DISTORTION OF THE BOUNDARY UNDER CONFORMAL MAPPING

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1. Introduction. Let Ω be a simply connected domain and a be a point in Ω. It has always been interesting to study the relation between the harmonic measure ω_Ω^a on $\partial\Omega$ with reference to a and the α-dimensional Hausdorff measure m_α on $\partial\Omega$.

We recall that in case Ω is a Jordan domain and f is a univalent conformal mapping from the unit disk Δ onto Ω, together with its continuous extension to $\partial\Delta$, satisfying $f(0) = a$, then $\omega_\Omega^a(E)$ is nothing but $\frac{1}{2\pi} \cdot$ length of $f^{-1}(E)$. If Ω is a Jordan domain with rectifiable boundary, F. and M. Riesz [17; p. 293 vol. I] proved in 1916 that ω_Ω^a and m_1 are mutually absolutely continuous. This is not true for Ω with nonrectifiable boundary. In [7; p. 830 and p. 18 in the translation] M. Lavrentiev constructed a Jordan domain Ω, and a set $E \subseteq \partial\Omega$ of zero length and of positive harmonic measure. In [8] Lohwater and Seidel constructed a Jordan domain Ω whose boundary meets a line segment in a set E of positive length and zero harmonic measure with respect to Ω. In [10] McMillan and Piranian simplify the original proof of Lavrentiev and give a stronger version of the result.

THEOREM (McMillan and Piranian). *There is a Jordan domain Ω and a set $E \subseteq \partial\Omega$, so that $m_1(E) = 0$ but $\omega_\Omega^a(E) = 1$. Moreover, every univalent conformal mapping of the unit disk Δ onto Ω has a power series absolutely convergent on $\bar{\Delta}$.*

For a general simply connected domain Ω, $a \in \Omega$ and a Borel set $E \subseteq \partial\Omega$, it follows from Beurling's projection theorem [1] that $m_{1/2}(E) = 0$ implies $\omega_\Omega^a(E) = 0$. The example by Lavrentiev or McMillan and Piranian says that $m_1(E) = 0$ does not imply $\omega_\Omega^a(E) = 0$.

QUESTION A. Whether $m_\alpha(E) = 0$ for some $\frac{1}{2} < \alpha < 1$ implies that $\omega_\Omega^a(E) = 0$.

Towards this long standing question, Carleson has proved the following theorem [2].

THEOREM (Carleson). (A) *There exists a number β, $\beta > \frac{1}{2}$, so that if E is a subset of the boundary of a Jordan domain Ω and if $m_\beta(E) = 0$ then $\omega_\Omega^a(E) = 0$.*

(B) *If $h(t) = t\exp\{(\log \frac{1}{t})^\epsilon\}$, $0 < \epsilon < \frac{1}{2}$, then there exist a Jordan domain Ω and $E \subseteq \partial\Omega$ with $\omega_\Omega^a(E) > 0$ and the h-Hausdorff measure $\Lambda_h(E) = 0$.*

For the definition of h-Hausdorff measure see [14]. When $\alpha > 0$ and $h(t) = t^\alpha$, Λ_h is the α-dimensional Hausdorff measure which we have denoted by m_α.

Because the original proof of Carleson's Theorem (B) is difficult, we give a simpler, constructive new proof. Our construction leads to a result slightly stronger

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than both McMillan and Piranian's Theorem and Carleson's Theorem (B), namely, we consider a bigger h function for Λ_h, and require $\partial\Omega$ to be a quasi-circle.

THEOREM 1. *There exist a Jordan domain Ω whose boundary is a quasi-circle, and a set $F \subseteq \partial\Omega$ so that for any $A > 0$, $\Lambda_h(F) = 0$, where $h(t) = t\exp\{A(\log \frac{1}{t})^{1/2}\}$ and $\omega_\Omega^a(F) = 1$, for $a \in \Omega$. Moreover, every univalent conformal mapping of the unit disk Δ onto Ω has a power series absolutely convergent on $\bar{\Delta}$.*

We also show that this cannot happen if $\partial\Omega$ is required to be starlike, namely

THEOREM 2. *If Ω is a starlike domain and $E \subseteq \partial\Omega$, then $m_1(E) = 0$ implies that $\omega_\Omega^a(E) = 0$.*

However, we can construct a Jordan domain Ω and a set $F \subseteq \partial\Omega$ having all the properties in Theorem 1, and Ω nearly starlike in a certain sense; see Theorem 4. We also obtain some other results on the relation between m_1 and ω_Ω^a on the boundary of a starlike domain.

Towards Question A, for a specifically situated set E, Øksendal has proved the following [12; p. 471]:

THEOREM (Øksendal). *Suppose D is a simply-connected domain, $a \in D$. Let K be a subset of ∂D, and assume that there exists a straight line L such that $K \subseteq L$. Then $m_1(K) = 0$ implies that $\omega_D^a(K) = 0$.*

The question that he left open is

QUESTION B. Can the straight line be replaced by other curves in the above theorem?

In view of Theorem 1, the above theorem does not hold if L is a quasi-circle, however, it holds when L is quasi-smooth.

THEOREM 3. *Suppose D is a simply-connected domain, $a \in D$. Let K be a subset of ∂D and assume that there exists a quasi-smooth curve L such that $K \subseteq L$. Then $m_1(K) = 0$ implies that $\omega_D^a(K) = 0$.*

2. Harmonic measure on quasi-smooth curves. A rectifiable Jordan curve L in $\mathbf{C}$ is called *quasi-smooth* if there exists $M < \infty$ so that for any $x, y \in L$

$$m_1(\text{the arc in } L\backslash\{x,y\} \text{ with smaller diameter}) < M|x-y|. \tag{2.1}$$

A Jordan curve L is called a *quasi-circle* if there exists $M < \infty$ so that for any $x, y \in L$, $\min(\text{diameters of the two arcs } L\backslash\{x,y\}) < M|x-y|$.

A Jordan domain Ω is called an (ϵ, ∞) *domain* if whenever $x, y \in \Omega$ there is a rectifiable arc $\gamma \subseteq \Omega$ joining x to y and satisfying

$$m_1(\gamma) \leqslant \epsilon|x-y| \quad \text{and} \tag{2.2}$$

$$\operatorname{dist}(z, \partial\Omega) \geqslant \frac{|x-z|\,|y-z|}{\epsilon|x-y|} \quad \text{for all} \quad z \in \gamma. \tag{2.3}$$

Clearly a quasi-smooth curve is a quasi-circle. L is a quasi-circle if and only if the two simply connected domains complementary to L are (ϵ, ∞) domains for some $\epsilon > 0$ (see [6] and [9]); in fact ϵ depends only on M.

