INVARIANT PSEUDODIFFERENTIAL OPERATORS ON TWO STEP NILPOTENT LIE GROUPS

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Introduction. Let G be a connected, simply connected two step nilpotent Lie group with Lie algebra $\mathcal{G}$. We shall develop a calculus of right invariant pseudodifferential operators on G which is based on the representation theory for G.

For $p \in \mathcal{S}^*(\mathcal{G}^*)$ define an operator $\lambda(p): \mathcal{S}(G) \to \mathcal{S}^*(G)$ by $\lambda(p)\phi = (Fp \circ \log) * \phi$, where $\mathcal{S}^*$ denotes the dual of the space of rapidly decreasing functions, $F: \mathcal{S}^*(\mathcal{G}^*) \to \mathcal{S}^*(\mathcal{G})$ is the Fourier transform, $\log: G \to \mathcal{G}$ is the inverse of the exponential map and $*$ is convolution as determined by the product on G. $\lambda(p)$ is the right invariant pseudodifferential operator with symbol p, $\lambda(p)$ being a partial differential operator if and only if p is a polynomial. Howe [5] showed that for arbitrary nilpotent Lie groups the calculus of invariant pseudodifferential operators "fibers" over $\mathcal{Z}^*$, where $\mathcal{Z}$ is the center of $\mathcal{G}$. In the case of the Heisenberg group this is essentially equivalent to saying that the calculus fibers over the orbits of the coadjoint action of G on $\mathcal{G}^*$. To be more explicit, if p and q are both in $\mathcal{S}(\mathcal{G}^*)$ let $p\#q$ be that element of $\mathcal{S}(\mathcal{G}^*)$ such that $\lambda(p\#q) = \lambda(p)\lambda(q)$. Then $p\#q(\xi)$ depends only on the values of p and q on that orbit of the coadjoint action that contains ξ. Furthermore, as was shown in Howe [4], after making the appropriate identification the calculus at the orbit level is the Weyl pseudodifferential operator calculus. In fact, as we show in Section 1, from this it follows easily that the calculus fibers over the orbits of the coadjoint action and the orbit level calculus is the Weyl calculus for any step two group G. If π is an irreducible unitary representation of G, the Kirillov theory then allows us to define an operator $\pi(p)$ for which the Weyl symbol is essentially p restricted to the orbit corresponding to π, and $\pi(p\#q) = \pi(p)\pi(q)$.

In Sections 2 and 3 it is shown that the calculus can be extended to classes of symbols $S^m(\Phi, \mathcal{G}^*)$, where Φ is some weight function as defined in Section 2, and $p \in S^m(\Phi, \mathcal{G}^*)$ if p satisfies estimates of the form

$$|D_1 \cdots D_k p(\xi)| \leqslant C\Phi(\xi)^{m-k}$$

where each D_j is a vector field which is tangent to the orbits of the coadjoint action. No assumptions are made about the differentiability of p except in directions tangent to the orbits. We prove that if $p \in S^m$ and $q \in S^k$, then $p\#q \in S^{m+k}$, and we give an asymptotic expansion for $p\#q$. In Section 4 it is shown that $\lambda(p)$ is a bounded operator on $L^2(G)$ if $p \in S^0(\Phi, \mathcal{G}^*)$. The main theorems of the calculus are proved by appealing to Hörmander [3] for the corresponding results in the Weyl calculus at the orbit level and showing that certain estimates are uniform over the orbits.

L^2 boundedness has been proved by Howe [5], without appealing to a standard operator calculus, for general nilpotent Lie groups in the case where $\Phi(\xi) = (1 + |\xi|)^\delta$ with $\delta > 1/2$ and where the derivatives of p satisfy the appropriate estimates in all

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directions. In [11] Strichartz developed a local theory of pseudodifferential operators for arbitrary Lie groups in which the symbols were functions defined on $\mathcal{G}^*$. In [8] the general calculus of Beals [1] was applied to step two nilpotent groups. The theory of Nagel and Stein [9] can also be applied to nilpotent Lie groups. The present approach is independent of these approaches and it differs from them in the use that is made of the Kirillov theory. This approach seems to be the most natural in dealing with problems that relate properties of the operator $\lambda(p)$ to properties of its representation theoretic "symbol", i.e. to properties of the operators $\pi(p)$ for irreducible unitary representations π. For example, in a future paper we will generalize Rockland's criteria for hypoellipticity [10] to the pseudodifferential case and give an orbitwise construction of the parametrix.

Some of the techniques used in this paper have also been used recently by Melin [7] in the case of the Heisenberg group.

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1. Orbits and the calculus for symbols in $\mathcal{S}$. In this section we present first a very brief review of some aspects of the Kirillov theory as it pertains to step two groups in particular. We then prove a proposition which implies that the calculus for symbols in $\mathcal{S}(\mathcal{G}^*)$ "fibers" over the orbits of the coadjoint action and that the calculus at the orbit level is the Weyl pseudodifferential operator calculus. This later fact is at least implicit in Howe [5]; however, our formulation of it is somewhat different. The formulation given here will be used in extending the calculus to more general symbol classes in later sections of the paper.

If π is any unitary representation of G, π will also denote $-i$ times the derived representation of $\mathcal{G}$, that is

$$\pi(x) = -i\frac{d}{dt}\pi(\exp tx)\Big|_{t=0}. \tag{1.1}$$

G is step two nilpotent throughout the paper.

Given $\xi \in \mathcal{G}^*$, a subspace $\tilde{V}$ of $\mathcal{G}$ is said to be maximally subordinate to ξ if $\tilde{V} = \{x: \langle \xi, [x,y]\rangle = 0 \text{ for all } y \in \tilde{V}\}$. Let V be any subspace of $\mathcal{G}$ such that $\mathcal{G} = V \oplus \tilde{V}$. Given $t \in V$ and $x \in \mathcal{G}$ there exist unique $\tilde{v}(t,x) \in \tilde{V}$ and $v(t,x) \in V$ such that $\exp t \exp x = \exp \tilde{v}(t,x) \exp v(t,x)$. Define the representation $\pi_{\xi,V,\tilde{V}}$ of G on $L^2(V)$ by

$$\pi_{\xi,V,\tilde{V}}(\exp x)f(t) = e^{i\langle \xi, \tilde{v}(t,x)\rangle} f(v(t,x)).$$

Whenever we refer to $\pi_{\xi,V,\tilde{V}}$ in this paper it is assumed that $\tilde{V}$ is maximally subordinate to ξ.

It is shown in Kirillov [6] that every irreducible unitary representation of G is unitarily equivalent to some $\pi_{\xi,V,\tilde{V}}$, and that $\pi_{\xi_1,V_1,\tilde{V}_1}$ is unitarily equivalent to $\pi_{\xi_2,V_2,\tilde{V}_2}$ if and only if there is an $x \in \mathcal{G}$ such that $(\exp \operatorname{ad} x)^*\xi_1 = \xi_2$. Since $\mathcal{G}$ is step two nilpotent, $(\exp \operatorname{ad} x)(y) = y + [x,y]$.

