ORIENTATION-REVERSING PL INVOLUTIONS ON ORIENTABLE TORUS BUNDLES OVER S^1

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0. Introduction. In this paper we characterize the orientation-reversing PL involutions on orientable torus bundles over S^1. Let $M(b)$ denote the Seifert manifold of type $\{b; (n_2, 2)\}$ (see P. Orlik [12] or P. Conner and F. Raymond [3]). The space $M(0)$ is the only Seifert manifold of this type which admits an orientation-reversing PL involution (see [11] or Theorem B below). We also obtain a complete classification of these involutions on $M(0)$. Crucial to our study is the following, as well as of interest in its own right.

THEOREM A. (1) *If an orientable torus bundle over S^1 admits a* PL *embedding of a Klein bottle K, then it is homeomorphic to a Seifert manifold $M(b)$ for some b.*

(2) *A union of two twisted I-bundles over K (as an adjunction space) with infinite first homology group is homeomorphic to $M(b)$ for some b.*

We remark that each Seifert manifold $M(b)$ contains a Klein bottle. Thus Theorem A shows that the family of orientable torus bundles over S^1 which contain Klein bottles is identified with the Seifert manifolds $M(b)$. Recall that $M(b)$ and $M(b')$ are homeomorphic if and only if $b' = \pm b$.

In Theorems B, C we characterize the orientation-reversing PL involutions on orientable torus bundles over S^1. In order to do this, we need to define some fibered 3-manifolds. Let $\mathbf{R}$ be the set of real numbers and S^1 the set of complex numbers with norm 1. The 2-dimensional torus T^2 may be represented as $T^2 = \{(z_1, z_2) \mid z_1, z_2 \in S^1\}$. Let φ be a homeomorphism of T^2. We let $T^2 \times \mathbf{R}/\varphi$ denote the torus fiber bundle over S^1 obtained from $T^2 \times \mathbf{R}$ by identifying (x, t) with $(\varphi(x), t+1)$ for each $(x, t) \in T^2 \times \mathbf{R}$. The elements of $T^2 \times \mathbf{R}/\varphi$ are denoted by $[x, t]$. Define a homeomorphism $\bar{\varphi}: T^2 \longrightarrow T^2$ by $\bar{\varphi}(z_1, z_2) = (z_1^{\lambda p+1} z_2^{\lambda q}, z_1^{\lambda r} z_2^{\lambda s+1})$ where p, q, r, s are integers with $\left|\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right| = -1$ and $\lambda = p+s$. Observe that $\left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right)^2 = \left(\begin{smallmatrix} \lambda p+1 & \lambda q \\ \lambda r & \lambda s+1 \end{smallmatrix}\right)$. We denote the space $T^2 \times \mathbf{R}/\bar{\varphi}$ by $\bar{M}\left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right)$. Then two spaces $\bar{M}\left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right)$ and $\bar{M}\left(\begin{smallmatrix} p' & q' \\ r' & s' \end{smallmatrix}\right)$, different from $T^2 \times S^1$, are homeomorphic if and only if the two matrices $\left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right)^2$ and $\left(\begin{smallmatrix} p' & q' \\ r' & s' \end{smallmatrix}\right)^2$ are similar (see Lemma 4.1).

Observe that $\bar{M}\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \approx T^2 \times S^1$. All PL involutions on $T^2 \times S^1$ are known (see K. Kwun and J. Tollefson [8]). Theorem C provides a complete classification of the orientation-reversing PL involutions on $M(0)$ (up to conjugation). Note that a space $\bar{M}\left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right)$ is not homeomorphic to $M(b)$ for any b (see Lemma 4.3).

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THEOREM B. *Suppose that a torus bundle* $M=T^2\times\mathbf{R}/\varphi$ *admits an orientation-reversing* PL *involution* h. *Then* M *is homeomorphic to either* $\bar{M}\binom{p\ q}{r\ s}$ *or* $M(0)$. *Furthermore, if* $M\approx\bar{M}\binom{p\ q}{r\ s}$ *and* $p+s\neq 0$ *(if* $p+s=0$, *then* $M\approx T^2\times S^1$*), then* h *is conjugate to a free* PL *involution* $\bar{h}$ *on* $T^2\times\mathbf{R}/\phi^2$ *defined by*

$$\bar{h}[x,\tau]=[\phi(x),\tau+\tfrac{1}{2}]\qquad \textit{for}\qquad (x,\tau)\in T^2\times\mathbf{R}$$

where ϕ *is an orientation-reversing homeomorphism of* T^2 *with* ϕ^2 *isotopic to a conjugate of* φ.

THEOREM C. *There exist exactly six orientation-reversing* PL *involutions on* $M(0)$, *up to conjugation. These may be distinguished by the fixed-point sets*: (1) *empty set,* (2) *one Klein bottle,* (3) *two Klein bottles,* (4) *one torus,* (5) *four points,* and (6) *one torus plus four points.*

The following is an immediate corollary of Theorem B, and essentially classifies all orientation-reversing PL involutions on the orientable torus bundles over S^1 with nonempty fixed-point set (see Theorem C and [8]).

