \mathfrak{R}_0$$

for all $n \geqslant 1$, and so $b_1,\ldots,b_i,u,b_{i+1}$ are an $\mathfrak{R}_0$-sequence, since b_{i+1} is not in any prime divisor of $u\mathfrak{R}_i$, by (2.2.4). Therefore $b_1,\ldots,b_{i+1},u$ are an $\mathfrak{R}_0$-sequence, by (2.2.2) $\Leftrightarrow$ (2.2.3), so it follows that (2.2.4) $\Rightarrow$ (2.2.2).

Finally, assume that (2.2.2) holds, fix i $(0 < i \leqslant s-1)$, and let p be a prime divisor of $u\mathfrak{R}_i$. Then there exists a homogeneous element xt^k in $\mathfrak{R}_i$ such that $u\mathfrak{R}_i : xt^k\mathfrak{R}_i = p$, so $u^{k+1}\mathfrak{R}_i : x\mathfrak{R}_i = p$, and so $(u^{k+1}\mathfrak{R}_i \cap \mathfrak{R}_0) : x\mathfrak{R}_0 = p \cap \mathfrak{R}_0$, by [12, p. 220]. Now $u^{k+1}\mathfrak{R}_i \cap \mathfrak{R}_0 = (B_i,u)^{k+1}\mathfrak{R}_0$, as in the previous paragraph, so $p \cap \mathfrak{R}_0$ is a prime divisor of $(B_i,u)^n\mathfrak{R}_0$ for some $n \geqslant 1$. But b_{i+1} is not in any prime divisor of $(B_i,u)^m\mathfrak{R}_0$ for all $m \geqslant 1$, by hypothesis and [5, (4.1)], so $b_{i+1} \notin p$, and so it follows that (2.2.2) $\Rightarrow$ (2.2.4). □

We now derive several corollaries of (2.2) which give some useful information concerning prime sequences over an ideal. The first of these extends (2.2.1) $\Rightarrow$ (2.2.4).

In the proof of (2.3) we use the following result proved in [11, (2.9)]: if each permutation of $c_0, c_1, \ldots, c_s$ is an R-sequence in a Noetherian ring R, then each permutation of $c_0, c_1/c_0, \ldots, c_i/c_0, c_{i+1}, \ldots, c_s$ is an A_i-sequence, where $A_i = R[c_1/c_0, \ldots, c_i/c_0]$ $(i = 1, \ldots, s)$.

(2.3) COROLLARY. (Cf. [6, (7.1.2)].) *With the notation of* (2.2), *if* $b_1,\ldots,b_s$ *are a prime sequence over I, then each permutation of* $u, tb_1, \ldots, tb_i, b_{i+1}, \ldots, b_s$ *is an* $\mathfrak{R}_i$*-sequence for* $i = 0, 1, \ldots, s$.

Proof. The elements $b_1,\ldots,b_s,u$ are an $\mathfrak{R}_0$-sequence, by (2.2.1) $\Rightarrow$ (2.2.2), so each permutation of $u, b_1, \ldots, b_s$ is an $\mathfrak{R}_0$-sequence, by (2.2.2) $\Leftrightarrow$ (2.2.3). Therefore, since $\mathfrak{R}_i = \mathfrak{R}_0[b_1/u, \ldots, b_i/u]$, the conclusion follows from [11, (2.9)]. □

(2.4) shows an interesting relationship between certain ideals related to I and $b_1,\ldots,b_s$; the asymptotic sequence version of (2.4) is not true. In (2.4) the usual convention that $J^n = R$ (where J is an ideal in R and $n \leqslant 0$) is used.

(2.4) COROLLARY. *With the notation of* (2.2), *for* $i = 0, 1, \ldots, s$, *for all* $n \geqslant 0$, *and for all positive integers* $e_1, \ldots, e_s$ *it holds that*

$$\sum_{j=1}^{i} b_j^{e_j}(I+B_i)^{n-e_j} + (b_{i+1}^{e_{i+1}}, \ldots, b_s^{e_s})(I+B_i)^n = (b_1^{e_1}, \ldots, b_s^{e_s})R \cap I^n.$$