We denote by $\Delta(a,r)$ the disk $\{|z-a|<r\}$. *From now on, we let L be a quasi-smooth curve, Ω_1 or Ω_2 be the interior or the exterior of L respectively,* M the constant associated with L in (2.1); and use c and C to denote constants depending only on M. It is immediate from (2.1), (2.2) and (2.3), that there exists $r_0 < \operatorname{diam} L/(16M^2)$, depending only on M and $\operatorname{diam} L$, so that for any $a \in L$, $0<r<r_0$, we can find a point $A = A(a,r) \in \Omega$, where $\Omega = \Omega_1$ or Ω_2, satisfying

$$A \in \Delta(a, Cr) \setminus \Delta(a, 2r) \quad \text{and} \tag{2.4}$$

$$C^{-1} r < \operatorname{dist}(A, L) < Cr \tag{2.5}$$

for some constant $C > 2$. Also there is a circle $|z-P| = \operatorname{diam} L/(16M^2)$ that is completely in Ω.

Let D be a regular domain. The unique Borel probability measure on ∂D, denoted ω_D^z, such that for all continuous functions f on ∂D, the solution of the Dirichlet problem $Hf(z) = \int_{\partial D} f(w)\, d\omega_D^z(w)$, is called the *harmonic measure* on ∂D, evaluated at z. See [4; p. 165] for definition of harmonic measure on the boundary of general domains.

Let $\Omega = \Omega_1$ or Ω_2. We quote two properties of the harmonic measure ω_Ω from the work of Jerison and Kenig [5], where a domain whose boundary is a quasi-smooth curve is called a chord-arc domain.

THEOREM A. (A_∞ *property of* ω_Ω). *For any* $0<\epsilon<1$, *there exists* δ, $0<\delta<1$, *depending on M only so that if* $a \in L$, $0<r<r_0$, $\Gamma = \Delta(a,r) \cap L$ *and E is a Borel subset of* Γ, *then*

$$\frac{m_1(E)}{m_1(\Gamma)} < \delta \Rightarrow \frac{\omega_\Omega^X(E)}{\omega_\Omega^X(\Gamma)} < \epsilon$$

for any $X \in \Omega \setminus \Delta(a, 2r)$.

THEOREM B. (*Doubling property of* ω_Ω). *Suppose* $a \in L$, $0<r<r_0$ *and* $X \in \Omega \setminus \Delta(a,2r)$. *Then there exists a constant C depending on M only, so that* $\omega_\Omega^X(\Delta(a,r) \cap L) \leqslant C\omega_\Omega^X(\Delta(a,r/2) \cap L)$.

Theorem A is basically Theorem (2.1) in [5]. We state it in a slightly stronger form by combining statement (2.2), Lemma (2.5), and the proof of Theorem (2.1) in [5].

Theorem B is the combination of Lemmas (4.8), (4.9) in [5] and Harnack's inequality.

LEMMA 1. *Let* $\Omega = \Omega_1$ *or* Ω_2. *Suppose* $a \in L$, $0<r<r_0$, $A = A(a,r)$ *as in* (2.4) *and* $E \subseteq L \cap \Delta(a,r)$. *If* $m_1(E) \geqslant r/2$ *then* $\omega_\Omega^A(E) > c > 0$ *for some constant c.*

Proof. When $\Omega = \Omega_2$, the proof follows from the case $\Omega = \Omega_1$ by a reflection of $\mathbf{C}$ with respect to the circle: $|z-P| = \operatorname{diam} L/(16M^2)$. Hence we may assume $\Omega = \Omega_1$. From Theorem A, we conclude that for any $0<\epsilon<1$, there exists δ, $0<\delta<1$, depending on M only so that whenever $\Gamma = \Delta(a,r) \cap L$ for some $a \in L$, $0<r<r_0$

and E is a Borel subset of Γ, then

$$\frac{m_1(E)}{m_1(\Gamma)} < \delta \Rightarrow \frac{\omega_\Omega^X(E)}{\omega_\Omega^X(\Gamma)} < \epsilon \tag{2.6}$$

for any $X \in \Omega \backslash \Delta(a, 2r)$, or

$$\frac{\omega_\Omega^X(E)}{\omega_\Omega^X(\Gamma)} < \alpha \Rightarrow \frac{m_1(E)}{m_1(\Gamma)} < \beta, \tag{2.7}$$

where $\alpha = 1 - \epsilon$ and $\beta = 1 - \delta$.

Because $A \notin \Delta(a, 2r)$, Lemma 1 follows immediately from (2.7) if $\beta < \frac{1}{2}$. If $\beta \geqslant \frac{1}{2}$, we shall use Lemma 2 to reduce β. The difficulty occurs when Γ is not a connected set.

Let I be the shortest subarc of L that contains $\Delta(a,r) \cap L$, γ be a subarc of I and E be a Borel set on γ. We claim that for any $\epsilon' > 0$, there exists δ', $0 < \delta' < 1$, depending on Ω and P so that

$$\frac{m_1(E)}{m_1(\gamma)} < \delta' \Rightarrow \frac{\omega_\Omega^A(E)}{\omega_\Omega^A(\gamma)} < \epsilon'. \tag{2.8}$$

From (2.1) we can find two disks $\Delta(b, \rho_1)$ and $\Delta(b, \rho_2)$ with $b \in \gamma$, $\rho_1 = \operatorname{diam} \gamma$, $\rho_2 = \operatorname{diam} \gamma / (16 M^2)$ and satisfying

$$\Gamma_2 \equiv \Delta(b, \rho_2) \cap L \subseteq \gamma \subseteq \Delta(b, \rho_1) \cap L \equiv \Gamma_1. \tag{2.9}$$

By (2.1) again there exists a constant $C_1 > 0$, so that

$$C_1^{-1} m_1(\Gamma_1) < m_1(\gamma) < C_1 m_1(\Gamma_2). \tag{2.10}$$

By (2.2), (2.3), (2.4), (2.5), (2.10), the Harnack principle and Theorem B, we can find a constant $C_2 > 0$ so that

$$\omega_\Omega^A(\Gamma_1) \leqslant C_2 \omega_\Omega^A(\Gamma_2) \leqslant C_2 \omega_\Omega^A(\gamma), \tag{2.11}$$

$$\omega_\Omega^{A(b,\rho_1)}(E) \leqslant C_2 \omega_\Omega^A(E) \quad \text{and} \quad \omega_\Omega^A(\Gamma_1) \leqslant C_2 \omega_\Omega^{A(b,\rho_1)}(E). \tag{2.12}$$