COROLLARY 1. *The spaces* $T^2\times S^1$ *and* $M(0)$ *are the only orientable torus bundles over* S^1 *which admit orientation-reversing* PL *involutions with nonempty fixed-point set.*

We divide the paper into four sections. In Section 1 we study representations of some fibered torus bundles. In Section 2 we give the proof of Theorem A. In Section 3 we list the (standard) orientation-reversing PL involutions on $M(0)$, and in Section 4 prove Theorems B and C.

Throughout the paper we work in the PL category exclusively. We refer to J. Hempel [5] for standard 3-manifold terms.

1. Representations of some fibered torus bundles. We always reserve the notations ξ,ρ for the standard elements of $\pi_1(T^2)$ represented by the paths $(e^{2\pi ti},1)$ and $(1,e^{2\pi\tau i})$, $0\le t,\ \tau\le 1$, respectively. Let $\varphi: T^2\longrightarrow T^2$ be a homeomorphism. Then the map φ induces an automorphism φ_* on $\pi_1(T^2)$ such that $\varphi_*(\xi)=\xi^p\rho^r$ and $\varphi_*(\rho)=\xi^q\rho^s$ for suitable integers p, q, r, s. Recall that the isotopy classes of homeomorphisms of T^2 is a group isomorphic to the multiplicative group of unimodular 2×2 matrices (see [10]). The matrix $\binom{p\ q}{r\ s}$ shall be called the matrix of φ.

We shall denote the torus bundle $T^2\times\mathbf{R}/\varphi$ by $M(\varphi)$. For each integer b we define a homeomorphism $\phi(b): T^2\longrightarrow T^2$ by $\phi(b)(z_1,z_2)=(z_1^{-1},z_1^{-b}z_2^{-1})$. Then the Seifert manifold $M(b)$ is homeomorphic to $M(\phi(b))$ (see [12]).

In the following we introduce a space $M^*(b)$ for each integer b which is a union of twisted I-bundles over K. We will see that these spaces can be also fibered over S^1 with fiber T^2. We let γ always denote a map $\gamma: T^2\longrightarrow T^2$ defined by $\gamma(z_1,z_2)=(-z_1,\bar{z}_2)$. Then the orbit space T^2/γ is a Klein bottle K. Let $N(\gamma)$ denote the twisted I-bundle over T^2/γ which is obtained from $T^2\times I$ by identifying $(x,0)$ with $(\gamma(x),0)$ for each $x\in T^2$. By a union of two twisted I-bundles over K we mean an adjunction space, denoted by $M^*(f)$, $M^*(f)=N(\gamma)\cup_f N(\gamma)$ where f is an attaching map of T^2 $(=\partial N(\gamma))$. We shall denote $M^*(f)$ by $M^*\binom{p\ q}{r\ s}$ in order to emphasize the matrix of f.

We may assume that $p\geq 0$ and $\begin{vmatrix} p & q \\ r & s \end{vmatrix}=1$. If $f: T^2 \longrightarrow T^2$ is defined by $f(z_1, z_2) = (z_1, z_1^b z_2)$, then $M^*(f)$ shall be simply denoted by $M^*(b)$. Let $g: T^2 \longrightarrow T^2/\gamma$ be the orbit map. We may assume $\pi_1(T^2/\gamma)$ is given as $\pi_1(T^2/\gamma) = \{\alpha, \beta \mid \alpha\beta\alpha^{-1}\beta = 1\}$ and $g_*(\xi) = \alpha^2$ and $g_*(\rho) = \beta$. Then, letting $M^* = M^*\begin{pmatrix} p & q \\ r & s \end{pmatrix}$, it follows from Van Kampen Theorem that the fundamental group $\pi_1(M^*)$ is given as

$$\pi_1(M^*) = \{\alpha_1, \beta_1, \alpha_2, \beta_2 \mid \alpha_1\beta_1\alpha_1^{-1}\beta_1 = 1, \alpha_2\beta_2\alpha_2^{-1}\beta_2 = 1, \alpha_1^2 = \alpha_2^{2p}\beta_2^r, \beta_1 = \alpha_2^{2q}\beta_2^s\}$$

(here, corresponding to α, β are α_i, β_i, respectively). In the following we study some properties of $M^*(b)$ and $M(b)$, which will be used in the later sections. For notations, if x is an element of a group G, then the image of x in $G/[G, G]$ under the natural homomorphism shall be denoted by $\bar{x}$. The proof of the following Proposition 1.1 uses a similar technique to that used in [13].