Proof. Fix i $(0 \leqslant i \leqslant s)$ and let $\mathfrak{R} = \mathfrak{R}(R, I+B_i)$. Then $tb_1, \ldots, tb_i, b_{i+1}, \ldots, b_s, u$ are an $\mathfrak{R}$-sequence, by (2.3), so $(tb_1)^{e_1}, \ldots, (tb_i)^{e_i}, b_{i+1}^{e_{i+1}}, \ldots, b_s^{e_s}, u$ are an $\mathfrak{R}$-sequence. Therefore $K\mathfrak{R}[1/u] \cap \mathfrak{R} = K$, where

$$K = ((tb_1)^{e_1}, \ldots, (tb_i)^{e_i}, b_{i+1}^{e_{i+1}}, \ldots, b_s^{e_s})\mathfrak{R}.$$

Also, $K\mathfrak{R}[1/u] = (b_1^{e_1}, \ldots, b_s^{e_s})R[t,u]$, so $K = H$, where

$$H = (b_1^{e_1}, \ldots, b_s^{e_s})R[t, u] \cap \mathcal{R},$$

since $H : u\mathcal{R} = H$. Therefore $\{r \in R; rt^n \in K\} = \{r \in R; rt^n \in H\}$ for all $n \geqslant 0$, and the conclusion readily follows from this and the definition of $\mathcal{R}$. □

(2.5) gives another characterization of a prime sequence over I, this one in terms of the form ring (= associated graded ring) of R with respect to I.

(2.5) COROLLARY. (Cf. [6, (5.10)].) *Let I be an ideal in a local ring (R, M) and let $b_1, \ldots, b_s$ be elements in M. Then $b_1, \ldots, b_s$ are a prime sequence over I if and only if the I-forms of $b_1, \ldots, b_s$ are an $\mathfrak{F}$-sequence in the form ring $\mathfrak{F} = \mathfrak{F}(R, I)$ of R with respect to I.*

Proof. By [8, Thm. 2.1], $\mathfrak{F} = \mathcal{R}/u\mathcal{R}$ and the I-form of $c \in R$ is $ct^k + u\mathcal{R}$ (where $c \in I^k, \notin I^{k+1}$) in $\mathfrak{F}$, where $\mathcal{R} = \mathcal{R}(R, I)$. Therefore, if $b_1, \ldots, b_s$ are a prime sequence over I, then $b_1, \ldots, b_s, u$ are an $\mathcal{R}$-sequence, by (2.2.1) ⇒ (2.2.2), so $u, b_1, \ldots, b_s$ are an $\mathcal{R}$-sequence, by (2.2.2) ⇔ (2.2.3), and so the I-forms of $b_1, \ldots, b_s$ are an $\mathfrak{F}$-sequence. The converse follows essentially by reading backwards, since u is regular in $\mathcal{R}$. □

(2.6) shows that a prime sequence over I gives rise to an A_i-sequence, where A_i is a certain monadic transformation ring of R. (Concerning (2.6), it should be noted that $R \subseteq A_i$, since $b_1, \ldots, b_s$ are an R-sequence (and hence each is regular) in R, by (2.11) below.)

(2.6) COROLLARY. (Cf. [6, (7.4)].) *Let $b_1, \ldots, b_s$ be a prime sequence over an ideal I in a local ring R and let $A_i = R[(I + B_i)/b_i]$ $(i = 1, \ldots, s)$. Then for $i = 1, \ldots, s$ each permutation of $b_1/b_i, \ldots, b_{i-1}/b_i, b_i, b_{i+1}, \ldots, b_s$ is an A_i-sequence.*

Proof. Let $\mathcal{R}_i = \mathcal{R}(R, I + B_i)$ and $\mathcal{S}_i = \mathcal{R}_i[1/tb_i]$. Then tb_i is a unit in $\mathcal{S}_i$, so $u\mathcal{S}_i = b_i\mathcal{S}_i$ and $tb_j\mathcal{S}_i = (b_j/b_i)\mathcal{S}_i$ $(j = 1, \ldots, i)$. Also, each permutation of $u, tb_1, \ldots, tb_i, b_{i+1}, \ldots, b_s$ is an $\mathcal{R}_i$-sequence, by (2.3), so it follows that each permutation of $tb_1, \ldots, tb_{i-1}, b_i, \ldots, b_s$ is an $\mathcal{S}_i$-sequence. Therefore the conclusion follows, since $\mathcal{S}_i = A_i[tb_i, 1/tb_i]$ and tb_i is transcendental over A_i. □

The next corollary of (2.2) shows that any two maximal prime sequences over an ideal in a local ring have the same length.