Given $\epsilon' > 0$, let $\epsilon = C_2^{-3} \epsilon'$, $\delta' = \delta$ be the number depending on ϵ as chosen in (2.6). If $m_1(E)/m_1(\gamma) < \delta'$, then $m_1(E)/m_1(\Gamma_1) < \delta$ by (2.9). It follows from (2.6), (2.11) and (2.12) that

$$\frac{\omega_\Omega^A(E)}{\omega_\Omega^A(\gamma)} \leqslant C_2^3 \frac{\omega_\Omega^{A(b,\rho_1)}(E)}{\omega_\Omega^{A(b,\rho_1)}(\Gamma_1)} < \epsilon C_2^3 = \epsilon'.$$

This proves (2.8), which implies that

$$\frac{\omega_\Omega^A(E)}{\omega_\Omega^A(\gamma)} < \alpha' \Rightarrow \frac{m_1(E)}{m_1(\gamma)} < \beta' \tag{2.13}$$

where $\alpha' = 1 - \epsilon'$ and $\beta' = 1 - \delta'$. Because harmonic measure and m_1 measure are mutually absolutely continuous on L [5], using arc length as the parameter and applying Lemma 2 n-times, we obtain

$$\frac{\omega_\Omega^A(E)}{\omega_\Omega^A(\gamma)} < \left(\frac{\alpha'}{2}\right)^{2n} \Rightarrow \frac{m_1(E)}{m_1(\gamma)} < \beta'^{2n}. \tag{2.14}$$

The set E in our lemma satisfies $m_1(E)/m_1(I) \geqslant (r/2)/(2rM) = 1/(4M)$; letting $n = [\log(\log(4M)/(-\log\beta'))] + 1$, $\gamma = I$ in (2.14) we conclude that $\omega_\Omega^A(E)/\omega_\Omega^A(I) > c > 0$ for some constant c. By [5; Lemma 4.2], $\omega_\Omega^A(I) \geqslant \omega_\Omega^A(L \cap \Delta(a, r)) > c > 0$ for some constant c. Hence $\omega_\Omega^A(E) > c > 0$ for some constant c. This completes the proof. □

LEMMA 2. *Suppose ω is a Borel measure on an interval I, which is mutually absolutely continuous with respect to m_1. Suppose there exist $0 < \alpha < 1$, $0 < \beta < 1$ so that for any Borel set E and subarc γ, satisfying $E \subseteq \gamma \subseteq I$,*

$$\frac{\omega(E)}{\omega(\gamma)} < \alpha \Rightarrow \frac{m_1(E)}{m_1(\gamma)} < \beta. \tag{2.15}$$

Then

$$\frac{\omega(E)}{\omega(\gamma)} < \frac{\alpha^2}{2} \Rightarrow \frac{m_1(E)}{m_1(\gamma)} < \beta^2.$$

Proof. The technique used in this proof is borrowed from Calderón–Zygmund decomposition [16; p. 17]. Suppose

$$\omega(E) < \frac{\alpha^2}{2}\omega(\gamma). \tag{2.16}$$

We subdivide γ into two subintervals of equal ω-measure. If there is a subinterval γ', on which $\omega(E \cap \gamma')/\omega(\gamma') \geqslant \alpha/2$, we select γ' in our collection S. Subdivide each remaining interval into two subintervals of equal ω-measure. Select each subinterval γ' with the property

$$\frac{\omega(E \cap \gamma')}{\omega(\gamma')} \geqslant \frac{\alpha}{2}. \tag{2.17}$$

Continue this process indefinitely or stop at a finite step if the ω-measure of E in each of the remaining intervals is zero. Write $S = \{\gamma_j\}$, where γ_j's are mutually disjoint. By a variant of Vitali's theorem

$$\omega\left(\bigcup \gamma_j \cap E\right) = \omega(E). \tag{2.18}$$

For each γ' in S, there is an interval γ'' containing γ', with $\omega(\gamma'') = 2\omega(\gamma')$, so that $\omega(E \cap \gamma'')/\omega(\gamma'') < \alpha/2$. Therefore

$$\frac{\omega(E \cap \gamma')}{\omega(\gamma')} \leqslant \frac{\omega(E \cap \gamma'')}{\omega(\gamma'')} \cdot \frac{\omega(\gamma'')}{\omega(\gamma')} < \frac{\alpha}{2} \cdot 2 = \alpha.$$

Hence

$$m_1(E \cap \gamma') < \beta m_1(\gamma'). \tag{2.19}$$

By (2.16), (2.17) and (2.18), $(\alpha/2)\,\Sigma\omega(\gamma_j) \leqslant \omega(\bigcup\gamma_j \cap E) = \omega(E) < (\alpha^2/2)\omega(\gamma)$, which says that $\omega(\bigcup\gamma_j) < \alpha\omega(\gamma)$. By (2.15), we have $m_1(\bigcup\gamma_j) < \beta m_1(\gamma)$. Combining with (2.19) we see that $m_1(\bigcup\gamma_j \cap E) < \beta^2 m_1(\gamma)$. Because $\omega(E\setminus\bigcup\gamma_j) = 0$, by absolute continuity of m_1 with respect to ω, $m_1(E\setminus\bigcup\gamma_j) = 0$. Therefore, $m_1(E) < \beta^2 m_1(\gamma)$. □

3. Proof of Theorem 3. Let Ω_1 and Ω_2 be the interior and exterior of L respectively. We let $J = L \cap D$, $D_i = D \cap \Omega_i$ and $\omega^z(E) = \omega_D^z(E)$, $\omega_i^z(E) = \omega_{D_i}^z(E)$ for $i = 1$ or 2.

Let x be any point in D_i ($i = 1$ or 2). Then by the Poisson integral in D_i, $\omega^x(K) = \int_{J\cup K} \omega^z(K)\, d\omega_i^x(z)$. Because $D_i \subseteq \Omega_i$, by the maximum principle and the F. and M. Riesz Theorem, we have $\omega_i^x(K) \leqslant \omega_{\Omega_i}^x(K) = 0$. Hence,

$$\omega^x(K) = \int_J \omega^z(K)\, d\omega_i^x(z). \tag{3.1}$$

Suppose we can find a constant $\beta < 1$ so that

$$\omega^z(K) < \beta < 1 \quad \text{for every} \quad z \in J. \tag{3.2}$$

Then by (3.1) and (3.2), we have $\omega^a(K) < \beta < 1$ for every $a \in D$. This is possible only when harmonic measure of K is zero with respect to D.