For $\xi \in \mathcal{G}^*$ the orbit of ξ under the coadjoint action of G, that is

$$\{(\exp \operatorname{ad} x)^*\xi : x \in \mathcal{G}\},$$

will be denoted $\mathcal{O}_\xi$, or $\mathcal{O}_\pi$ where $\pi = \pi_{\xi,V,\tilde V}$. $\mathcal{O}_\xi$ is an affine subspace of $\mathcal{G}^*$ and has a natural associated symplectic form σ which can be defined as follows: Let η and ζ be elements of $\mathcal{G}^*$ which are parallel to $\mathcal{O}_\xi$. Then there exist y and z in $\mathcal{G}$ such that $(\exp \operatorname{ad} y)^*\xi = \xi + \eta$ and $(\exp \operatorname{ad} z)^*\xi = \xi + \zeta$. Let $\sigma(\eta, \zeta) = \langle \xi, [y, z]\rangle$. It can be easily checked that σ is well-defined and non-degenerate.

Given an orbit $\mathcal{O}$ of the coadjoint action we shall briefly describe a choice of $\xi \in \mathcal{O}$, $\tilde V$ and V such that the irreducible unitary representation $\pi_{\xi,V,\tilde V}$ has a particularly simple form. Let $\mathcal{G} = \mathcal{G}_1 \oplus \mathcal{G}_2$ where $\mathcal{G}_2 \supset [\mathcal{G}, \mathcal{G}]$. We may assume that an inner product has been given on $\mathcal{G}_1$. It is shown in [8] that there is an orthonormal basis $\mathcal{B} = \{y_1, \ldots, y_d; \ldots, y_{2d}; \ldots, y_N\}$ for $\mathcal{G}_1$, depending on $\mathcal{O}$, such that for all $\xi \in \mathcal{O}$, $\langle \xi, [y_j, y_{j+d}]\rangle = \lambda_j > 0$ for $j \leqslant d$ and $\langle \xi, [y_j, y_k]\rangle = 0$ for all other choices of $j \leqslant k$. Here $d = \dim \mathcal{O}/2$. Let $V = \operatorname{span}\{y_1, \ldots, y_d\}$, $\tilde V = \mathcal{G}_2 \oplus \operatorname{span}\{y_{d+1}, \ldots, y_N\}$ and $\mathcal{R} = \mathcal{G}_2 \oplus \operatorname{span}\{y_{2d+1}, \ldots, y_N\}$. For any $\xi \in \mathcal{O}$, $\mathcal{R} = \{x: \langle \xi, [x, y]\rangle = 0, \text{ for all } y \in \mathcal{G}\}$. Hence if ξ and ξ_1 are both in $\mathcal{O}$, then $\xi|_{\mathcal{R}} = \xi_1|_{\mathcal{R}}$. As shown in [8], there is a $\xi \in \mathcal{O}$ such that if $\pi = \pi_{\xi,V,\tilde V}$, then

$$
\begin{aligned}
\pi(y_j)f(t) &= -i\partial f/\partial t_j, && j \leqslant d;\\
\pi(y_{j+d})f(t) &= \lambda_j t_j f(t), && j \leqslant d;\\
\pi(x)f(t) &= \langle \xi, x\rangle f(t), && x \in \mathcal{R}.
\end{aligned}
\tag{1.2}
$$

Here $(t_1, \ldots, t_d)$ are the coordinates of $t \in V$ with respect to the basis $\{y_1, \ldots, y_d\}$.

Let $\pi = \pi_{\xi,V,\tilde V}$, for $\tilde V$ any subspace maximally subordinate to ξ (not necessarily of the form of the preceding paragraph). Let $\operatorname{sym}\pi(x)$ denote the Weyl symbol of the differential operator $\pi(x)$. The correspondence between symbol q and operator $Q = q^w(t, D)$ in the Weyl calculus is given by

$$
Qu(t) = \int e^{i\langle \tau, t-s\rangle} q(\tfrac{1}{2}s + \tfrac{1}{2}t, \tau)\, u(s)\, đs\, đ\tau
$$

where $đs = (2\pi)^{-k/2} ds$, $k = \dim V$. Since G is step two nilpotent, $\operatorname{sym}\pi(x)$ is a real valued affine function defined on $V \times V^*$. (Hence the Weyl symbol of $\pi(x)$ is the same as the standard symbol.)

Define a map $\psi_\pi: V \times V^* \to \mathcal{G}^*$ by

$$
\langle \psi_\pi(t, \tau), x\rangle = \operatorname{sym}\pi(x)(t, \tau).
$$

PROPOSITION 1.1. *ψ_π is an affine symplectomorphism from $V \times V^*$ onto the orbit $\mathcal{O}_\pi$.*

Proof. Let $\psi = \psi_\pi$. If $v \in V$ and $w \in \tilde V$, then it follows from the Campbell–Baker–Hausdorff formula that $\langle \psi(t, \tau), v + w\rangle = \langle \psi'(t, \tau), v + w\rangle + \langle \xi, w\rangle$, where

$$
\langle \psi'(t, \tau), v + w\rangle = \langle \xi, [t, w] + \tfrac{1}{2}[t, v]\rangle + \langle \tau, v\rangle. \tag{1.3}
$$

Thus ψ is affine with linear part ψ'.

For $w \in \tilde V$ define $f_w \in V^*$ by $f_w(v) = \langle \xi, [v, w]\rangle$. Since $\tilde V$ is maximally subordinate to ξ, $\{f_w: w \in \tilde V\} = V^*$. Hence given $(t, \tau) \in V \times V^*$ there is an $s \in \tilde V$ such that

$$
\langle \xi, [s, v]\rangle = -\tfrac{1}{2}\langle \xi, [t, v]\rangle + \langle \tau, v\rangle, \quad \text{for all } v \in V. \tag{1.4}
$$

Then $(\exp \operatorname{ad}(s+t))^*\xi = \xi + \psi'(t,\tau)$, so $\psi'(t,\tau)$ is parallel to $\mathcal{O}_\pi$ for all (t,τ). Choosing $w_0 \in \tilde{V}$ so that $\langle \xi, v\rangle = \langle \xi, [v, w_0]\rangle$ for all $v \in V$, we see that $\psi(0,0) = (\exp \operatorname{ad} w_0)^*\xi \in \mathcal{O}_\pi$. Thus $\psi_\pi : V \times V^* \to \mathcal{O}_\pi$. The bijectivity of ψ_π follows from the fact that $\dim \mathcal{O}_\pi = 2 \dim V$. For $(t,\tau) \in V \times V^*$ and $(y,\eta) \in V \times V^*$ choose $s \in \tilde{V}$ so that (1.4) holds and $z \in \tilde{V}$ so that $\langle \xi, [z, v]\rangle = -\frac{1}{2}\langle \xi, [y, v]\rangle + \langle \eta, v\rangle$, for all $v \in V$. Then

$$\sigma(\psi'(t,\tau), \psi'(y,\eta)) = \langle \xi, [s+t, y+z]\rangle = \langle \tau, y\rangle - \langle \eta, t\rangle,$$

so ψ is a symplectomorphism. □

Since G is nilpotent we may, and throughout the paper shall, take for Haar measure on G the push forward via the exponential map of the measure dx on $\mathcal{G}$. Thus if $\phi \in \mathcal{S}(G)$ and π is a unitary representation of G,

$$\pi(\phi) = \int_{\mathcal{G}} \phi(\exp x)\pi(\exp x)\, dx.$$

DEFINITION. If $p \in \mathcal{S}(\mathcal{G}^*)$ and π is a unitary representation of G, define $\pi(p)$ to be $\pi(\hat{p}\circ \log)$, where

$$\hat{p}(\xi) = F(p)(\xi) = \int_{\mathcal{G}^*} e^{-i\langle \xi, x\rangle} p(\xi)\, d\xi.$$

