ON THE CLASSIFICATION OF STATIONARY SETS

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1. INTRODUCTION

In this paper, a *cardinal* is an initial ordinal and a cardinal α is *regular* provided it is not the sum of fewer, smaller cardinals. We will reserve the symbol κ for a fixed regular uncountable cardinal. Viewing the ordinal κ as the set of smaller ordinals we will speak of "subsets of κ" instead of "subsets of $[0,\kappa)$." (However, see the comments on special notations at the end of this section.)

The set κ will always carry the usual order topology and subsets S of κ will always be endowed with the relative topology inherited from κ. We define

$$\operatorname{cub}(\kappa) = \{S \subset \kappa : S \text{ is closed and unbounded (equivalently, cofinal in } \kappa)\}$$

We say that a set $S \subset \kappa$ is *stationary* if $S \cap C \neq \emptyset$ for each $C \in \operatorname{cub}(\kappa)$ and that S is *bistationary* if both S and $\kappa - S$ are stationary.

Regular cardinals and their stationary subsets have long been important tools in topology, especially as indexing and constructive devices, *e.g.*, in the recent papers [4], [5], [7], [8]. They have received less attention as topological objects in their own right, except in the theory of ordered spaces where stationary sets in regular cardinals are viewed as the archetypical non-paracompact ordered spaces. More precisely, it is proved in [3] that if X is (a subspace of) a linearly ordered topological space, then X is *not* paracompact if and only if some closed subspace of X is homeomorphic to a stationary set in an uncountable regular cardinal. That theorem raises the question of whether two stationary subsets of a fixed κ can be distinguished topologically from each other. Phrased in that way, the question has an immediate affirmative answer since, if S is a bistationary set in κ, then no member of $\operatorname{cub}(\kappa)$ can even be mapped continuously onto a cofinal subset of S. (Further, the existence of bistationary sets is guaranteed by the Ulam-Solovay theorem cited as Theorem E below.) However it is more difficult to determine whether two bistationary sets in κ, *e.g.*, a bistationary set and its complement, are of different topological types, especially if $\kappa = \omega_1$. In this paper we settle that question by proving

A. THEOREM. *If* S *and* T *are disjoint bistationary subsets of* κ, *then there is no continuous mapping of* S *onto a cofinal subset of* T.

Indeed we show that there are many topologically incomparable types of stationary subsets of κ by proving

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B. THEOREM. *There is a collection* $\mathscr{B}$ *of bistationary subsets of* κ *such that:*

(1) $\mathscr{B}$ *has cardinality* 2^κ;

(2) *if* $S \neq T$ *belong to* $\mathscr{B}$ *then there is no continuous mapping from* S *onto a cofinal subset of* T.

The key step in proving Theorems A and B is

C. THEOREM. *If* S *is a stationary subset of* κ *and if* T *is a cofinal subset of* κ *which is a continuous image of* S, *then* T *is stationary and* $S - T$ *is* not *stationary.*

The condition in the conclusion of Theorem C, when made suitably symmetric, gives a useful necessary condition for homeomorphism of two stationary subsets S and T of κ, namely, that the set $S \Delta T = (S - T) \cup (T - S)$ must be nonstationary, so that $S \cap T$ must be very large indeed. Furthermore, even though the condition "$S \Delta T$ is nonstationary" is not sufficient for homeomorphism of S and T, it is possible to translate the condition into a functional relationship between S and T provided one is willing to consider the class of measurable functions (see Section 4 for precise details) instead of the more restrictive class of continuous functions.

The σ-algebra with respect to which our functions are measurable is a generalization of the Borel σ-algebra. To avoid cluttering this Introduction with new terminology, we preview our results on measurable functions by considering the case where $\kappa = \omega_1$; in this case no such generalization is required. Recall that the *Borel σ-algebra* on a space X, denoted $\mathscr{B}\ell(X)$, is the smallest σ-algebra of subsets of X to which each open set belongs, and that two spaces X and Y are *Borel-isomorphic* if there is a bijection $f: X \to Y$ such that $B \in \mathscr{B}\ell(X)$ if and only if $f[B] \in \mathscr{B}\ell(Y)$. We prove

D. THEOREM. *Let* S *and* T *be stationary subsets of* ω_1. *Then the following are equivalent:*

(1) $S \Delta T$ *is nonstationary;*

(2) S *and* T *are Borel isomorphic;*

(3) *there are continuous functions* $f: S \to T$ *and* $g: T \to S$, *not necessarily surjective, such that both* $f[S]$ *and* $g[T]$ *are uncountable.*