To complete the proof of the theorem, we need only to show (3.2).

For a fixed $z \in J$, let J_z be the component of J that contains z and let a, b be the end points of J_z, so labeled that $|a - z| \leqslant |b - z|$. Let M be the constant associated with the quasi-smooth curve L, $r = \min\{|z-a|/(16M), r_0\}$, $\Delta_z = \Delta(z, r)$, $C_z = \{w : |w - z| = r\}$. It follows from the definition of quasi-smoothness that

$$\Delta(z, 4r) \cap (L\setminus J_z) = \varnothing. \tag{3.3}$$

By the properties (2.2) and (2.3) of Ω_i, there are a number δ, $0 < \delta < 1$, depending on M only, and subarcs T_z^i of C_z in Ω_i such that

$$m_1(T_z^i) = \alpha\pi r \qquad \text{and} \tag{3.4}$$

$$\operatorname{dist}(T_z^i, L) > \delta r. \tag{3.5}$$

We denote $T_z^1 \cup T_z^2$ by T_z and observe that $T_z^1 \cap D_2 = \varnothing$, $T_z^1 \cap D_1 = \varnothing$ and $T_z \cap L = \varnothing$. However, T_z or Δ_z may meet D^c.

For $z \in J$, by (3.4) and the maximum principle, and the fact, following from (3.3), that $\partial D \cap K \cap \overline{\Delta(z, r)} = \varnothing$,

(3.6)

$$\omega^z(K) \leqslant \int_{C_z \cap D} \omega^w(K)\, d\omega_{\Delta_z}^z(w) \leqslant 1 - \alpha + \frac{\alpha}{2} \sup_{w \in T_z \cap D_1} \omega^w(K) + \frac{\alpha}{2} \sup_{w \in T_z \cap D_2} \omega^w(K).$$

Because $T_z \cap L = \varnothing$, we have $T_z \cap D = (T_z \cap D_1) \cup (T_z \cap D_2)$. Suppose $w \in T_z \cap D_i$, then by (3.1)

$$\omega^w(K) = \int_J \omega^x(K)\, d\omega_i^w(x). \tag{3.7}$$

Suppose we can prove that there is a constant $\beta_0 < 1$, so that

$$\text{(3.8)} \qquad \operatorname{Min}\left\{ \sup_{w \in T_z \cap D_1} \omega_1^w(J), \ \sup_{w \in T_z \cap D_2} \omega_2^w(J) \right\} < \beta_0.$$

Letting $\beta = 1 - (\alpha/2) + (\alpha\beta_0/2)$, we conclude (3.2) from (3.8). Now it remains to show (3.8).

Let w be a point in $T_z \cap D_i$. Let V be the component of $D \backslash (L \backslash J_z)$ that contains z. V is simply connected.

First, we assume that w is not in V. We see from (3.3) that $\Delta(w, 2r) \cap (L \backslash J_z) = \varnothing$. Let D_w be the component of D_i that contains w. D_w is simply connected and

$$\text{(3.9)} \qquad \partial D_w \cap L \cap \Delta(w, 2r) = \varnothing.$$

Therefore, there is a cross cut γ in $\Delta(w, 2r)$ that separates w from z_1 and $\gamma \subseteq \partial D_w \backslash L$. Hence, by (3.9), the maximum principle and a solution to the Carleman–Milloux problem [11; p. 107] we obtain $\omega_{D_w}^w(\partial D_w \backslash L) > \eta_1 > 0$ for some absolute constant η_1. Hence,

$$\text{(3.10)} \qquad \omega_i^w(\partial D_i \backslash J) \geqslant \omega_{D_w}^w(\partial D_w \backslash L) > \eta_1 > 0.$$

Next, we assume that $w \in V$, $V \cap D_i$ is simply connected and $\partial(D_i \cap V)$ is formed by disjoint arcs $\{J_{i,j}\}_j$ of J of maximal lengths and a closed subset of ∂D. By the maximum principle,

$$\omega_i^w(J) = \omega_{D_i \cap V}^w\left(\bigcup_j J_{i,j} \right) \leqslant \omega_{\Omega_i}^w\left(\bigcup_j J_{i,j} \right).$$

Let $S_i = L \backslash \bigcup_j J_{i,j}$. We have

$$\text{(3.11)} \qquad \omega_i^w(\partial D_i \backslash J) \geqslant \omega_{\Omega_i}^w(S_i).$$

By the simple-connectedness of D, each point $y \in J \backslash J_z$ can be a boundary point of, at most, one of the regions $D_1 \cap V$ or $D_2 \cap V$. This says that $(\bigcup_j J_{1,j} \backslash J_z) \cap (\bigcup_j J_{2,j} \backslash J_z) = \varnothing$ or

$$\text{(3.12)} \qquad S_1 \cup S_2 = L \backslash J_z.$$

Recall that a is one of the end points of J_z on L. It follows from (3.2) that

$$\text{(3.13)} \qquad m_1(\Delta(a, r) \cap S_1) \geqslant r/2 \quad \text{or}$$

$$\text{(3.14)} \qquad m_1(\Delta(a, r) \cap S_2) \geqslant r/2.$$

Let i_0 be the index of S for which (3.13) or (3.14) holds. From (2.2), (2.3), (3.5), (3.11), (3.13), (3.14), Lemma 1 and the Harnack principle, it follows that

$$\text{(3.15)} \qquad \omega_{i_0}^w(\partial D_{i_0} \backslash J) \geqslant \omega_{\Omega_{i_0}}^w(S_{i_0}) > \eta_2 > 0$$

for some constant η_2 depending, at most, on M and r_0. Letting $\beta_0 = 1 - \max\{\eta_1, \eta_2\}$, we conclude (3.8) from (3.10) and (3.15). This completes the proof. □