(2.7) COROLLARY. (Cf. [2].) *If I is an ideal in a local ring (R, M), then any two maximal prime sequences over I have the same length.*

Proof. Let $b_1, \ldots, b_s$ and $c_1, \ldots, c_t$ be two maximal prime sequences over I, let $s \leqslant t$, and let $\mathcal{R} = \mathcal{R}(R, I)$. Then $b_1, \ldots, b_s, u$ and $c_1, \ldots, c_t, u$ are $\mathcal{R}$-sequences contained in $(u, M)\mathcal{R}$, by (2.2.1) ⇒ (2.2.2). Also, there exists a prime divisor P of $(u, b_1, \ldots, b_s)\mathcal{R}$ such that $M\mathcal{R} \subseteq P$. For if not, then there exists $b_{s+1} \in M$ such that $b_1, \ldots, b_s, u, b_{s+1}$ are an $\mathcal{R}$-sequence, so $b_1, \ldots, b_{s+1}, u$ are an $\mathcal{R}$-sequence, by (2.2.2) ⇔ (2.2.3), hence $b_1, \ldots, b_{s+1}$ are a prime sequence over I, by (2.2.2) ⇒ (2.2.1), and this contradicts the fact that $b_1, \ldots, b_s$ are a maximal prime sequence over I. Therefore grade $P_P = s + 1$ and the $\mathcal{R}$-sequence $c_1, \ldots, c_t, u$ is contained in P, so it follows that $t \leqslant s$. Therefore $s = t$, and so the conclusion follows. □

(2.8) globalizes (2.7) to the case where the prime sequences over I are contained in the Jacobson radical of R.

(2.8) COROLLARY. *Any two prime sequences over an ideal I in a Noetherian ring R which are maximal with respect to being contained in the Jacobson radical of R have the same length.*

Proof. Suppose not and let $b_1, \ldots, b_s$ and $c_1, \ldots, c_t$ be two prime sequences over I which are maximal with respect to being contained in the Jacobson radical J of R and are such that $s < t$. Then, since $b_1, \ldots, b_s$ is a maximal prime sequence over I that is contained in J, there exists a prime divisor P of $(I, b_1, \ldots, b_s)^n R$ (for some $n \geqslant 1$) such that $J \subseteq P$. (By [1], $\{P \in \operatorname{Spec} R;\ P$ is a prime divisor of $(I, b_1, \ldots, b_s)^n R$ for some $n \geqslant 1\}$ is a finite set.) Then the images in R_P of $b_1, \ldots, b_s$ are a maximal prime sequence over I_P. However, $c_1, \ldots, c_t$ are in J and $J \subseteq P$, so the images of $c_1, \ldots, c_t$ in R_P are a longer prime sequence over I_P, and this contradicts (2.7). Therefore the conclusion follows. □

(2.9) shows that each permutation of a prime sequence over I is again a prime sequence over I.

(2.9) COROLLARY. (Cf. [6, (6.2)].) *If $b_1, \ldots, b_s$ are a prime sequence over an ideal I in a local ring R, then each permutation of $b_1, \ldots, b_s$ is a prime sequence over I.*

Proof. This follows immediately from (2.2.1) $\Leftrightarrow$ (2.2.3). □

(2.10) globalizes (2.9) to the case where the b_i are in the Jacobson radical of R.

(2.10) COROLLARY. (Cf. [6, (6.3)].) *Let $b_1, \ldots, b_s$ be a prime sequence over an ideal I in a Noetherian ring R. If $b_1, \ldots, b_s$ are in the Jacobson radical of R, then each permutation of $b_1, \ldots, b_s$ is a prime sequence over I.*

Proof. Suppose not and let $c_1, \ldots, c_s$ be a permutation of $b_1, \ldots, b_s$ such that $c_1, \ldots, c_i$ $(0 \leqslant i < s)$ are a prime sequence over I and there exists a prime divisor P of $C_i = (I, c_1, \ldots, c_i)^n R$ (for some $n \geqslant 1$) such that $c_{i+1} \in P$. Let M be a maximal ideal in R that contains P. Then the hypothesis implies that the images of $b_1, \ldots, b_s$ in R_M are a prime sequence over I_M, so the images of $c_1, \ldots, c_s$ in R_M are a prime sequence over I_M, by (2.9). However, P_M is a prime divisor of C_{iM} and the image of c_{i+1} is in P_M, and this implies that the images of $c_1, \ldots, c_{i+1}$ are not a prime sequence over I_M. This contradiction thus implies that the conclusion holds. □

(2.11) shows that a prime sequence over I is also an R-sequence; concerning this, see (2.16) below.