PROPOSITION 1.1. *Suppose that $M(\varphi)$ is homeomorphic to $M(\phi(b))$. Then φ is isotopic to a conjugate of $\phi(b)$.*

Proof. Let $M = M(\phi(b))$ and $M' = M(\varphi)$. Then $\pi_1(M)$ can be represented by $\pi_1(M) = \{\xi, \rho, t \mid [\xi, \rho] = 1,\ t\xi t^{-1} = \xi^{-1},\ t\rho t^{-1} = \xi^{-b}\rho^{-1}\}$ where t is generated by a simple closed curve which meets each fiber in a single point. Similarly,

$$\pi_1(M') = \{\xi', \rho', t' \mid [\xi', \rho'] = 1, \quad t'\xi' t'^{-1} = \varphi_*(\xi'), t'\rho' t'^{-1} = \varphi_*(\rho')\}.$$

Let $k: M \longrightarrow M'$ be a homeomorphism preserving the base points. Then k_* is an isomorphism of $\pi_1(M)$ to $\pi_1(M')$. Let J and J' be the subgroups of $\pi_1(M)$ and $\pi_1(M')$ generated by $\{\xi, \rho\}$ and $\{\xi', \rho'\}$, respectively. Let $w \in J$. Suppose that $k_*(w) = w' t'^n$ for some $w' \in J'$. Observe that $\bar{w}$ is an element of $\mathrm{Tor}(H_1(M))$. Thus we see that $\bar{w}' \bar{t}'^n \in \mathrm{Tor}(H_1(M'))$. Considering the group structure of $H_1(M')$, it is impossible unless $n = 0$. Thus one sees that $k_*(J) = J'$. Now it is easy to show that $k_*(\phi(b))_*(w) = \varphi_* k_*(w)$, and therefore $\varphi_* = k_*(\phi(b))_* k_*^{-1}$. Thus φ must be isotopic to a conjugate of $\phi(b)$. □

PROPOSITION 1.2. *Let M^* be a union of two twisted I-bundles over K. If M^* can be fibered over S^1, then M^* is homeomorphic to $M^*\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}$ for some b.*

Proof. By the above, $\pi_1(M^*)$ may be represented as

$$\pi_1(M^*) = \{\alpha_1, \beta_1, \alpha_2, \beta_2 \mid \alpha_1\beta_1\alpha_1^{-1}\beta_1 = 1, \alpha_2\beta_2\alpha_2^{-1}\beta_2 = 1, \alpha_1^2 = \alpha_2^{2p}\beta_2^r, \beta_1 = \alpha_2^{2q}\beta_2^s\}.$$

Abelianizing the group of $\pi_1(M^*)$, then

$$H_1(M^*) = \{\alpha_1, \beta_1, \alpha_2, \beta_2 \mid \beta_1^2 = 1, \quad \beta_2^2 = 1, \quad \alpha_1^2 = \alpha_2^{2p}\beta_2^r, \quad \beta_1 = \alpha_2^{2q}\beta_2^s,$$
$$[\alpha_1, \alpha_2] = [\beta_1, \beta_2] = [\alpha_i, \beta_j] = 1\}$$

where $i, j = 1, 2$. Since $H_1(M^*)$ is infinite, one can argue that $q = 0$, and M^* is homeomorphic to $M^*\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}$ for some b. □

REMARK. It follows from the above proof that if $M^* = M^*\begin{pmatrix} p & q \\ r & s \end{pmatrix}$ is fibered over S^1, then $q = 0$. We use this fact in Section 4.

PROPOSITION 1.3. *$M^*\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}$ is homeomorphic to $M(b)$.*

Proof. The twisted I-bundle $N(\gamma)$ may be viewed as the orientable annulus bundle over S^1 with connected boundary. Let A be a fibered annulus. Observe that each boundary component of A is isotopic to the simple closed curve which generates $\rho \in \pi_1(T^2)$ $(T^2 = \partial N(\gamma))$. Let f be the attaching map $f: \partial N(\gamma) \longrightarrow \partial N(\gamma)$. Since $f_*(\rho) = \rho$, we see that the two simple closed curves of $f(\partial A)$ also bound a fibered annulus A' in the other $N(\gamma)$ in $M^* = M^*\left(\begin{smallmatrix} 1 & 0 \\ b & 1 \end{smallmatrix}\right)$. Thus $S = A \cup A'$ is a nonseparating torus in M^*. Cut M^* along S. Let φ be the map of S repairing the cut. Observe that the fiber structures of the two $N(\gamma)$ (as annulus bundles) agree on the boundaries up to isotopy (note that $f_*(\rho) = \rho$), and therefore we see that $M^* \approx S \times I/\varphi$. The torus S may be parametrized in terms of T^2, so that the generator $\rho \in \pi_1(T^2)$ can be represented by a boundary component of $A \subset S$ (with an orientation). Recall that since $\varphi \mid A: A \longrightarrow A$ is orientation-preserving and interchanges the two boundary components, ρ must be sent to ρ^{-1} under the automorphism $\varphi_*: \pi_1(T^2) \longrightarrow \pi_1(T^2)$. Thus the matrix of φ is given as $\left(\begin{smallmatrix} a' & 0 \\ c' & -1 \end{smallmatrix}\right)$ for some a' and c'. Since $\varphi: S \longrightarrow S$ is orientation-preserving, we see that $a' = -1$. On the other hand, we see that

$$\pi_1(M^*) = \{\alpha_1, \alpha_2, \beta \mid \alpha_1\beta\alpha_1^{-1}\beta = 1, \quad \alpha_2\beta\alpha_2^{-1}\beta = 1, \quad \alpha_1^2 = \alpha_2^2\beta^b\}.$$