4. Proof of Theorem 1.

LEMMA 3. *Given a sequence of integers* $0 < n_1 < n_2 < \cdots$, *let* f *and* f_N *be the analytic functions on* $\Delta \equiv \Delta(0, 1)$ *satisfying* $f(0) = f_N(0) = 0$ *and*

$$f'(z) = \exp\left(\epsilon \sum_{k=1}^{\infty} z^{2^{n_k}}\right) \tag{4.1}$$

$$f_N'(z) = \exp\left(\epsilon \sum_{k=1}^{N} z^{2^{n_k}}\right). \tag{4.2}$$

If $0 < \epsilon < 1/16$, *then* f *and* f_N *are univalent,* $\partial f(\Delta)$ *is a quasi-circle and* $\partial f_N(\Delta)$ *is analytic.*

Proof. It is a result of Becker [13; p. 172 and p. 294] that if $f'(0) \neq 0$ and $(1-|z|^2)|zf''(z)/f'(z)| \leqslant 1$, $z \in \Delta$, then $f(z)$ is univalent in Δ; and that if $f(0)=0$ and $f'(0)=1$ and

$$(1-|z|^2)|zf''(z)/f'(z)| < k < 1, \qquad z \in \Delta \tag{4.3}$$

then $\partial f(\Delta)$ is a quasi-circle.

To prove this lemma, we need only to show (4.3) if $0 < \epsilon < 1/16$.

$$\left|\frac{zf''(z)}{f'(z)}\right| = \left|\epsilon \sum_{k=1}^{\infty} 2^{n_k} z^{2^{n_k}}\right| < 2\epsilon \sum_{m=1}^{\infty} |z|^m < \frac{1}{4(1-|z|^2)}.$$

This proves (4.3). Similarly we can prove (4.3) for f_N, and these inequalities give the lemma. □

LEMMA 4. *Let* $0 < n_1 < n_2 < n_3 < \cdots$, *and* f, f_N *be the same as in Lemma* 3. *If* $0 < \epsilon < 1/16$, *then for* $z, w \in \Delta$,

$$\begin{aligned} &|f'(z)| < (1-|z|)^{-1/3} < (1-|z|)^{-1/2}, \\ &|f_N'(z)| < (1-|z|)^{-1/3} < (1-|z|)^{-1/2} \quad \textit{and} \end{aligned} \tag{4.4}$$

$$\begin{aligned} &|f(z)-f(w)| \leqslant 25|z-w|^{2/3}, \qquad |f_N(z)-f_N(w)| \leqslant 25|z-w|^{2/3} \\ &|f(z)-f(w)| \leqslant 25|z-w|^{1/2}, \qquad |f_N(z)-f_N(w)| \leqslant 25|z-w|^{1/2}. \end{aligned} \tag{4.5}$$

Proof. Let $z = re^{i\theta}$ and obtain

$$\log|f'(z)| \leqslant \epsilon \sum_{k=1}^{\infty} r^{2^{n_k}} \leqslant 2\epsilon \sum_{m=1}^{\infty} \frac{r^m}{m} < \frac{1}{3} \log \frac{1}{1-|z|}.$$

Hence, $|f'(z)| < (1-|z|)^{-1/3}$. The proof of the second inequality in (4.4) is similar. (4.5) follows from (4.4) by a classical theorem of Hardy and Littlewood [17; p. 263 vol. I], the constant 25 can be found by straightforward calculation. □

LEMMA 5. *Let* $0 < \epsilon < 1/16$ *and* f_N, f *be the functions defined in Lemma* 3. *Given* η, $0 < \eta < 1/4$, *there is a number* $K(\eta) = 20 + 6\log(1/\eta)$ *so that if* $n_{N+1} > \max\{n_N, K(\eta)\}$ *then* $|f_N - f| < \eta$ *uniformly on* $|z| = 1$.

Proof. Because $0 < \epsilon < 1/16$, by Lemma 3, f and f_N can be extended to be homeomorphisms on $\bar{\Delta}$. Suppose $n_{N+1} > \max\{n_N, K(\eta)\}$ and $r = 1 - \eta^2/10000$. It is easy to check that $r^{2^{n_{N+1}}} < \eta < 1/4$ and $\epsilon \sum_{k=1}^{\infty} r^{2^{n_k}} < 1/48$. From Lemma 4, it follows that

$$
\begin{aligned}
|f_N(e^{i\theta})-f(e^{i\theta})| &\leqslant |f_N(e^{i\theta})-f_N(re^{i\theta})|+|f_N(re^{i\theta})-f(re^{i\theta})|+|f(re^{i\theta})-f(e^{i\theta})| \\
&\leqslant 50\sqrt{1-r}\ +\int_0^r |f_N'(te^{i\theta})-f'(te^{i\theta})|\,dt \\
&\leqslant \frac{\eta}{2}+\int_0^r |f_N'(te^{i\theta})|\left|1-\exp\left(\epsilon\sum_{k=N+1}^{\infty}(te^{i\theta})^{2^{n_k}}\right)\right|dt \\
&\leqslant \frac{\eta}{2}+\int_0^r |f_N'(te^{i\theta})|\,dt\cdot 2\epsilon r^{2^{n_{N+1}}}\leqslant \frac{\eta}{2}+\frac{\eta}{8}<\eta. \qquad \square
\end{aligned}
$$

Proof of Theorem 1. Fix a sequence of integers $0<n_1<n_2<n_3<\cdots$, such that on an infinite subsequence n_{N_j}, we have

$$2N>n_N \tag{4.6}$$

$$n_{N+1}>\max\{n_N, K(10^{-N})\} \tag{4.7}$$

for each $N=N_j$, $j=1,2,\ldots$, where $K(\eta)=20+6\log(1/\eta)$. Such a sequence can be constructed explicitly as the sequence of integers in the following blocks:

$$1, [40, 2\cdot 40-1], [40^2, 2\cdot 40^2-1], [40^3, 2\cdot 40^3-1], \ldots$$

By the central limit theorem on lacunary trigonometric series of Salem and Zygmund [15; p. 333], it follows that for any Borel set $F\subseteq[0,2\pi)$, with $m_1(F)>0$, $-\infty<x<\infty$,

$$m_1\{\theta\in F: \sum_1^N \cos(2^{n_k}\theta)<x\sqrt{N/2}\ \}\to a(x)m_1(F),\ N\to\infty, \tag{4.8}$$

where $a(x)$ is the Gaussian distribution with mean value 0 and variance 1; we note that $0<a(x)<1$. Let $0<\epsilon<1/16$.

$$E_N=\{\theta\in[0,2\pi): \sum_{k=1}^N \cos(2^{n_k}\theta)<-3(A+1)\sqrt{N}/\epsilon\} \tag{4.9}$$

By (4.8), $m_1(E_N)>0$, for large N.