PROPOSITION 1.2. *For any irreducible unitary representation $\pi = \pi_{\xi, V, \tilde{V}}$, the Weyl symbol of $\pi(p)$ is $p \circ \psi_\pi$.*

Before proving this we show that it yields immediately the fibering of the pseudodifferential operator calculus for symbols in $\mathcal{S}$ as described in the introduction. Given p and q in $\mathcal{S}(\mathcal{G}^*)$, if $p\#q$ is the function for which $\lambda(p\#q) = \lambda(p)\lambda(q)$ then $F(p\#q)\circ \log = (F(p)\circ \log) * (F(q)\circ \log)$, hence $\pi(p\#q) = \pi(p)\pi(q)$. If $\mathcal{O}$ is any orbit of the coadjoint action, let $\pi = \pi_{\xi, V, \tilde{V}}$ be a representation corresponding to $\mathcal{O}$. Then

$$p\#q|_{\mathcal{O}} = [(p\circ \psi_\pi)\#(q\circ \psi_\pi)]\circ \psi_\pi^{-1} \tag{1.5}$$

where the # on the right-hand side refers to the composition product of symbols in the Weyl pseudodifferential operator calculus on V.

Proof of Proposition 1.2. If G is a Heisenberg group the result is well-known ([4], [12]), at least in slightly different guises. Indeed, if π is a one-dimensional representation, i.e. $\mathcal{O} = \{\xi\}$, then the result is trivial, $\pi(p)$ being multiplication by $p(\xi)$; while, if π is a representation satisfying (1.2) with $\lambda_1 = \cdots = \lambda_d = \lambda > 0$, and $\mathcal{G}$ and $\mathcal{G}^*$ are identified with R^{2d+1} by means of the corresponding basis $\{y_1, \ldots, y_{2d}, [y_1, y_{d+1}]\}$ and its dual, then $\psi_\pi(t,\tau) = (\tau, \lambda t, \lambda)$ and

$$\pi(\hat{p}\circ \log) = \int e^{i\lambda(b\cdot t + c + a\cdot b/2)} \hat{p}(a,b,c) f(t+a)\, da\, db\, dc$$

$$= \int e^{i\tau(t-s)} p(\tau, \lambda(s+t)/2, \lambda) f(s)\, ds\, d\tau.$$

For general step two G let $\mathcal{R} = \{x: \langle \xi, [x,y] \rangle = 0, \text{ for all } y \in \mathcal{G}\}$ and let $\mathcal{G}_0 = \{x \in \mathcal{R}: \langle \xi, x \rangle = 0\}$. Then $\mathcal{G}/\mathcal{G}_0$ is either Abelian or a Heisenberg algebra. Let $\rho: \mathcal{G} \to \mathcal{G}/\mathcal{G}_0$ be the natural projection. Define $\eta \in (\mathcal{G}/\mathcal{G}_0)^*$ by $\eta(x+\mathcal{G}_0) = \xi(x)$, and let $\pi_1 = \pi_{\eta, \rho V, \rho \tilde{V}}$. Then $\pi = \pi_1 \circ \rho$ (identifying V and ρV), and hence $\psi_\pi = \rho^* \circ \psi_{\pi_1}$. By the Fourier inversion theorem, for any $x \in \mathcal{G}$ and $p \in \mathcal{S}(\mathcal{G}^*)$, $\int_{\mathcal{G}_0} \hat{p}(x+y)\, dy = (p \circ \rho^*)\hat{\ }(\rho x)$, from which it follows that $\pi(p) = \pi_1(p \circ \rho^*)$. Since the proposition is true for π_1 it is true for π. □

2. Classes of symbols. Let $|\ \ |$ be a norm on the step two nilpotent Lie algebra $\mathcal{G} = \mathcal{G}_1 \oplus \mathcal{G}_2$, where $\mathcal{G}_2 \supset [\mathcal{G}, \mathcal{G}]$. $|\ \ |$ will also denote the dual norm on $\mathcal{G}^*$. For $\xi \in \mathcal{G}^*$, let $\xi' = \xi|_{\mathcal{G}_2}$. A weight function on $\mathcal{G}^*$ is defined to be a continuous function Φ on $\mathcal{G}^*$ for which there exist constants $c > 0$ and $C \geqslant 1$ such that

$$(2.1) \qquad |\xi - \eta| \leqslant c\Phi(\xi) \quad \text{implies} \quad \Phi(\xi) \leqslant C\Phi(\eta), \quad \text{and}$$

$$(2.2) \qquad c(1+|\xi'|)^{1/2} \leqslant \Phi(\xi), \quad \text{for all } \xi \in \mathcal{G}^*.$$

Note that if (2.1) holds, then $|\xi - \eta| \leqslant cC^{-1}\Phi(\xi)$ implies $|\xi - \eta| \leqslant c\Phi(\eta)$ which implies $\Phi(\eta) \leqslant C\Phi(\xi)$. Also if (2.1) holds, then $|\xi| \leqslant c\Phi(\xi)$ implies $\Phi(\xi) \leqslant C\Phi(0)$, so $\Phi(\xi) \leqslant c^{-1}|\xi| + C\Phi(0)$. Thus (2.1) and (2.2) are equivalent to the following (with new c and C):

$$(2.1)' \qquad |\xi - \eta| \leqslant c\Phi(\xi) \quad \text{implies} \quad c\Phi(\xi) \leqslant \Phi(\eta) \leqslant C\Phi(\xi),$$

$$(2.2)' \qquad c(1+|\xi'|)^{1/2} \leqslant \Phi(\xi) \leqslant C(1+|\xi|).$$

Note that the definition of weight function is independent of the choice of the norm. We shall usually assume that the norm is associated with an inner product on $\mathcal{G}$ and also satisfies

$$(2.3) \qquad |[x,y]| \leqslant |x||y|.$$

Two weight functions Φ and Φ_1 are equivalent if there exist C and $c > 0$ such that $c\Phi(\xi) \leqslant \Phi_1(\xi) \leqslant C\Phi(\xi)$ for all ξ. As in [2], given a weight function Φ_1 there is an equivalent weight function Φ that is smooth and satisfies

$$(2.1)'' \qquad |d^k\Phi(\xi)| \leqslant C_k \Phi^{1-k}(\xi),$$

where $d^k\Phi$ is the kth derivative of Φ.

Also, given a norm on $\mathcal{G}$ and a weight function Φ_1 satisfying (2.2)′ there is an equivalent weight function $\Phi(\xi) = c^{-1}\Phi_1(\xi)$ satisfying

$$(2.2)'' \qquad (1+|\xi'|)^{1/2} \leqslant \Phi(\xi) \leqslant C(1+|\xi|).$$

Given a continuous function p on $\mathcal{G}^*$, $x \in \mathcal{G}$ and $\xi \in \mathcal{G}^*$, define $D_x p(\xi)$ by

$$D_x p(\xi) = \lim_{h \to 0} h^{-1}[p(\xi + h \operatorname{ad} x^* \xi) - p(\xi)],$$

when the limit exists. Note that $D_x p(\xi)$ is a derivative of p in a direction parallel to the orbit $\mathcal{O}_\xi$.