Our paper is organized as follows. Section 2 contains characterizations of stationary sets in regular cardinals in terms of a certain σ-algebra which generalizes the usual Borel σ-algebra. Section 3 describes another generalization of the Borel σ-algebra which can help in understanding the σ-algebra used in section 2. In section 4 we introduce measurable and strongly measurable functions and study their interrelations; these mappings, and not the continuous mappings, are appropriate for the study of stationary sets. Section 5 presents our main results on the classification of stationary sets. In section 6 we characterize stationary sets using mappings into metrizable spaces; these results are not directly related to the problem of classifying stationary sets, but the methods employed to obtain them are similar to the techniques used in earlier sections of the paper. Section 7 displays examples to which earlier sections refer and section 8 presents a list of open questions.

The main part of the paper is self-contained except for the following important theorem due (essentially) to Ulam [12] for successor cardinals and to Solovay [11] for the general case.

E. THEOREM. *Let* S *be a stationary subset of a regular uncountable cardinal* κ. *Then there is a family* $\mathscr{D}$ *of pairwise disjoint stationary subsets of* κ, *each contained in* S, *such that* $\mathscr{D}$ *has cardinality* κ.

The *cardinality* of a set S is denoted by $|S|$ and the *power set* of S is denoted by $\mathscr{P}(S)$. Further, since κ will always denote a *regular* uncountable cardinal, the statements "S is a cofinal subset of κ" and "$S \subset \kappa$ has $|S| = \kappa$" are equivalent. Throughout the paper we will identify κ with the set $[0,\kappa)$ of ordinals less than κ. However, when forced to consider initial subspaces of κ we will write $[0,\lambda)$ instead of merely λ so that the reader can always determine whether we mean a large subspace of κ or just the point λ of the set κ. Having given that forewarning, we hope that our notation will clarify rather than obfuscate the exposition. Finally, *unless otherwise noted, all closures are taken in the space* κ.

All of our theorems involve the *regular* cardinal κ, but that does not cause any essential loss of generality since, for any cardinal (*i.e.*, initial ordinal) λ, there is a closed cofinal copy of the regular cardinal $\mathrm{cf}(\lambda)$ in λ and the stationary subsets of λ are entirely classified (modulo nonstationary difference) by their traces on $\mathrm{cf}(\lambda)$. The case where $\mathrm{cf}(\lambda) = \omega_0$ is uninteresting and the cases in which $\mathrm{cf}(\lambda) > \omega_0$ are covered by our theorems.

Our terminology and notation concerning linearly ordered sets and spaces generally follows that of [3].

2. MEASURABLE SETS AND CHARACTERIZATIONS OF STATIONARY SETS

The collection $\mathrm{cub}(\kappa)$ and the notions of stationary and bistationary subsets of the regular uncountable cardinal κ were defined in the Introduction. We now generalize these notions in a natural manner.

2.1 *Definition.* Let S be a cofinal subset of κ. Define

$$\begin{aligned}
\mathrm{cub}(S) &= \{A \subset S\colon A \text{ is relatively closed and } |A| = \kappa\};\\
\mathscr{S}(S) &= \{A\colon A \subset S \text{ and } A \text{ is stationary in } \kappa\};\\
\mathscr{M}(S) &= \{A \subset S\colon \text{either } A \text{ or } S - A \text{ contains a member of } \mathrm{cub}(S)\};\\
\mathscr{M}_+(S) &= \{A \subset S\colon A \text{ contains a member of } \mathrm{cub}(S)\}.
\end{aligned}$$

Members of $\mathscr{M}(S)$ are called *measurable* subsets of S.

2.2 *Remarks.* If S is cofinal in κ, then every relatively open subset of S belongs to $\mathscr{M}(S)$ and $\mathscr{M}(S)$ is a σ-algebra by (2.3c) or (2.5), depending upon whether or not $S \in \mathscr{S}(\kappa)$. Hence $\mathscr{B}\ell(S) \subset \mathscr{M}(S)$ where $\mathscr{B}\ell(S)$ is the Borel σ-algebra of the space S. For successor cardinals κ, Proposition (5.1a) explains exactly when $\mathscr{B}\ell(S) = \mathscr{M}(S)$ for a stationary subset S of κ. The result that $\mathscr{B}\ell(S) = \mathscr{M}(S)$ for each stationary set S in ω_1 is due independently to Rao and Rao [9] and to the two authors of the present paper. If $S \in \mathscr{S}(\kappa)$ then

$$\mathrm{cub}(S) = \{S \cap C: C \in \mathrm{cub}(\kappa)\} \subset \mathscr{S}(S).$$

And if $S \notin \mathscr{S}(\kappa)$ then $\mathscr{S}(S) = \emptyset$ even though cub(S) and $\mathscr{M}(S)$ will be large collections; see (2.5).