Define f and f_N as in Lemma 3. Let $\Omega=f(\Delta)$,

$$E=\bigcap_{m=1}^{\infty}\bigcup_{j=m}^{\infty}E_{N_j}. \tag{4.10}$$

From Lemma 3, we see that $\partial\Omega$ is a quasi-circle and f can be extended to a homeomorphism on $\bar{\Delta}$. From Lemma 4, we see that f and $f\circ T$ are in Lipschitz class $\Lambda_{2/3}$, for any Möbius transformation of Δ onto Δ. It follows from a theorem of S. Bernstein [17; p. 240 vol. I] that f and $f\circ T$ have power series converging absolutely on $\bar{\Delta}$.

We claim that $m_1(E)=2\pi$. Otherwise, the set $\Phi=\{\theta\in[0,2\pi):\theta$ is in finitely many E_{n_j}'s$\}$ has positive m_1 measure. Hence, for some integer i, the set $\Phi_i=\{\theta\in[0,2\pi):\theta\notin E_{N_j},\ j\geqslant i\}$ has positive m_1 measure. Applying (4.8) to $F=\Phi_i$, $x=-3(A+1)\sqrt{2}/\epsilon$, we can easily get a contradiction. This proves that $m_1(E)=2\pi$. Because f is a homeomorphism on $\bar{\Delta}$, by the conformal invariance of harmonic measure, $\omega_\Omega(f(E))=1$. Let $F=f(E)$; it remains to show that $\Lambda_h(F)=0$.

For each integer N, consider a covering S_N of E_N by those intervals I_m of the form $2\pi(m-1)4^{-N} \leqslant \theta \leqslant 2m\pi 4^{-N}$ that meet E_N. There are most 4^N of them. For any positive integer J, $\bigcup_{j\geqslant J} S_{N_j}$ is a covering of E by (4.10), hence, $\bigcup_{j\geqslant J}\bigcup_{I\in S_{N_j}} f(I)$ is a covering of F.

Let $N=N_j$ for some $j\geqslant 1$, $\theta\in I$ in S_N. There is a point $\theta_0\in I\cap E_N$. Hence,

$$\begin{aligned}\sum_{k=1}^{N}\cos(2^{n_k}\theta) &\leqslant \sum_{k=1}^{N}\cos(2^{n_k}\theta_0)+\frac{2\pi}{4^N}\sup_I\left|\frac{d}{d\theta}\sum_{k=1}^{N}\cos(2^{n_k}\theta)\right|\\ &\leqslant -3(A+1)\sqrt{N}/\epsilon+\frac{2\pi}{4^N}2\cdot 2^{n_N}\\ &\leqslant -3(A+1)\sqrt{N}/\epsilon+4\pi< -2(A+1)\sqrt{N}/\epsilon.\end{aligned}$$

In the above inequalities we used (4.6) and (4.9). Therefore, $|f_N'(e^{i\theta})|<e^{-2(A+1)\sqrt{N}}$, which shows that $m_1(f_N(I))<2\pi 4^{-N}e^{-2(A+1)\sqrt{N}}$. From (4.7) and Lemma 5, we see that $|f_N(e^{i\theta})-f(e^{i\theta})|<10^{-N}$. Hence,

$$\operatorname{diam}(f(I))<2\pi 4^{-N}e^{-2(A+1)\sqrt{N}}+2\cdot 10^{-N}<4^{-N+3}e^{-2(A+1)\sqrt{N}}$$

for sufficiently large N.

For large integer J, we have

$$\begin{aligned}\Lambda_h(F) &\leqslant \sum_{j=J}^{\infty}\sum_{I\in S_{N_j}}\operatorname{diam}(f(I))\exp(A\,|\log\operatorname{diam} f(I)|^{1/2})\\ &\leqslant \sum_{j=J}^{\infty}4^{N_j}(4^{-N_j+3}e^{-2(A+1)\sqrt{N_j}})\exp(A\sqrt{\log(4^{N_j-3}e^{2(A+1)\sqrt{N_j}}}\ )\\ &\leqslant \sum_{j=J}^{\infty}e^{-\sqrt{N_j}}\end{aligned}$$

Letting $J\to\infty$, we obtain $\Lambda_h(F)=0$, which implies that $m_1(F)=0$. This completes the proof. □

5. Harmonic measure on starlike domains. First we give the proof of Theorem 2. Let the origin be the center of the starlike domain Ω, and $f:\Delta\to\Omega$ be a univalent function with $f(0)=0$. Because Ω is starlike about 0, we have [13; p. 42]

$$\operatorname{Re}\frac{zf'(z)}{f(z)}>0.$$

Hence, by a theorem of Lusin and Privalov [17; p. 253, vol. I], $zf'(z)/f(z)$ has nonzero nontangential limit m_1 – a.e. on $\partial\Delta$. Because $f(\{\frac{1}{2}<|z|<1\})$ omits a neighborhood of the origin, by a theorem of Plessner [17; p. 203, vol. II], f has nonzero nontangential limit m_1 – a.e. on $\partial\Delta$. Hence, $f'(z)$ also has nonzero nontangential limit m_1 – a.e. on $\partial\Delta$. Let S be a set on $\partial\Delta$ with $m_1(S)=2\pi$, on which f and f' both have nonzero nontangential limits.

Let G be any compact subset of $f(S)\cap E$. Let T_θ be the smallest convex set containing $|z|<\frac{1}{2}$ and having $e^{i\theta}$ as a boundary point. Given any $\epsilon>0$, by the Egorov theorem there is a compact subset F of $f^{-1}(G)$, so that $m_1(f^{-1}(G)\setminus F)<\epsilon$ and f, f' converge uniformly as $z\in U\equiv\bigcup_{e^{i\theta}\in F}T_\theta$ approaches F. Hence, f can be extended to a C^1 function on $\bar{U}$ [17; pp. 199-201, vol. II]. Because U is a Lipschitz domain, by the Whitney extension theorem [16; Chap. VI] f can be extended to a C^1 function g on $\mathbf{C}$. Suppose $m_1(F) > 0$,

$$0 < \int_F \left| \frac{d}{d\theta} g(e^{i\theta}) \right| d\theta < m_1(g(F)) = m_1(f(F)) \leqslant m_1(E)$$

which contradicts the assumption $m_1(E) = 0$. Therefore, $m_1(F) = 0$. Since ϵ is arbitrary, $m_1(f^{-1}(G)) = 0$.