If Φ is a weight function on $\mathcal{G}^*$ we say that $p \in S^m(\Phi, \mathcal{G}^*)$ if for every k and every choice of $x_1, \ldots, x_k$ in $\mathcal{G}$, $D_{x_1} \cdots D_{x_k} p(\xi)$ exists and is continuous on $\mathcal{G}^*$ and $\rho_k(p, S^m(\Phi, \mathcal{G}^*))$ is finite, where $\rho_k(p, S^m(\Phi, \mathcal{G}^*))$ is defined as

$$\sup \Phi^{k-m}(\xi) |D_{x_1} \cdots D_{x_k} p(\xi)| \prod_1^k |(\operatorname{ad} x_j)^* \xi|^{-1},$$

the supremum being taken over all $\xi \in \mathcal{G}^*$ and all $x_1, \ldots, x_k \in \mathcal{G}$. Note that if p is k times continuously differentiable with kth derivative $d^k p$, then $D_{x_1} \cdots D_{x_k} p(\xi) = d^k p(\xi; \operatorname{ad} x_1^* \xi, \ldots, \operatorname{ad} x_k^* \xi)$.

If Φ is equivalent to Φ_1, then

$$S^m(\Phi, \mathcal{G}^*) = S^m(\Phi_1, \mathcal{G}^*).$$

PROPOSITION 2.1. *Let an inner product be given on $\mathcal{G}$ with norm satisfying* (2.3). *Let Φ be a weight function satisfying* (2.2)″. *Let $\pi = \pi_{\xi, V, \tilde{V}}$ be any of the irreducible unitary representations defined in Section* 1, *let $\psi_\pi \colon W = V \times V^* \to \mathcal{O}_\pi$ be the affine map defined in Section* 1 *and let ψ_π' be its linear part. For each $x \in W$ let $g_{\pi,x}$ be the quadratic form on W given by*

$$g_{\pi,x}(w) = |\psi_\pi'(w)|^2 \Phi(\psi_\pi(x))^{-2}.$$

For any real number m, let $\tilde{m}_\pi(x) = \Phi(\psi_\pi(x))^m$. Then in the terminology of Hörmander [3], *g_π is slowly varying, σ temperate, $\tilde{m}_\pi$ is σ, g_π temperate and $g_\pi \leqslant g_\pi^\sigma$. Furthermore the constants c and C in equations* (2.4), (2.5), (4.5) *and Definition* 4.1 *of* [3] *can be chosen to depend only on the constants in* (2.1) *and* (2.2) *above and hence may be assumed to be independent of π. If $p \in S^m(\Phi, \mathcal{G}^*)$, then*

$$\sup_\pi \rho_k(p \circ \psi_\pi; S(\tilde{m}_\pi, g_\pi)) = \rho_k(p; S^m(\Phi, \mathcal{G}^*)),$$

where $\rho_k(q; S(\tilde{m}, g)) = \sup |d^k q(x; t_1, \ldots, t_k)| \tilde{m}(x)^{-1} \prod g_x(t_j)^{-1/2}$.

Proof. The last assertion is an immediate consequence of the definitions of the seminorms ρ_k, the fact that each $\psi_\pi' t_j$ is parallel to $\mathcal{O}_\pi$ and hence for any $y \in W$, $\psi_\pi' t_j = \operatorname{ad} x_j^* \psi_\pi y$ for some $x_j \in \mathcal{G}$, and the fact that then

$$d^k(p \circ \psi_\pi)(y; t_1, \ldots, t_k) = D_{x_1} \cdots D_{x_k} p(\psi_\pi y).$$

Also that g_π is slowly varying and $\tilde{m}_\pi$ is g_π continuous are immediate consequences of (2.1)′ and (2.2)′ above.

Suppose that $\pi_1 = \pi_{\xi_1, V_1, \tilde{V}_1}$ and $\pi_2 = \pi_{\xi_2, V_2, \tilde{V}_2}$ are unitarily equivalent. Let $W_j = V_j \times V_j^*$, $\psi_j = \psi_{\pi_j}$, $g_j = g_{\pi_j}$ and $\tilde{m}_j = \tilde{m}_{\pi_j}$ for $j = 1, 2$. By Proposition 1.1 there is an affine map $S = S' + S_0 \colon W_2 \to W_1$, where S' is linear and symplectic and S_0 is constant, such that $\psi_2 = \psi_1 \circ S$. Then $g_{2x}(w) = g_{1Sx}(S'w)$, $g_{2x}^{\sigma_2}(w) = g_{1Sx}^{\sigma_1}(S'w)$ and $\tilde{m}_2 = \tilde{m}_1 \circ S$. It is easily seen that if g_1 is σ_1 temperate and $\tilde{m}_1$ is σ_1, g_1 temperate then g_2 is σ_2 temperate and $\tilde{m}_2$ is σ_2, g_2 temperate with the same constants c and C. Thus to complete the proof of the proposition it suffices to consider only those representations satisfying (1.2).

Let $\mathcal{B} = \{y_1, \ldots, y_N\}$, V, $\tilde{V}$, $\mathcal{R}$, λ and ξ be as described in the paragraph preceding (1.2). Let $\{y_1^*, \ldots, y_N^*\}$ be the dual basis on $\mathcal{G}_1^*$. For $t \in V$ let λt denote that element of V with coordinates $(\lambda_1 t_1, \ldots, \lambda_d t_d)$ with respect to the basis $\{y_1, \ldots, y_d\}$. Define $\phi \colon V \to \mathcal{G}_1^*$ by $\phi(y_j) = y_{j+d}^*$. Let $\psi = \psi_\pi$, $g = g_\pi$, $\tilde{m} = \tilde{m}_\pi$. Then (1.2) implies that

$$\psi(t, \tau) = \tau + \phi(\lambda t) + \xi|_{\mathcal{R}}; \tag{2.4}$$

so $\psi'(t, \tau) = \tau + \phi(\lambda t)$. Since $\{y_1, \ldots, y_N\}$ is orthonormal,

$$|\psi'(t,\tau)|^2 = |\tau|^2 + |\lambda t|^2 = \sum \tau_j^2 + \lambda_j^2 t_j^2.$$

Thus $g_{x,\eta}(t,\tau) = (|\tau|^2 + |\lambda t|^2)\Phi(\psi(x,\eta))^{-2}$ and $g_{x,\eta}^\sigma(t,\tau) = (|\lambda^{-1}\tau|^2 + |t|^2)\Phi(\psi(x,\eta))^2$.

To prove that $g \leqslant g^\sigma$ it suffices to show that

$$\lambda_j \leqslant \Phi(\psi(x,\eta))^2, \quad \text{for all } (x,\eta) \in W, \text{ all } j \leqslant d. \tag{2.5}$$

Let $\psi_2(t,\tau)$ be the restriction of $\psi(t,\tau)$ to $\mathcal{G}_2$. Since $\psi(t,\tau) \in \mathcal{O}_\xi$, $\psi_2(t,\tau) = \xi|_{\mathcal{G}_2}$. Thus $\langle \psi_2(t,\tau), [y_j, y_{j+d}] \rangle = \lambda_j$ for $j \leqslant d$. Consequently, by (2.2)″ and (2.3),

$$\lambda_j \leqslant |\psi_2(t,\tau)||y_j||y_{j+d}| = |\psi_2(t,\tau)| \leqslant \Phi(\psi(t,\tau))^2.$$

Thus $g \leqslant g^\sigma$.