Thus $\pi_1(M^*)$ is isomorphic to $\pi_1(M(b))$ (see [12]). Now we conclude that $c' = \pm b$, and therefore the result follows. □

COROLLARY 1.4. (1) *$M^*(b)$ is homeomorphic to $M(b)$.* (2) *If $M^*(b')$ is homeomorphic to $M^*(b)$, then $b' = \pm b$.*

Proof. Immediate from Proposition 1.3. □

2. Proof of Theorem A. Let M be an orientable torus bundle over S^1 and K a Klein bottle in M. Crucial to the proof is the following theorem.

THEOREM 2.1. *There exists a nonseparating torus S in M such that $S \cap K$ is a simple closed curve which is two-sided and nonseparating in K.*

LEMMA 2.2. [4]. *There are exactly five isotopy classes of simple closed curves in K. If $\pi_1(K) = \{\alpha, \beta \mid \alpha\beta\alpha^{-1}\beta = 1\}$, then these can be represented by $\{1\}$, α, β, $\alpha\beta$, α^2.*

LEMMA 2.3. *Let S be a nonseparating torus in M. Then S splits M into a space $\bar{M}$ homeomorphic to $T^2 \times I$.*

Proof. Since any compressible torus in an orientable irreducible 3-manifold is separating, S must be incompressible. Now it is not difficult to see that the inclusion $i: S \longrightarrow \bar{M}$ induces an isomorphism $i_*: \pi_1(S) \longrightarrow \pi_1(\bar{M})$ (view $S \subset \bar{M}$ in the obvious manner). Thus the result follows from [2]. □

The following must be well known (as the proof is easy). One may use a theorem of [2] as in Lemma 2.3.

LEMMA 2.4. *Let A be an annulus properly embedded in $T^2 \times I$ such that the two boundary components are not contained in a single boundary component of $T^2 \times I$. If A is not contractible in $T^2 \times I$, then A splits $T^2 \times I$ into a space homeomorphic to $A \times I$.*

Assume Theorem 2.1 which will be proved at the end of this section. We split M along S so as to obtain a space $\bar{M}$ homeomorphic to $T^2 \times I$ (see Lemma 2.3). Let A be the annulus obviously obtained from K by the splitting.

LEMMA 2.5. *Let N be a regular neighborhood of K in M. Then* $\mathrm{cl}(M-N)$ *is an orientable twisted I-bundle over K.*

Proof. Take a regular neighborhood U of A in $\bar{M}$ such that $U \cap \partial\bar{M}$ is a union of two annuli A_1 and A_2. Since the two boundary components of A are the same in M after repairing the cut, we may assume that A_1 meets A_2 in this repairing process so as to obtain a regular neighborhood $\bar{U}$ of K in M. It follows from Lemma 2.4 that the closure of the complement of $\bar{U}$ in M must be homeomorphic to a space $B \times I/\varphi$ where B is an annulus and $\varphi : B \longrightarrow B$ is an attaching map. Since M is orientable, φ must be orientation-preserving. Since φ interchanges the two boundary components of B, we see that $B \times I/\varphi$ is homeomorphic to the orientable annulus bundle over S^1 with connected boundary. Thus the result follows. □

Now it follows from Lemma 2.5 and Propositions 1.2 and 1.3 that the space M is homeomorphic to $M(b)$ for some b. Thus the proof of Theorem A will follow by the completion of Theorem 2.1.

Proof of Theorem 2.1. Let S be a fiber in M. Put S in general position with respect to K. Then the intersection curves are all simple closed curves without branch points. We let S be so chosen that the number of simple closed curves in $S \cap K$ is minimal among all such possible tori S. Let $c(S)$ be the number of components of $S \cap K$. Since $T^2 \times I$ does not admit a Klein bottle, we see that $c(S) \neq 0$ (cf. Lemma 2.3). We divide the proof into several cases according to the types of simple closed curves in K and S. Let σ be a simple closed curve in $S \cap K$.

Case 1). σ is one-sided in K. Since S is two-sided in M, this case cannot occur.

Case 2). σ bounds a disk in K or S. Since S and K are incompressible, σ bounds a disk in both S and K. Thus one may find another nonseparating torus S' with $c(S') < c(S)$.