Suppose $u(w) = \omega_\Omega^w(G) > 0$ for $w \in \Omega$. Let $v(z) \equiv u \circ f(z)$ on Δ. We must have $v > 0$ on Δ. Suppose v has positive radial limit at some point $e^{i\theta} \in S$. Because f has radial limit at $e^{i\theta}$, $u(w)$ has a positive limit as $w \to f(e^{i\theta})$ along the image of the ray $\overline{0, e^{i\theta}}$. This is possible only when $f(e^{i\theta}) \in G$, because G is compact. Hence, $e^{i\theta} \in f^{-1}(G)$. However, $m_1(f^{-1}(G)) = 0$, which implies that $v \equiv 0$, a contradiction. Therefore, $\omega_\Omega(G) = 0$, and thus $\omega_\Omega(f(S) \cap E) = 0$.

Arguing as in the last paragraph, we can show that $\omega_\Omega(E \backslash f(S)) = 0$. Hence, $\omega_\Omega(E) = 0$. This completes the proof of Theorem 2. □

It is pointed out to us by J. Brennan and Chr. Pommerenke that Theorem 2 also follows from McMillan's Nontwist Point Theorem [13; p. 326].

Suppose Ω is a starlike domain. Theorem 2 says that sets of linear measure zero on $\partial\Omega$ must have harmonic measure zero. We can think of Ω as a domain so that $\partial\Omega$ meets each line in a certain family $\mathfrak{F} \equiv \{$lines passing through the center of $\Omega\}$ at exactly two connected components. In this sense, $\partial\Omega$ is not too big. Parametrizing lines on $\mathfrak{F}$ by θ, where $0 \leqslant \theta < \pi$ is the angle between l and the x-axis, we denote lines in $\mathfrak{F}$ by l_θ. Simple examples show that there exists starlike domain Ω, so that $m_1(\partial\Omega \cap l_\theta) > 0$ on a set of θ with positive m_1 measure. In this sense, $\partial\Omega$ is not too small.

Given any Hausdorff measure Λ_τ with $\tau(t)\uparrow$, $\tau(0) = 0$, we call a simply connected domain Ω *nearly starlike* corresponding to $\tau(t)$ if $\Lambda_\tau(\partial\Omega \cap l) = 0$ for every line l in the complex plane.

In general, the boundary of a nearly starlike domain Ω can be bigger than that of a starlike domain regarding the number of components of $l \cap \partial\Omega$, but much smaller than that of a starlike domain regarding the Hausdorff measures of $l \cap \partial\Omega$. We ask whether a set F on the boundary of a nearly starlike domain with zero m_1-measure must have zero harmonic measure. The answer is negative and we have the following example.

THEOREM 4. *Given any increasing function $\tau(t)$ for $0 \leqslant t < \infty$ with $\tau(0) = 0$, there exist a Jordan domain Ω and a set $F \subseteq \partial\Omega$ with all the properties in Theorem* 1, *and*

$$\Lambda_\tau(l \cap \partial\Omega) = 0 \qquad (5.1)$$

for every line l in the plane.

QUESTION. Is Theorem 4 still true if we require (5.1) to hold for all curves l with some uniform smoothness property, for example C^2 or real-analytic?

A closed set of Hausdorff measure Λ_τ zero for every increasing τ with $\tau(0) = 0$, must be countable, [14, p. 67].

QUESTION. Can we find a Jordan domain Ω, a set $F \subseteq \partial\Omega$ with all the properties in Theorem 1 and yet $\partial\Omega \cap l$ is countable for every line l in the plane?

Proof of Theorem 4. Suppose finite sequences $0 < N_1 < N_2 < \cdots < N_j$ and $0 < n_1 < n_2 < \cdots < n_{N_J}$ are found so that

(5.2) $$2N_{j+1} > n_{N_{j+1}}$$

(5.3) $$n_{N_j+1} > \max\{n_{N_j}, K(\eta_{N_j})\}, \quad \text{where} \quad K(\eta) = 20 + 6\log(1/\eta),$$

for $1 \leqslant j \leqslant J-1$ and given η_{N_j}, $1 \leqslant j \leqslant J-1$.

Let $N = N_J$ and f_N be as in (4.2). We shall define N_{J+1} and n_k, $N_J + 1 \leqslant k \leqslant N_{J+1}$.

Since $\partial f_N(\Delta)$ is analytic, we can subdivide $\partial f_N(\Delta)$ into $P = P(N)$ arcs so that any subarc $\Gamma = \Gamma(p)$, $1 \leqslant p \leqslant P(N)$, after a rotation if necessary, can be represented as the graph of a real analytic function: $y = g(x)$, $a \leqslant x \leqslant b$ and

(5.4) $$|g'(x)| < 2, \qquad a \leqslant x \leqslant b.$$

Because $\partial f_N(\Delta)$ is not a line, $g(x)$ is not linear. Hence, g'' has, at most, finitely many zeros on $a \leqslant x \leqslant b$; call them x_m, $1 \leqslant m \leqslant M = M(p)$. Let $x_0 = a$, $x_{M+1} = b$. Given $\eta > 0$, the exact size to be determined later, let $S^\eta = \{z : \operatorname{dist}(z, S) < \eta\}$ for any set S. For any given line l, we shall study the set $l \cap \Gamma^\eta$. In case l is a vertical line

(5.5) $$m_1(l \cap \Gamma^\eta) \leqslant 2\eta.$$

So we shall assume that l is defined by $y = Ax + B$. Choose $c > 0$ so small that the arcs $\gamma_m = g((x_m - c, x_m + c) \cap [a, b])$ are disjoint, the exact choice of c be determined later. There exists $\delta > 0$, such that

(5.6) $$|g''(x)| > \delta > 0 \quad \text{on} \quad \Gamma \backslash \{\gamma_m\}_{m=0}^{M+1}.$$

Let τ_n be the components of $\Gamma \backslash \{\gamma_0, \gamma_1, \ldots, \gamma_{M+1}\}$. It is clear that $\bigcup_m \gamma_m^\eta \cup \bigcup_n \tau_n^\eta = \Gamma^\eta$.