To prove that g is σ temperate, i.e.

$$g_{x,\eta}^\sigma \leqslant C g_{t,\tau}^\sigma (1 + g_{t,\tau}^\sigma(t-x, \eta-\tau))^{2N} \quad \text{for some } N,$$

it suffices to prove

$$\Phi(\psi(x,\eta))^2 \leqslant C^2 \Phi(\psi(t,\tau))^2 (1 + \Phi(\psi(t,\tau))^2 (|x-t|^2 + |\lambda^{-1}(\eta-\tau)|^2)). \tag{2.6}$$

If $|\psi(x,\eta) - \psi(t,\tau)|^2 \leqslant c^2 \Phi(\psi(x,\eta))^2$, then $\Phi(\psi(x,\eta))^2 \leqslant C^2 \Phi(\psi(t,\tau))^2$; otherwise

$$\begin{aligned} c^2 \Phi(\psi(x,\eta))^2 &\leqslant |\psi(x,\eta) - \psi(t,\tau)|^2 = |\lambda x - \lambda t|^2 + |\eta - \tau|^2 \\ &\leqslant \Phi(\psi(t,\tau))^4 (|x-t|^2 + |\lambda^{-1}(\eta-\tau)|^2) \end{aligned}$$

by (2.5). This proves (2.6). The same argument gives

$$\Phi(\psi(x,\eta))^2 \leqslant C^2 \Phi(\psi(t,\tau))^2 (1 + \Phi(\psi(x,\eta))^2 (|x-t|^2 + |\lambda^{-1}(\eta-\tau)|^2)), \tag{2.7}$$

which implies

$$\tilde{m}(w_1) \leqslant C_m \tilde{m}(w)(1 + g_{w_1}^\sigma(w - w_1))^{1/2|m|} \tag{2.8}$$

for $m \geqslant 0$, while (2.6) implies (2.8) for $m \leqslant 0$. Thus $\tilde{m}$ is σ, g temperate. This completes the proof of Proposition 2.1. □

DEFINITION. Given $p \in S^m(\Phi, \mathcal{G}^*)$ and a sequence $\{p_j\}$ in $S^m(\Phi, \mathcal{G}^*)$ we say that $\{p_j\}$ converges weakly to p in $S^m(\Phi, \mathcal{G}^*)$, if $\{p_j\}$ is bounded in $S^m(\Phi, \mathcal{G}^*)$ (with respect to the seminorms ρ_k), and if for all $x_1, \ldots, x_k \in \mathcal{G}$ and every compact $K \subset \mathcal{G}^*$, $D_{x_1} \cdots D_{x_k} p_j$ converges to $D_{x_1} \cdots D_{x_k} p$ uniformly on K.

Let $\tilde{\mathfrak{D}}(\mathcal{G}^*)$ denote the vector space of all continuous functions p defined on $\mathcal{G}^*$ with compact support for which $D_{x_1}\cdots D_{x_k}p$ exists and is continuous for all $x_1,\ldots,x_k$ in $\mathcal{G}$.

PROPOSITION 2.2. *Given $p\in S^m(\Phi,\mathcal{G}^*)$ there is a sequence $\{p_j\}$ in $\tilde{\mathfrak{D}}(\mathcal{G}^*)$ converging weakly to p in $S^m(\Phi,\mathcal{G}^*)$.*

Proof. Let $\phi\in C_c^\infty(\mathcal{G}^*)$, $\phi_j(\xi)=\phi(j^{-1}\xi)$. Since $\Phi(\xi)\leqslant C_1(|\xi|+1)\leqslant Cj$ on the support of $D^\alpha\phi_j(\xi)=j^{-|\alpha|}D^\alpha\phi(j^{-1}\xi)$, there is a C_α such that $D^\alpha\phi_j(\xi)\leqslant C_\alpha\Phi(\xi)^{-|\alpha|}$ for all ξ and all j. Thus $\{\phi_j p\}$ converges weakly to p in $S^m(\Phi,\mathcal{G}^*)$. □

3. The symbol product. If $p\in S^m(\Phi,\mathcal{G}^*)$ and $q\in S^k(\Phi,\mathcal{G}^*)$ it is shown in this section that there is a $p\#q\in S^{m+k}(\Phi,\mathcal{G}^*)$ such that $\lambda(p)\lambda(q)u=\lambda(p\#q)u$ (for u in a certain space of functions containing $\mathcal{S}(G)$). Theorem 3.3 deals with an asymptotic formula for $p\#q$.

Let $\pi=\pi_{\xi,V,\tilde{V}}$ be any of the irreducible unitary representations described in Section 1, let $\psi_\pi\colon V\times V^*\to\mathcal{O}_\pi$ be the map defined in Section 1, and define g_π and $\tilde{m}_\pi$ as in Proposition 2.1.

DEFINITION 3.1. If $p\in S^m(\Phi,\mathcal{G}^*)$, $\pi(p)$ is the pseudodifferential operator on V with Weyl symbol $p\circ\psi_\pi$.

By Proposition 1.2 this is consistent with the previous definition, $\pi(p)=\pi(\hat{p}\circ\log)$, when $p\in\mathcal{S}(\mathcal{G}^*)$. Let $q\in S^k(\Phi,\mathcal{G}^*)$. By Theorem 4.2 of [3] and Proposition 2.1 above there is an element, denoted $(p\circ\psi_\pi)\#(q\circ\psi_\pi)$, in $S((m+k)_\pi^{\tilde{}},g_\pi)$ which is the symbol in the Weyl calculus for the operator $\pi(p)\pi(q)$.

DEFINITION 3.2. If $p\in S^m(\Phi,\mathcal{G}^*)$ and $q\in S^k(\Phi,\mathcal{G}^*)$, $p\#q$ is that function on $\mathcal{G}^*$ which is defined on each orbit $\mathcal{O}_\pi$ by

$$p\#q|_{\mathcal{O}_\pi}=((p\circ\psi_\pi)\#(q\circ\psi_\pi))\circ\psi_\pi^{-1}.$$

We must first show that for any orbit $\mathcal{O}$ the definition of $p\#q|_{\mathcal{O}}$ is independent of the choice of the representation $\pi_{\xi,V,\tilde{V}}$ for the orbit $\mathcal{O}$. If $\pi_1=\pi_{\xi_1,V_1,\tilde{V}_1}$ and $\mathcal{O}_{\pi_1}=\mathcal{O}_\pi$, then $\chi=\psi^{-1}\circ\psi_1\colon V_1\times V_1^*\to V\times V^*$ is an affine symplectic map. Using the formula

$$a_1\#a_2(x,\xi)=\iiiint a_1(x+z,\xi+\zeta)a_2(x+t,\xi+\tau)\exp\{2i\sigma(t,\tau;z,\zeta)\}\,dz\,d\zeta\,dt\,d\tau$$

given in [3] for the symbol of the composition of two pseudodifferential operators on V in the Weyl calculus (σ being the symplectic form on $V\times V^*$), it is easily seen that $(a_1\#a_2)\circ\chi=(a_1\circ\chi)\#(a_2\circ\chi)$. Thus $p\#q$ as defined by Definition 3.2 is a well-defined function on $\mathcal{G}^*$.