Thus $S \cap K$ consists of only two-sided noncontractible simple closed curves in both K and S. If $S \cap K$ contains two simple closed curves σ_1 and σ_2, it follows from Lemma 2.2 that the two curves must be parallel in K (and also in S).

Case 3). The union $\sigma_1 \cup \sigma_2$ bounds an annulus A in K. Clearly $\sigma_1 \cup \sigma_2$ separates S into two annuli, say A_1, A_2. Let $S_i = A \cup A_i$ $(i=1,2)$. Since S is nonseparating in M, at least one of S_1 and S_2 cannot separate M, say S_1. Let U be a small neighborhood of A in M such that $U \cap K$ is an annulus and $U \cap S$ is a regular neighborhood of $\sigma_1 \cup \sigma_2$ in S.

Subcase 1). $S_1 \cap U$ meets both sides of $U - (K \cap U)$. Let J, J' be the two components of $\partial U \cap S_1$. Let E be an annulus in U such that (i) $\partial E = J \cup J'$, (ii) $\mathrm{Int}(E) \cap S = \emptyset$, and (iii) $E \cap K$ $(= E \cap A)$ is a simple closed curve. Let $S' = (S_1 - (U \cap S_1)) \cup E$. Then we see that $c(S') < c(S)$.

Subcase 2). $S_1 \cap U$ meets only one side of $U - (K \cap U)$. Pulling S_1 away in U from K, one may find a new nonseparating S' with $c(S') < c(S)$.

Thus the only possibility compatible with our choice of S is that $S \cap K$ is a noncontractible two-sided simple closed curve σ in K.

Case 4). σ is separating in K. Let M' be the space obtained by splitting M along S. Then $M' \approx T^2 \times I$. Note that σ splits K into two möbius bands B_1, B_2 and each B_i $(i=1,2)$ is properly embedded in M'. Let F_1 and F_2 be the two boundary components of M'. By using the identity maps of each F_i, we attach two copies of M' in the obvious manner so as to obtain a space $\bar{M}$ homeomorphic to $T^2 \times S^1$. Then obviously $\bar{M}$ contains two Klein bottles. However, $T^2 \times S^1$ does not admit an embedding of K (see [1], [9]).

The only remaining possibility is that σ be two-sided and nonseparating, which completes the proof. □

3. Orientation-reversing involutions on $M(0)$. We list the (standard) six orientation-reversing involutions on $M(0)$ as claimed in Theorem C. In the following list we view $M(0)$ as $T^2 \times I/\phi(0)$ (see Section 1 for $\phi(0)$). We denote by $[z_1, z_2, \tau]$ the image of $(z_1, z_2, \tau) \in T^2 \times I$ under the identification map of $T^2 \times I$ to $T^2 \times I/\varphi$. Let $F_i = \mathrm{Fix}(h_i)$.

(1) $h_1[z_1, z_2, \tau] = [\bar{z}_1, -z_2, \tau]$; $F_1 = \phi$,

(2) $h_2[z_1, z_2, \tau] = [z_2, z_1, \tau]$; $F_2 = K$,

(3) $h_3[z_1, z_2, \tau] = [\bar{z}_1, z_2, \tau]$; $F_3 = K \dot{\cup} K$,

(4) $h_4[z_1, z_2, \tau] = [z_1, z_2, 1-\tau]$; $F_4 = T^2 \dot{\cup} S^0 \dot{\cup} S^0$,

(5) $h_5[z_1, z_2, \tau] = [-\bar{z}_1, z_2, \tau]$; $F_5 = T^2$,

(6) $h_6[z_1, z_2, \tau] = [-\bar{z}_1, \bar{z}_2, 1-\tau]$; $F_6 = S^0 \dot{\cup} S^0$.

4. Proofs of Theorems B and C. We use the same notations as in Section 1. Let ϕ be an orientation-reversing homeomorphism of T^2. Then the matrix of ϕ may be given as $\begin{pmatrix} p & q \\ r & s \end{pmatrix}$ with $\begin{vmatrix} p & q \\ r & s \end{vmatrix} = -1$.

LEMMA 4.1. *If a torus bundle $M(\varphi)$ is homeomorphic to $M(\phi^2)$ $(\not\approx T^2 \times S^1)$, then φ is isotopic to a conjugate of ϕ^2.*

Proof. By computation, the matrix of ϕ^2 is given as $\begin{pmatrix} \lambda p+1 & \lambda q \\ \lambda r & \lambda s+1 \end{pmatrix}$ where $\lambda = p+s$. Thus we see that $\pi_1(M(\phi^2))$ can be represented by

$$\pi_1(M(\phi^2)) = \{\xi, \rho, t \mid [\xi, \rho] = 1, \quad t\xi t^{-1} = \xi^{\lambda p+1}\rho^{\lambda r}, \quad t\xi t^{-1} = \xi^{\lambda q}\rho^{\lambda s+1}\}.$$

Observe that $\bar{\xi}^{\lambda} = 1 = \bar{\rho}^{\lambda}$ where $\bar{\xi}$, $\bar{\rho}$ are the images of ξ and ρ under the natural homomorphism of $\pi_1(M(\phi^2))$ to $H_1(M(\phi^2))$. Obviously if $\lambda = 0$, then $M(\phi^2) \approx T^2 \times S^1$. Thus $\lambda \neq 0$, and one completes the proof by doing in the similar manner to the proof of Proposition 1.1. □

The proof of the following is elementary and is omitted.