(2.11) COROLLARY. (Cf. [6, (6.4)].) *If $b_1, \ldots, b_s$ are a prime sequence over an ideal I in a local ring R, then $b_1, \ldots, b_s$ are an R-sequence.*

Proof. By (2.2.1) $\Rightarrow$ (2.2.2), $b_1, \ldots, b_s, u$ are an $\mathcal{R}_0$-sequence, where $\mathcal{R}_0 = \mathcal{R}(R, I)$. Therefore $b_1, \ldots, b_s$ are an $\mathcal{R}_0[1/u]$-sequence, and $\mathcal{R}_0[1/u] = R[t, u]$. Therefore, since t is an indeterminate and $u = 1/t$, it follows that $b_1, \ldots, b_s$ are an R-sequence. □

(2.12) globalizes (2.10) to the case where I is contained in the Jacobson radical of R.

(2.12) COROLLARY. (Cf. [6, (6.5)].) *Let $b_1, \ldots, b_s$ be an asymptotic sequence over an ideal I in a Noetherian ring R. If I is contained in the Jacobson radical of R, then $b_1, \ldots, b_s$ are an R-sequence.*

Proof. Suppose not and choose i $(0 \leqslant i < s)$ such that $b_1, \ldots, b_i$ are an R-sequence and there exists a prime divisor P of $B_i = (b_1, \ldots, b_i)R$ such that $b_{i+1} \in P$. Let M be a maximal ideal in R that contains P, so $I + B_i \subseteq M$ and the image of b_{i+1} in R_M is in P_M, so the images in R_M of $b_1, \ldots, b_{i+1}$ are not an R_M-sequence. However, $b_1, \ldots, b_{i+1}$ are a prime sequence over I, so their images in R_M are a prime sequence over I_M. But this contradicts (2.11), so the conclusion follows. □

The graded ring case is briefly considered in (2.13).

(2.13) REMARK. (Cf. [6, (6.6)].) Let R be a graded Noetherian ring and let I be a homogeneous ideal in R. Then the following statements hold:

(2.13.1) Any two homogeneous prime sequences over I (that is, prime sequences over I consisting of homogeneous elements) which are maximal with respect to being contained in all maximal homogeneous ideals in R have the same length.

(2.13.2) If $b_1, \ldots, b_s$ are a homogeneous prime sequence over I and are in all maximal homogeneous ideals in R, then each permutation of $b_1, \ldots, b_s$ is a prime sequence over I.

(2.13.3) If I is contained in all maximal homogeneous ideals in R and $b_1, \ldots, b_s$ are a homogeneous prime sequence over I, then $b_1, \ldots, b_s$ are an R-sequence.

Proof. The proofs are similar to the proofs of (2.8), (2.10), and (2.12), respectively, but also use the fact that every homogeneous ideal in R is contained in a maximal homogeneous ideal in R. □

The next result shows that the prime divisors of powers of $(b_1, \ldots, b_i)R$ are contained in the prime divisors of powers of $(I, b_1, \ldots, b_i)R$.

(2.14) COROLLARY. (Cf. [7, (4.2)].) *Let $b_1, \ldots, b_s$ be a prime sequence over an ideal I in a Noetherian ring R, let $B_i = (b_1, \ldots, b_i)R$ $(i = 1, \ldots, s)$, and let $B_0 = (0)$. Assume that I is contained in the Jacobson radical of R, fix i, and let p be a prime divisor of B_i^n for some $n \geqslant 1$. Then there exists a prime divisor P of $(I + B_i)^m$ for some $m \geqslant 1$ such that $p \subseteq P$.*

Proof. Suppose not. Now $\mathcal{P} = \{P;\ P$ is a prime divisor of $(I + B_i)^m$ for some $m \geqslant 1\}$ is a finite set, by [1], so the supposition implies that there exists $x \in p$, $\notin \bigcup\{P; P \in \mathcal{P}\}$. Therefore $b_1, \ldots, b_i, x$ are a prime sequence over I and they are not an R-sequence, since $B_i : xR \neq B_i$, by the choice of x and [5, (4.1)]. However, this contradicts (2.12), so the conclusion follows. □

(2.15) shows that the images of $b_{i+1}, \ldots, b_s$ in R/B_i are a prime sequence over $(I + B_i)/B_i$. This does not follow immediately from the definition.

(2.15) COROLLARY. (Cf. [7, (3.3)].) *With the notation of* (2.14), *the images modulo* B_i *of* $b_{i+1},\ldots,b_s$ *are a prime sequence over* $(I+B_i)/B_i$ *for* $i=1,\ldots,s-1$.