THEOREM 3.1. *The map $(p,q)\mapsto p\#q$ is a continuous bilinear function from $S^m(\Phi,\mathcal{G}^*)\times S^k(\Phi,\mathcal{G}^*)$ to $S^{m+k}(\Phi,\mathcal{G}^*)$. Furthermore if $p_j\in\tilde{\mathfrak{D}}(\mathcal{G}^*)$ converges weakly to p in $S^m(\Phi,\mathcal{G}^*)$ and $q_j\in\tilde{\mathfrak{D}}(\mathcal{G}^*)$ converges weakly to q in $S^k(\Phi,\mathcal{G}^*)$, then $p_j\#q_j$ converges weakly to $p\#q$ in $S^{m+k}(\Phi,\mathcal{G}^*)$.*

Proof. We begin by deriving some integral formulas for $p\#q$ when p and q are in $\tilde{\mathfrak{D}}(\mathcal{G}^*)$. Note that Proposition 1.2 holds for $p\in\tilde{\mathfrak{D}}(\mathcal{G}^*)$, thus $\pi(p)=\pi(\hat{p}\circ\log)$, $p\#q=F^{-1}((\hat{p}\circ\log*\hat{q}\circ\log)\circ\exp)$ and therefore

$$p\#q(\eta) = \int_{\mathcal{G}}\int_{\mathcal{G}} e^{i\langle\eta,y\rangle}\hat{p}(x)\hat{q}(\log(\exp(-x)\exp y))\,dx\,dy.$$

Making the change of variables $\exp z = \exp(-x)\exp y$ we obtain

$$p\#q(\eta) = \int_{\mathcal{G}}\int_{\mathcal{G}} e^{i\langle\eta,x+\exp\operatorname{ad}(1/2x)(z)\rangle}\hat{p}(x)\hat{q}(z)\,dx\,dz. \tag{3.1}$$

If p is defined on $\mathcal{G}$ and $y \in \mathcal{G}$ let $D_y p(x) = \frac{d}{dt}p(x+t\operatorname{ad}yx)|_{t=0}$ when the derivative exists. If $p \in \tilde{\mathfrak{D}}(\mathcal{G}^*)$, then $D_y Fp(x)$ exists for all x and y in $\mathcal{G}$ and

$$\begin{aligned} FD_y p(x) &= \frac{d}{dt}\int e^{-i\langle\xi,x\rangle}p(\xi+t\operatorname{ad}y^*\xi)\,d\xi\bigg|_{t=0} \\ &= \frac{d}{dt}\int e^{-i\langle\xi+t\operatorname{ad}(-y)^*\xi,x\rangle}p(\xi)\,d\xi\bigg|_{t=0} = D_{-y}Fp(x). \end{aligned} \tag{3.2}$$

Let $f(\eta,x,z) = \exp i\langle\eta, x+\exp\operatorname{ad}(\frac{1}{2}x)z\rangle$ and let $D_{y(\eta)}f(\eta,x,z)$ denote the result of applying the operator D_y to f as a function of η. Define $D_{y(x)}$ and $D_{y(z)}$ similarly. Because of the step two nilpotence of G,

$$D_{y(\eta)}f(\eta,x,z) = i\langle\eta,[y,x+z]\rangle f(\eta,x,z) = D_{y(x)}f(\eta,x,z) + D_{y(z)}f(\eta,x,z).$$

Thus (3.1) and an integration by parts imply that

$$D_y(p\#q)(\eta) = \iint f(\eta,x,z)[D_{-y}\hat{p}(x) + D_{-y}\hat{q}(z)]\,dx\,dz.$$

It now follows from (3.2) that

$$D_y(p\#q) = (D_y p)\#q + p\#D_y q \tag{3.3}$$

for p and q in $\tilde{\mathfrak{D}}(\mathcal{G}^*)$.

Applying the Fourier inversion theorem to (3.1) yields

$$p\#q(\eta) = \int_{\mathcal{G}^*}\int_{\mathcal{G}} e^{i\langle\eta-\xi,x\rangle}p(\xi)\,q(\exp\operatorname{ad}(\tfrac{1}{2}x)^*\eta)\,dx\,d\xi. \tag{3.4}$$

Given $\eta \in \mathcal{G}^*$, let $\mathcal{R} = \{x: \langle\eta,[x,y]\rangle = 0 \text{ for all } y \in \mathcal{G}\}$. Let $\mathcal{G}_0$ be a subspace so that $\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{R}$. We may identify $\mathcal{G}_0^*$ with $\mathcal{R}^\perp$, the annihilator of $\mathcal{R}$ in $\mathcal{G}^*$, and $\mathcal{R}^*$ with $\mathcal{G}_0^\perp$. Let $\eta = \eta_1 + \eta_2$ with $\eta_1 \in \mathcal{R}^*$ and $\eta_2 \in \mathcal{G}_0^*$. If $x_1 \in \mathcal{R}$, then

$$q(\exp\operatorname{ad}(\tfrac{1}{2}(x+x_1))^*\eta) = q(\exp\operatorname{ad}(\tfrac{1}{2}x)^*\eta).$$

so applying the Fourier inversion formula on $\mathcal{R}$ and using Fubini's Theorem, (3.4) implies

$$p\#q(\eta) = \int_{\mathcal{G}_0^*}\int_{\mathcal{G}_0} e^{i\langle\eta_2-\xi,x\rangle}p(\xi+\eta_1)q(\exp\operatorname{ad}(\tfrac{1}{2}x)^*\eta)\,dx\,d\xi.$$

Since $\mathcal{O}_\eta = \mathcal{G}_0^* + \{\eta\}$, this can be written as

$$p\#q(\eta) = \int_{\mathcal{O}_\eta}\int_{\mathcal{G}/\mathcal{R}_\eta} e^{i\langle \eta-\xi, x\rangle} p(\xi) q(\exp \operatorname{ad}(\tfrac{1}{2}x)^*\eta)\, dx\, d\xi. \tag{3.5}$$

Integration by parts shows that for any $M \in \mathbf{N}$,

$$\begin{aligned} p\#q(\eta) = \int_{\mathcal{G}_0^*}\int_{\mathcal{G}_0} & e^{i\langle \eta_2-\xi, x\rangle}(1+|x|^2)^{-M}(I-\Delta_\xi)^M[p(\xi+\eta_1)(1+|\eta_2-\xi|^2)^{-M}] \\ & \times (I-\Delta_x)^M q(\exp \operatorname{ad}(\tfrac{1}{2}x)^*\eta)\, dx\, d\xi. \end{aligned} \tag{3.6}$$

Note that $\Delta_x q(\exp \operatorname{ad}(\frac{1}{2}x)^*\eta)$ involves derivatives of q only in directions parallel to $\mathcal{O}_\eta$. Consequently if M is sufficiently large, the integral on the right-hand side of (3.6) converges absolutely for any $p \in S^m(\Phi, \mathcal{G}^*)$ and any $q \in S^k(\Phi, \mathcal{G}^*)$.

Suppose $\{p_j\} \in \tilde{\mathfrak{D}}(\mathcal{G}^*)$ converges weakly to p in $S^m(\Phi, \mathcal{G}^*)$ and $\{q_j\} \in \tilde{\mathfrak{D}}(\mathcal{G}^*)$ converges weakly to q in $S^k(\Phi, \mathcal{G}^*)$. Let $\mathcal{O}_\pi$ be any orbit of the coadjoint action of G on $\mathcal{G}^*$ and let $\psi = \psi_\pi$. By Theorem 4.2 of [3], $(p_j\#q_j)\circ\psi = (p_j\circ\psi)\#(q_j\circ\psi)$ converges uniformly on compact subsets of $V\times V^*$ to $(p\circ\psi)\#(q\circ\psi) = (p\#q)\circ\psi$. Thus for any orbit $\mathcal{O}$, $p_j\#q_j$ converges uniformly on compact subsets of $\mathcal{O}$ to $p\#q$. We want to show that $p_j\#q_j$ converges uniformly on compact subsets of $\mathcal{G}^*$ to $p\#q$. To show this it suffices to show that $p_j\#q_j$ converges uniformly on compact subsets of $\mathcal{G}^*$ to some function. But it can be seen that $p_j\#q_j$ converges uniformly on compact subsets of $\mathcal{G}^*$ to the function given by the integral on the right-hand side of (3.6). Using (3.3) and repeating the above argument inductively shows that for any $x_1,\dots,x_k$ in $\mathcal{G}$, $D_{x_1}\cdots D_{x_k}(p_j\#q_j)$ converges uniformly on compact subsets of $\mathcal{G}^*$ to $D_{x_1}\cdots D_{x_k}(p_j\#q_j)$. In particular $D_{x_1}\cdots D_{x_k}(p\#q)$ is continuous. Also (3.3) holds for all $p \in S^m(\Phi, \mathcal{G}^*)$ and $q \in S^k(\Phi, \mathcal{G}^*)$.