LEMMA 4.2. *Let A, B be any unimodular 2×2 matrices and $C = \begin{pmatrix} -1 & 0 \\ -b & -1 \end{pmatrix}$. If $A^2 = BCB^{-1}$, then $b = 0$.*

LEMMA 4.3. *$M(\phi^2)$ is not homeomorphic to $M(b)$.*

Proof. If $M(\phi^2)\approx T^2\times S^1$, the result is obvious. So assume that $M(\phi^2)\not\approx T^2\times S^1$. Suppose the contrary that they were homeomorphic. Then it follows from Lemma 4.1 that the matrix of ϕ^2 is similar to $\begin{pmatrix} -1 & 0 \\ -b & -1 \end{pmatrix}$. Furthermore, by Lemma 4.2, we see that $b=0$. Thus $\begin{pmatrix} p & q \\ r & s \end{pmatrix}^2=\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. Now one can easily argue that $\begin{vmatrix} p & q \\ r & s \end{vmatrix}\neq -1$, which is a contradiction. □

Let g be an involution on T^2. Then g is conjugate to one of the following involutions (as is well-known):

$$g_1(z_1,z_2) = (z_1,-z_2),\quad g_2(z_1,z_2) = (-z_1,\bar{z}_2)$$

$$g_3(z_1,z_2) = (\bar{z}_1,\bar{z}_2),\quad g_4(z_1,z_2) = (z_1,\bar{z}_2)$$

$$g_5(z_1,z_2) = (z_2,z_1).$$

The following lemma is a special case of [7].

LEMMA 4.4. *Let k be an involution on $T^2\times I$. Then there exists a product structure on $T^2\times I$ and a map g with $g^2=1$ such that k is given by $k(x,t)=(g(x),t)$ or $(g(x),1-t)$ for each $(x,t)\in T^2\times I$.*

LEMMA 4.5. *Let h be an involution on M. Then there exists a nonseparating torus S in M such that either $h(S)\cap S=\emptyset$ or $h(S)=S$ and S is in a general position with respect to* Fix(h).

Proof. This essentially follows from the proof of Theorem B of [8] (also see [7]). Note that the torus in each construction in the proof there can be chosen to be nonseparating. □

Let h be an orientation-reversing involution on $M=M(\varphi)$. If Fix$(h)\neq\emptyset$ then it follows from Lemmas 2.3, 4.4 and 4.5 that Fix(h) contains T^2, K, isolated points, or their combinations. In case each component of Fix(h) is 2-dimensional, Fix(h) contains at most two components. We will prove Theorems B and C simultaneously. The proof will be divided into several cases according to the types of fixed-point sets.

Case 1). Fix(h) contains a separating torus in M. It follows from Lemma 4.5 that there exists a nonseparating torus S in M such that either $h(S)\cap S=\emptyset$ or $h(S)=S$ and S is in general position with respect to Fix(h). Since Fix(h) contains a separating torus we must have $h(S)=S$. By Lemmas 2.3 and 4.4 there exists a fiber structure $T^2\times\mathbf{R}/\varphi$ for some equivariant map φ such that h is given by $h[x,\tau]=[g(x),\tau]$ where g is an involution on T^2. Observe that Fix(g) is invariant under φ. Since Fix(h) contains a separating torus, we see that Fix(g) contains two components, and g is equivalent to g_4. Thus, we may assume that $M=T^2\times\mathbf{R}/\bar{\varphi}$ and $h[x,\tau]=[g_4(x),\tau]$ for a suitable equivariant map $\bar{\varphi}: T^2\longrightarrow T^2$. Let c_1,c_2 be the components of Fix(g_4). Then $c_1\cup c_2$ separates T^2 into two components A_1,A_2 (we view $c_1,c_2\subset M$ in the obvious manner). Since Fix(h) contains a separating torus, we see that $\bar{\varphi}(c_1)=c_2$ (therefore, Fix(h) is the torus). By the same reason, one sees that $\bar{\varphi}(A_i)=A_i$ $(i=1,2)$. Thus we may assume that $\bar{\varphi}_*(\rho)=\rho^{-1}$, and the matrix of $\bar{\varphi}$ is given as

$\begin{pmatrix} -1 & 0 \\ b' & -1 \end{pmatrix}$ for some integer b'. On the other hand, since $g_4\bar{\varphi}=\bar{\varphi}g_4$ is required, one sees that $b'=0$. Now, given two such involutions, one may easily find an equivalence t between them by lifting a suitable homeomorphism between their orbit spaces in a usual way. Thus h is conjugate to h_5.