From (5.4) it is clear that

(5.7) $$\operatorname{diam}(l \cap \gamma_m^\eta) < 4c + 2\eta.$$

To estimate $l \cap \tau_n^\eta$ we consider points $(x, g(x))$ on τ_n, whose distances to the line l are at most η, that is $|g(x) - Ax - B| < \eta(1 + A^2)^{1/2}$. We suppose first that $|A| \geqslant 3$, so that $|g'(x) - A|$ is never less than $|A|/3$ but $(1 + A^2)^{1/2} < \frac{4}{3}A$. Hence, by the mean-value theorem, the set of x's in question is an arc of length $< 8\eta$, and

(5.8) $$\operatorname{diam}(\tau_n^\eta \cap l) < 34\eta, \quad \text{if} \quad |A| \geqslant 3.$$

If $|A| < 3$, we divide the arc τ_n into three arcs (some may be void), defined by the inequalities (i) $g'(x) - A \leqslant -(\eta\delta)^{1/2}$, (ii) $|g'(x) - A| < (\eta\delta)^{1/2}$, (iii) $g'(x) - A \geqslant (\eta\delta)^{1/2}$; we call the arcs τ_{n1}, τ_{n2}, τ_{n3}. The set of x's in τ_{n2} has length $< 2\eta^{1/2}\delta^{-1/2}$; for τ_{n1} and τ_{n3} we use the inequality $(1 + A^2)^{1/2} < 4$ to find an upper bound $8\eta^{1/2}\delta^{-1/2}$. Thus

(5.9) $$\operatorname{diam}(\tau_{ni}^\eta \cap l) \leqslant 32\eta^{1/2}\delta^{-1/2} + 2\eta, \qquad i = 1, 2, 3.$$

We recall that c and δ are independent of η; δ depends on c. We may choose c, η in this order so that

$$(5.10) \qquad 0<\eta<10^{-N} \quad \text{and}$$

$$(5.11) \qquad \tau(6c)+\tau(34\eta)+\tau(32\eta^{1/2}\delta^{-1/2}+2\eta)<2^{-N}\left(\sum_{p=1}^{P(N)}(2M(p)+4)\right)^{-1}.$$

We recall that $N=N_J$; let $\eta_{N_J}=\eta$ as defined in (5.10). It is easy to choose N_{J+1} and n_k, $N_J+1\leqslant k\leqslant N_{J+1}$ to satisfy the conditions (5.2) and (5.3). Continuing this process indefinitely, we may find $0<n_1<n_2<\cdots$ with properties (5.2) and (5.3), η_{N_J} with properties (5.10) and (5.11) for all sufficiently large J.

Let f, $\Omega=f(\Delta)$, and F be defined in the same way as in the proof of Theorem 1, after $0<n_1<n_2\cdots$ are fixed. The properties of $\partial\Omega$ and F in Theorem 1 follow by the same proof.

It remains to show (5.1). Because of (5.3) and Lemma 5, $|f_N-f|<\eta_N$ uniformly on $z=1$. Hence, for large $N=N_J$,

$$(5.12) \qquad \partial\Omega\subseteq\bigcup_{p=1}^{P(N)}\{z:\operatorname{dist}(z,\Gamma(p))<\eta_N\}.$$

It follows from (5.5), (5.7), (5.8), (5.9), (5.11) and (5.12) that $\Lambda_\tau(l\cap\partial\Omega)\leqslant 2^{-N}$. Hence, $\Lambda_\tau(l\cap\partial\Omega)=0$. This completes the proof. □

THEOREM 5. *There exist a starlike Jordan domain Ω whose boundary is given by $r=r(\theta)$, and a set $E\subseteq\partial\Omega$ of angular measure* 0 *and harmonic measure* 1.

Proof. Let λ be a singular probability measure on $[0,2\pi]$, positive on every interval, and Hölder-continuous, e.g., $\lambda(a,b)\leqslant c(b-a)^{1/2}(0\leqslant a<b\leqslant 2\pi)$. Define

$$w(z)=u(z)+iv(z)=\int_0^{2\pi}(e^{i\theta}+z)(e^{i\theta}-z)^{-1}d\lambda(\theta);$$

$$zf'(z)/f(z)=w(z), \quad \text{or} \quad f'(z)/f(z)=z^{-1}(w(z)-1)+z^{-1}.$$

Then $f(0)=0$, and f' has no zeroes, since the right side in the last equation has residue 1 at $z=0$. From the equation $\partial/\partial\theta\arg f(re^{i\theta})=u(re^{i\theta})>0$ and our remark that $f'\neq 0$, we conclude that f is a conformal mapping onto a starlike domain Ω. We proceed to show that f is continuous on the boundary and that $\partial\Omega$ is given by $r=r(\theta)$.

The first point is a consequence of the Hölder-continuity of λ, which yields the bound $f'(z)/f(z)=O(1-|z|)^{-1/2}$, and so f is Hölder-continuous for $|z|\leqslant 1$. For the second point, we observe that to each ϵ there is a $\delta>0$ so that $\lambda(\theta,\theta+\epsilon)\geqslant\delta$ for every θ; the interval $(\theta,\theta+\epsilon)$ is, of course, placed on the circle of length 2π. From this it follows that $\int_\theta^{\theta+\epsilon}u(re^{it})\,dt>\delta$ when $0<r<1$, and therefore, $f(e^{i\theta}e^{i\epsilon})/f(e^{i\theta})$ has argument at least δ. This shows that $\partial\Omega$ is a Jordan curve $r=r(\theta)$; it is not difficult to obtain measures λ for which $r(\theta)$ is Hölder-continuous.

Now λ is singular, so there is a set $F\subseteq[0,2\pi]$, of measure 2π, but $\arg f(e^{i\theta})$ maps F onto a set of length 0; we take $E=r(F)$. Observe that E must have positive length, so that r must be of infinite variation. □

6. Concluding remark. What are the metric properties of sets of zero harmonic measure on a quasi-circle or on the boundary of a starlike domain?

REMARK. For any γ, $1 \leqslant \gamma < 2$, there exist a Jordan domain Ω, whose boundary is a quasi-circle (or which is starlike), and a set $E \subseteq \partial\Omega$ so that $m_\gamma(E) > 0$ but $\omega_\Omega(E) = 0$.

The reason is simple. Gehring and Väisälä [3] proved that there exists a quasi-circle Γ of Hausdorff dimension $\gamma + \epsilon$, $0 < \epsilon < 2 - \gamma$. Hence, Γ has non-σ-finite m_γ measure. By a theorem of Besicovitch, generalized by Davies and Larman [14; p. 124], Γ contains uncountably many compact subsets each of positive m_γ measure. Therefore, at least one of these subsets must have zero harmonic measure with respect to the interior of Γ. The starlike case is similar.

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