To prove the second statement of the theorem it now suffices to prove the first statement, since continuity will imply that $\{p_j\#q_j\}$ is bounded in $S^{m+k}(\Phi, \mathcal{G}^*)$ if $\{p_j\}$ and $\{q_j\}$ are bounded in $S^m(\Phi, \mathcal{G}^*)$and $S^k(\Phi, \mathcal{G}^*)$ respectively.

As in Section 2, ρ_j denotes the jth seminorm either in $S^m(\Phi, \mathcal{G}^*)$ or in one of the spaces $S(\tilde{m}_\pi, g_\pi)$. Let π be any of the representations $\pi_{\xi, V, \tilde{V}}$. By Theorem 4.2 of [3] and Proposition 2.1 above, given i there exist J and C such that

$$\begin{aligned} & \rho_i(p\circ\psi_\pi\#q\circ\psi_\pi; S((m+k)\tilde{}_\pi, g_\pi)) \\ & \leqslant C\sum_0^J \rho_j(p\circ\psi_\pi; S(\tilde{m}_\pi, g_\pi)) \sum_0^J \rho_j(q\circ\psi_\pi; S(\tilde{k}_\pi, g_\pi)). \end{aligned} \tag{3.7}$$

The J and C in (3.7) can be chosen independently of π, since the c and C in (2.4), (2.5), (4.5), and Definition 4.1 of [1] can be chosen independently of π. By Proposition 2.1

$$\rho_i(p\#q; S^{m+k}) = \sup_\pi \rho_i(p\circ\psi_\pi\#q\circ\psi_\pi; S((m+k)\tilde{}_\pi, g_\pi)).$$

Thus (3.7) implies

$$\rho_i(p\#q; S^{m+k}) \leqslant C\sum_0^J \rho_j(p; S^m) \sum_0^J \rho_j(q; S^k),$$

which completes the proof of Theorem 3.2. □

We wish now to show that $\lambda(p\#q) = \lambda(p)\lambda(q)$, but there is a problem in that $\lambda(q)$ does not map $\mathcal{S}(G)$ into $\mathcal{S}(G)$. Therefore we introduce the following larger space of functions: we say that $u \in S^{-\infty}(\Phi, G)$ if

$$\tilde{u} = F^{-1}(u \circ \exp) \in \bigcap_k S^k(\Phi, \mathcal{G}^*).$$

For $p \in S^m(\Phi, \mathcal{G}^*)$ and $u \in S^{-\infty}(\Phi, G)$ define

$$\lambda(p)u = F(p\#\tilde{u}) \circ \log.$$

By Theorem 3.1, $\lambda(p): S^{-\infty}(\Phi, G) \to S^{-\infty}(\Phi, G)$. We wish to show that this definition of $\lambda(p)u$ agrees with the original definition, $\lambda(p)u = \hat{p} \circ \log * u$, when $u \in \mathcal{S}(G)$. This is clear if $p \in \tilde{\mathfrak{D}}(\mathcal{G}^*)$, since Proposition 2.2 holds for $p \in \tilde{\mathfrak{D}}(\mathcal{G}^*)$. If $p \in S^m(\Phi, \mathcal{G}^*)$, let $\{p_j\}$ be a sequence in $\tilde{\mathfrak{D}}(\mathcal{G}^*)$ converging weakly to p in $S^m(\Phi, \mathcal{G}^*)$. Then $p_j\#\tilde{u} = (\hat{p}_j \circ \log * u)^{\sim}$ converges to $(\hat{p} \circ \log * u)^{\sim}$ in $\mathcal{S}^*(\mathcal{G}^*)$ as $j \to \infty$. By Theorem 3.1 $p_j\#\tilde{u}$ converges uniformly on compacts to $p\#\tilde{u}$. Hence $F(p\#\tilde{u}) \circ \log = \hat{p} \circ \log * u$, for $u \in \mathcal{S}(G)$.

PROPOSITION 3.2. *If $p \in S^m(\Phi, \mathcal{G}^*)$ and $q \in S^k(\Phi, \mathcal{G}^*)$, then $\lambda(p\#q)u = \lambda(p)\lambda(q)u$ for all $u \in S^{-\infty}(\Phi, G)$.*

Proof. The # product is associative, so $\lambda(p\#q)u = F(p\#q\#\tilde{u}) \circ \log = \lambda(p)\lambda(q)u$. □

Let V and V_1 be finite dimensional vector spaces, $W = V \oplus V^*$, $W_1 = V_1 \oplus V_1^*$, and let ψ be an affine symplectic map from W to W_1. For $F = F(x, \xi; y, \eta) \in C^\infty(W \oplus W)$ let

$$\sigma(D)F = \sum \left(\frac{\partial^2}{\partial \xi_j \partial y_j} - \frac{\partial^2}{\partial x_j \partial \eta_j} \right) F.$$

Then $\sigma(D)(F \circ (\psi \times \psi)) = (\sigma(D)F) \circ \psi \times \psi$, and consequently,

$$\sigma(D)^k(F \circ (\psi \times \psi)) = (\sigma(D)^k F) \circ (\psi \times \psi) \tag{3.8}$$

for all $k \in \mathbf{N}$. For f and g in $C^\infty(W)$ let

$$\{f, g\}_j(x, \xi) = \sigma(D)^j (f \otimes g)(x, \xi; x, \xi).$$

(3.8) implies that

$$\{f \circ \psi, g \circ \psi\}_j = \{f, g\}_j \circ \psi. \tag{3.9}$$

Note that $\{f, g\}_1$ is the usual Poisson bracket and $\{f, g\}_0 = fg$.