Case 2). $\mathrm{Fix}(h)\approx K$. Let N be a regular neighborhood of K. Then the complement N' of N in M is a twisted I-bundle over K (see Lemma 2.5). Then it may be assumed that $M=N(\gamma)\cup_{f_0}N(\gamma)$, where f_0 is an equivariant attaching map. Let $\bar{h}_1=h\,|\,N(\gamma)$ $(=N)$ and $\bar{h}_2=h\,|\,N(\gamma)$ $(=N')$. We let $[x,\tau]$ denote the image of (x,τ) under the identification map of $T^2\times I$ to $N(\gamma)$. Since $\mathrm{Fix}(\bar{h}_1)=K$ and $\mathrm{Fix}(\bar{h}_2)=\phi$, we may assume (see [6], [14]) that M and $\bar{h}_i$ are given as $M=N(\gamma)\cup_f N(\gamma)$ and $\bar{h}_1[x,\tau]=[\gamma(x),\tau]$, $\bar{h}_2[x,\tau]=[\gamma g_1(x),\tau]$ where $f\colon T^2\longrightarrow T^2$ is a suitable equivariant map. Here, we may further assume that the matrix of f is given as $\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}$ (see Section 1). Since f is an equivariant map, we see that $b=0$. Observe that the orbit space is homeomorphic to a twisted I-bundle over K. Thus, given two such involutions, one may find an equivalence between them by lifting an appropriate homeomorphism between their orbit spaces. Therefore h is conjugate to h_2.

Case 3). $\mathrm{Fix}(h)\approx K\dot{\cup}K$. As before, we may assume that M is given as $M=N(\gamma)\cup_f N(\gamma)$ and h is split into two involutions $\bar{h}_1,\bar{h}_2$ on $N(\gamma)$ such that

$$\bar{h}_1[x,\tau]=[\gamma(x),\tau],\quad \bar{h}_2[x,\tau]=[\gamma(x),\tau]$$

where $f\colon T^2\longrightarrow T^2$ is a suitable equivariant map. Since $f\gamma=\gamma f$, we may assume that the matrix of f is given as $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Now one sees that the orbit space is a product I-bundle whose boundary is the projection of $\mathrm{Fix}(h)$. Thus it is easy to show that any two involutions h on M with $\mathrm{Fix}(h)\approx K\dot{\cup}K$ are conjugate. Therefore h is conjugate to h_3.

Case 4). $\mathrm{Fix}(h)$ contains either a nonseparating torus or isolated points (or both). In any case it follows from Lemmas 2.3, 4.4 and 4.5 that there exists a fiber structure $T^2\times I/\varphi$ for an attaching map $\varphi\colon T^2\longrightarrow T^2$ such that h is given by

$$h[x,\tau]=\begin{cases} [k_1(x),\tfrac{1}{2}-\tau] & 0\le\tau\le\tfrac{1}{2} \\ [k_2(x),\tfrac{3}{2}-\tau] & \tfrac{1}{2}\le\tau\le 1 \end{cases}$$

where k_i $(i=1,2)$ is an involution on T^2 (if $\mathrm{Fix}(h)$ contains a nonseparating torus T, then split M along T. Note that h interchanges the sides of T). Obviously $\varphi=k_2k_1$. Since φ is orientation-preserving and h is orientation-reversing, we may assume that (1) $\mathrm{Fix}(k_1)=T^2$ and $\mathrm{Fix}(k_2)=\emptyset$, (2) $\mathrm{Fix}(k_1)=4$ points and $\mathrm{Fix}(k_2)=\emptyset$, (3) $\mathrm{Fix}(k_1)=T^2$ and $\mathrm{Fix}(k_2)=4$ points, or (4) $\mathrm{Fix}(k_1)=T^2$ and $\mathrm{Fix}(k_2)=T^2$.

Subcase 1). $\mathrm{Fix}(k_1)=T^2$ and $\mathrm{Fix}(k_2)=\emptyset$. Since $\varphi=k_2k_1$, we see that the matrix of φ is given as $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (note that k_1 is the identity and k_2 is conjugate to g_1). Thus the space M in this case is homeomorphic to $T^2\times S^1$.