Proof. Assume first that R is local, let $\mathcal{R}=\mathcal{R}(R,I)$, and let $B_i^*=B_iR[t,u]\cap\mathcal{R}$. Then $B_i\mathcal{R}:u\mathcal{R}=B_i\mathcal{R}$, by (2.21)$\Rightarrow$(2.2.2), so it follows that $B_i\mathcal{R}=B_i^*$. Therefore, since $b_1,\ldots,b_i,b_{i+1},\ldots,b_s,u$ are an $\mathcal{R}$-sequence, by (2.2), it follows that the B_i^*-residue classes of $b_{i+1},\ldots,b_s,u$ are an $\mathcal{R}/B_i^*$-sequence. But $\mathcal{R}/B_i^*\cong\mathcal{R}(R/B_i,(I+B_i)/B_i)$, by [9, Thm. 2.1], so the images of $b_{i+1},\ldots,b_s$ in R/B_i are a prime sequence over $(I+B_i)/B_i$, by (2.2.2)$\Rightarrow$(2.2.1).

Now, for the general Noetherian ring case, suppose that there exist i and j such that the images of $b_{i+1},\ldots,b_j$ are a prime sequence over $(I+B_i)/B_i$ and the image of b_{j+1} is in some prime divisor P' of $(I+B_j)^n/B_i$ for some $n\geqslant 1$. Then it may be assumed that R is local with maximal ideal P, where P is the pre-image in R of P'. But then it readily follows that the supposition contradicts what was shown in the previous paragraph, so the conclusion follows. □

(2.16) shows that the images of $b_1,\ldots,b_s$ in R/I^n are an R/I^n-sequence for all $n\geqslant 1$. This also does not follow immediately from the definition.

(2.16) COROLLARY. (Cf. [8, (4.2)].) *If* $b_1,\ldots,b_s$ *are a prime sequence over an ideal* I *in a Noetherian ring* R, *then their images in* R/I^n *are an* R/I^n*-sequence for all* $n\geqslant 1$.

Proof. By definition, b_1 is not in any prime divisor of I^n for all $n\geqslant 1$, so the conclusion follows if $s=1$. If $s>1$, then note that $(R/I^n)/(b_1(R/I^n))\cong R/((I^n,b_1)R)\cong R'/I'^n$ for all $n\geqslant 1$, where the $'$ denotes residue class modulo b_1R. Then $b_2',\ldots,b_s'$ are a prime sequence over I', by (2.15), so their images in R'/I'^n are an R'/I'^n-sequence for all $n\geqslant 1$, by induction on s. Therefore the conclusion follows from the isomorphism, since b_1+I^n is regular in R/I^n for all $n\geqslant 1$. □

(2.17) shows an interesting fact concerning the prime divisors of ideals of the form $((I,b_1,\ldots,b_j)^n,b_{j+1},\ldots,b_i)R$.

(2.17) COROLLARY. (Cf. [7, (4.3)].) *Let* $b_1,\ldots,b_s$ *be a prime sequence over an ideal* I *in a Noetherian ring* R *and let* $B_i=(b_1,\ldots,b_i)R$ $(i=0,1,\ldots,s)$. *Then for* $0\leqslant j\leqslant i\leqslant s$ *it holds that if* p *is a prime divisor of* $((I,b_1,\ldots,b_j)^n,b_{j+1},\ldots,b_i)R$ *for some* $n\geqslant 1$, *then there exists a prime divisor* P *of* $(I,b_1,\ldots,b_i)^mR$ *for some* $m\geqslant 1$ *such that* $p\subseteq P$.

Proof. Suppose not and let c in p such that c is not in any prime divisor of $(I,b_1,\ldots,b_i)^mR$ for all $m\geqslant 1$. Then $b_{j+1},\ldots,b_i,c$ are a prime sequence over $C=(I,b_1,\ldots,b_j)R$ but their images modulo C^n are not a prime sequence in R/C^n. However, this contradicts (2.16), so the conclusion follows. □

This paper will be closed with the following remark which extends the usefulness of the preceding results.

(2.18) REMARK. It follows from (2.2.1) ⇒ (2.2.2) that if $b_1, \ldots, b_s$ are a prime sequence over an ideal I in a local ring (R, M) and $e_1, \ldots, e_s$ are positive integers, then $b_1^{e_1}, \ldots, b_s^{e_s}$ are a prime sequence over I. Then it follows from (2.17) that they are a prime sequence over I^m for all $m \geqslant 1$. Therefore the results in this paper hold with I^m in place of I and $b_1^{e_1}, \ldots, b_s^{e_s}$ in place of $b_1, \ldots, b_s$.

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