Let $p \in S^m(\Phi, \mathcal{G}^*)$ and $q \in S^k(\Phi, \mathcal{G}^*)$ and let $\mathcal{O}$ be an orbit of the coadjoint action of G on $\mathcal{G}^*$. Let $\pi = \pi_{\xi, V, \tilde{V}}$ where $\xi \in \mathcal{O}$. Define $\{p, q\}_j$ on $\mathcal{O}$ by

$$\{p, q\}_j|_{\mathcal{O}} = \{p \circ \psi_\pi, q \circ \psi_\pi\}_j \circ \psi_\pi^{-1}.$$

Proposition 1.1 and (3.9) imply that $\{p, q\}_j|_{\mathcal{O}}$ is defined independently of the choice of the particular representation π corresponding to the orbit $\mathcal{O}$. The asymptotic expansion for the symbol of a composition in the Weyl calculus (Theorem 4.2 of [3]) should therefore yield the following well-defined asymptotic expansion in our calculus:

$$p\#q \sim \sum (2i)^{-j}\{p,q\}_j/j!.$$

We wish to obtain estimates for the remainder

$$r_J = p\#q - \sum_{j<J} (2i)^{-j}\{p,q\}_j/j!. \tag{3.10}$$

Let $\xi \in \mathcal{G}^*$, let $\pi = \pi_{\xi,V,\tilde{V}}$ and let $\psi = \psi_\pi$, $g = g_\pi$, $\tilde{m} = \tilde{m}_\pi$, $\tilde{k} = \tilde{k}_\pi$ be defined as in Sections 1 and 2. According to Theorem 4.2 of [3], $r_J \circ \psi \in S(\tilde{m}\tilde{k}h^J, g)$ where

$$h(x,\eta)^2 = \sup g_{x,\eta}/g^\sigma_{x,\eta}.$$

Using the coordinate system introduced in the proof of Proposition 2.1 and the formulas given for $g_{x,\eta}$ and $g^\sigma_{x,\eta}$ just after equation (2.5) we obtain

$$\begin{aligned} h(x,\eta)^2 &= \Phi(\psi(x,\eta))^{-4}\sup(|\tau|^2 + |\lambda t|^2)/(|\lambda^{-1}\tau|^2 + |t|^2) \\ &= \Phi(\psi(x,\eta))^{-4}\max_{j\leqslant d}|\lambda_j|^2. \end{aligned}$$

Here $\lambda_j = \langle \xi, [y_j, y_{j+d}]\rangle$ where y_j and y_{j+d} are certain unit vectors in $\mathcal{G}$ as in the paragraph preceding (1.2). Using (2.3) we see that $\max|\lambda_j| \leqslant |\xi'|$, where $\xi' = \xi|_{\mathcal{G}_2}$. Thus

$$h(x,\eta) \leqslant \Phi(\psi(x,\eta))^{-2}|\xi'|.$$

The techniques used to prove Theorem 3.1 can be applied again to prove the following:

THEOREM 3.3. *Let $p \in S^m(\Phi, \mathcal{G}^*)$ and $q \in S^k(\Phi, \mathcal{G}^*)$. For $J \in \mathbf{N}$ define r_J by* (3.10). *Then given $M \in \mathbf{N}$ there is a C_M such that*

$$|D_{x_1}\cdots D_{x_M} r_J(\xi)| \leqslant C_M |\xi'|^J \Phi(\xi)^{m+k-2J-M}\prod_1^M |\operatorname{ad} x_j^* \xi|$$

for all $\xi \in \mathcal{G}^*$ and all $x_1, \ldots, x_M \in \mathcal{G}$.

Note that if $\xi' = 0$, then $\mathcal{O}_\xi = \{\xi\}$, which implies that $p\#q(\xi) = p(\xi)q(\xi)$ and hence $r_J(\xi) = 0$ for all $J \in \mathbf{N}$.

If x is linear on $\mathcal{G}^*$, i.e. if $x \in \mathcal{G}$, and λ is the left regular representation of G, then $\lambda(x)$ as defined by $\lambda(x)u = (Fx \circ \log) * u$ is the same as $\lambda(x)$ as defined by (1.1). Also if $\pi = \pi_{\xi,V,\tilde{V}}$, then $\pi(x)$ as defined by Definition 3.1 is the same as $\pi(x)$ as defined by (1.1). If x_1 and x_2 are in $\mathcal{G}$, then Proposition 3.2 and Theorem 3.3 imply that

$$\lambda(x_1x_2) = \lambda(x_1)\lambda(x_2) + \tfrac{1}{2}i\lambda(\{x_1,x_2\})$$

is a right invariant partial differential operator on G and

$$\pi(x_1x_2) = \pi(x_1)\pi(x_2) + \tfrac{1}{2}i\pi(\{x_1,x_2\}).$$

Proceeding by induction and making use of the Birkhoff–Poincaré–Witt Theorem it follows that:

PROPOSITION 3.4. *The map $p \to P = \lambda(p)$ is a bijection from the vector space of polynomial functions on $\mathcal{G}^*$ onto the space of right invariant partial differential operators on G. If π is any of the representations $\pi_{\xi,V,\tilde{V}}$ then $\pi(p)$ as defined by Definition 3.1 is the same as the operator $\pi(P)$ (as defined, say, in [10]).*

4. L^2 boundedness.

THEOREM 4.1. *If $p \in S^0(\Phi, \mathcal{G}^*)$, then there is a C such that $\|\lambda(p)u\| \leqslant C\|u\|$ for all $u \in \mathcal{S}(G)$, where $\| \ \|$ is the L^2 norm.*

Proof. Let π be any of the irreducible unitary representations $\pi_{\xi,V,\tilde{V}}$. If $p \in \tilde{\mathfrak{D}}(\mathcal{G}^*)$ and $u \in \mathcal{S}(G)$,

$$\pi(\lambda(p)u) = \pi(\hat{p}\circ\log * u) = \pi(\hat{p}\circ\log)\pi(u) = \pi(p)\pi(u). \tag{4.1}$$

By the Plancherel Theorem for G there is a measure d on the set of equivalence classes of irreducible unitary representations of G such that

$$\|u\|^2 = \int |\!|\!|\pi(u)|\!|\!|^2 \, d\pi, \quad u \in L^2(G), \tag{4.2}$$

where $|\!|\!| \ |\!|\!|$ is the Hilbert–Schmidt norm on the Hilbert space of the representation π.

By Proposition 2.1 $\pi(p) \in S(1, g_\pi)$ for all $\pi = \pi_{\xi,V,\tilde{V}}$, with seminorms independent of π and where the constants in (2.4), (2.5), and (4.5) of [3] are independent of π. By Theorem 5.3 of [3] there is a C independent of π such that

$$\|\pi(p)\phi\| \leqslant C\|\phi\|, \tag{4.3}$$

for all $\phi \in L^2(V_\pi)$. By (4.1) $\|\pi(\lambda(p)u)\phi\| \leqslant C\|\pi(u)\phi\|$ for all $u \in \mathcal{S}(G)$ and $\phi \in L^2(V_\pi)$. Summing over an orthonormal basis for $L^2(V_\pi)$ gives $|\!|\!|\pi(\lambda(p)u)|\!|\!| \leqslant C|\!|\!|\pi(u)|\!|\!|$, where C is independent of π. It now follows from (4.2) that if $p \in \tilde{\mathfrak{D}}(\mathcal{G}^*)$, then there is a C such that $\|\pi(p)u\| \leqslant C\|u\|$, for all $u \in \mathcal{S}(G)$.

If p is an arbitrary element of $S^0(\Phi, \mathcal{G}^*)$ and $\{p_j\}$ in $\tilde{\mathfrak{D}}(\mathcal{G}^*)$ converges weakly to p in $S^0(\Phi, \mathcal{G}^*)$, then there is a C such that $\|\lambda(p_j)u\| \leqslant C\|u\|$ for all $u \in \mathcal{S}(G)$ and all j, for the C in (4.3) can be chosen independently of p for p in a bounded subset of $S^0(\Phi, \mathcal{G}^*)$. For $u \in \mathcal{S}(G)$, hence $\tilde{u} = F^{-1}(u\circ\exp) \in \mathcal{S}(\mathcal{G}^*)$, it can be seen using (3.6) that $p_j \# \tilde{u}$ converges to $p \# \tilde{u}$ in L^2. Hence $\lambda(p_j)u$ converges to $\lambda(p)u$ in L^2, so $\|\lambda(p)u\| \leqslant C\|u\|$. □

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