Subcase 2). $\mathrm{Fix}(k_1)=4$ points and $\mathrm{Fix}(k_2)=\emptyset$. Since $\varphi=k_2k_1$, one can compute to see that the matrix of φ is given as $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ (note that k_1,k_2 are conjugate to g_3,g_1, respectively). By suitable choices of k_2 and φ, we may assume that $k_1=g_3$ (observe that the matrix of φ does not change (see Proposition 1.1)). Since k_2 and g_1 are conjugate, there exists an equivalence $\beta: T^2 \longrightarrow T^2$ such that $k_2\beta = \beta g_1$. It is easy to find a homeomorphism $\beta': T^2 \longrightarrow T^2$ isotopic to β such that β' commutes with g_1. Let J be an ambient isotopy with $J_1 = \beta\beta'^{-1}$. We let M_0 denote the space $T^2 \times I/g_1g_3$, and h_0 the involution on M_0 defined by

$$h_0[x,\tau] = \begin{cases} [g_3(x), \frac{1}{2} - \tau] & 0 \leq \tau \leq \frac{1}{2}, \\ [g_1(x), \frac{3}{2} - \tau] & \frac{1}{2} \leq \tau \leq 1. \end{cases}$$

We define a homeomorphism $\Phi: M_0 \longrightarrow M$ by

$$\Phi[x,\tau] = \begin{cases} [x,\tau] & 0 \leq \tau \leq \frac{1}{2} \\ [J(x, 4\tau - 2), \tau] & \frac{1}{2} \leq \tau \leq \frac{3}{4} \\ [k_2 J(g_1(x), 4 - 4\tau), \tau] & \frac{3}{4} \leq \tau \leq 1. \end{cases}$$

It is checked that Φ is well defined and it is, in fact, an equivalence between h_0 and h (recall that $k_1 = g_3$). Therefore h is conjugate to h_6.

Subcase 3). $\mathrm{Fix}(k_1) = T^2$ and $\mathrm{Fix}(k_2) = 4$ points. By doing in the same spirit as in the above (Subcase 2), one can show that the matrix of φ is given as $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and h is conjugate to h_4.

Subcase 4). $\mathrm{Fix}(k_1) = T^2$ and $\mathrm{Fix}(k_2) = T^2$. Since $\varphi = k_2k_1$, one sees that φ is isotopic to the identity, and M is homeomorphic to $T^2 \times S^1$.

Case 5). $\mathrm{Fix}(h) = \emptyset$. It follows from Lemma 3.3 that there exists a nonseparating torus S in M such that either $h(S) \cap S = \emptyset$ or $h(S) = S$.

Subcase 1). $h(S) \cap S = \emptyset$. Then $S \cup h(S)$ separates M into two components Q_1, Q_2, each of which is homeomorphic to $T^2 \times I$. If $h(Q_1) = Q_2$, then we see that there exists a fiber structure $T^2 \times \mathbf{R}/\varphi$ for a map $\varphi: T^2 \longrightarrow T^2$ such that h is given by

$$h[x,\tau] = [k(x), \tau + \tfrac{1}{2}] \qquad \text{for} \qquad (x,\tau) \in T^2 \times \mathbf{R}$$

where k is a map of T^2 with $k^2 = \varphi$. Since k is orientation-reversing, it follows from the proof of Lemma 4.1 that M is homeomorphic to $\bar{M}\left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right)$ for some integers p, q, r, s with $\left|\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right| = -1$. Therefore it follows from Lemma 4.3 that $M \not\approx M(b)$ for any b in this case, and the Seifert manifolds $M(b)$ do not admit orientation-reversing involutions of this type. Now we assume that $h(Q_i) = Q_i$ $(i = 1, 2)$. We can treat this case as an additional subcase of Case 4. That is, $\mathrm{Fix}(k_i) = \emptyset$. Since h is orientation-reversing and φ is orientation-preserving, one sees that k_1 k_2 are orientation-preserving. Thus the matrix of k_i can be given as $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, and φ is isotopic to the identity, and M is homeomorphic to $T^2 \times S^1$.

Subcase 2). $h(S) = S$. We may assume that h does not interchange the sides of S (if so, one may return to the Subcase 1). By Lemma 4.4, one sees that there exists a fiber structure $T^2 \times \mathbf{R}/\varphi$ for an equivariant map $\varphi: T^2 \longrightarrow T^2$ such that h is given by $h[x,\tau] = [g(x), \tau]$ where g is an involution on T^2. Furthermore, we may assume that g is given by $g(z_1, z_2) = (-z_1, \bar{z}_2)$ for each $(z_1, z_2) \in T^2$. Since $\varphi g = g\varphi$, we see that the matrix of φ must be given as $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. Thus, the classification of these involutions depends on the various possible equivariant maps φ. However, since the matrices of such φ must be $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, any two such equivariant maps are isotopic. Thus it follows from [15] that h is conjugate to h_1 (note that $\mathrm{Fix}(g)=\emptyset$).

Observe that if $M \approx \bar{M}\begin{pmatrix} p & q \\ r & s \end{pmatrix}$ $(p+s \neq 0)$, then subcase 1 of Case 5 can only occur. Consequently, the above completes the proofs of Theorems B and C.

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