We begin with two easy lemmas which will be needed in later sections.

2.3. LEMMA. *Let* $S \in \mathscr{S}(\kappa)$. *Then:*

(a) $\mathrm{cub}(S) \subset \mathscr{S}(S) \subset \mathscr{S}(\kappa)$;

(b) *the set of non-isolated points of* S *belongs to* $\mathrm{cub}(S) \subset \mathscr{S}(S)$;

(c) *if* $\{C_\alpha: \alpha \in A\} \subset \mathrm{cub}(S)$ *and if* $|A| < \kappa$ *then* $\bigcap \{C_\alpha: \alpha \in A\} \in \mathrm{cub}(S)$;

(d) *if* $N_\alpha \notin \mathscr{S}(\kappa)$ *for each* $\alpha \in A$ *and if* $|A| < \kappa$, *then* $\bigcup \{N_\alpha: \alpha \in A\} \notin \mathscr{S}(\kappa)$.

Proof. Assertions (a) and (b) are clear, and (c) follows from the corresponding well-known result for members of cub(κ) while (d) follows from (c).

Using (2.3c) we can easily prove the following well known "diagonal intersection lemma" which is needed at several points in the sequel.

2.4 LEMMA. *Let* $S \in \mathscr{S}(\kappa)$ *and for each* $x \in S$ *let* $C_x \in \mathrm{cub}(\kappa)$. *Then the set* $D = \{x \in S: x \in C_y \text{ for each } y \in S \cap [0,x)\}$ *belongs to* cub(S).

Proof. For each $x \in \kappa - S$ let $C_x = \kappa$. Consider the set

$$E = \{x \in \kappa: x \in C_y \text{ for each } y < x\}.$$

The set E is closed in κ. For suppose $x \in \bar{E}$ and let $y < x$ be arbitrary. Then $x \in (E \cap (y,x])^-$. But clearly $E \cap (y,x] \subset C_y$ so that $x \in C_y$ since C_y is closed. Next, E is cofinal in κ. For let $w \in \kappa$. Because of (2.3c) we can inductively define a sequence $\langle x_n \rangle$ in κ by

$$x_0 = w \quad \text{and} \quad x_{n+1} = \min\left(\bigcap \{C_y: y < x_n\} \cap (x_n, \kappa)\right).$$

Then the ordinal $z = \sup\{x_n: n \in \omega\}$ belongs to E and is greater than w. Therefore $E \in \mathrm{cub}(\kappa)$ so that $E \cap S \in \mathrm{cub}(S)$. But $E \cap S = D$.

Our next result characterizes stationary subsets of κ and is used repeatedly in later sections. The equivalence of (a) and (f) is the contrapositive of the well known Pressing Down Lemma.

2.5 THEOREM. *Let* $S \subset \kappa$ *have cardinality* κ. *The following are equivalent:*

(a) S *is* not *stationary;*

(b) *if* $T \subset S$ *has* $|T| = \kappa$, *then some relatively closed discrete subspace* D *of* S *has* $|D| = \kappa$ *and* $D \subset T$;

(c) *there are two disjoint members of* $\mathrm{cub}(S)$;

(d) $\mathscr{P}(S) = \mathscr{M}(S)$;

(e) *there is a function* $f\colon S \to \kappa$ *such that* if $x \in S - \{0\}$ *t* $f(x) < x$ *and for each* $y \in \kappa$, $|f^{-1}[\{y\}]| < \kappa$;

(f) *there is a function* $g\colon S \to \kappa$ *such that if* $x \in S - \{0\}$ *then* $g(x) < x$ *and for each* $y \in \kappa$, $g^{-1}[\{y\}] \notin \mathscr{S}(\kappa)$.

Proof. We show that (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (a), then that (b) $\Rightarrow$ (d) $\Rightarrow$ (a) and finally that (a) $\Rightarrow$ (e) $\Rightarrow$ (f) $\Rightarrow$ (a).

(a) $\Rightarrow$ *(b)*: Since S is non-stationary some $C \in \mathrm{cub}(\kappa)$ has $S \cap C = \emptyset$. Let $\{I_\gamma\colon \gamma \in \Gamma\}$ be the family of maximal convex subsets of $\kappa - S$. Then $\Gamma_0 = \{\gamma \in \Gamma\colon I_\gamma \cap T \neq \emptyset\}$ has cardinality κ; for each $\gamma \in \Gamma_0$ choose $d_\gamma \in T \cap I_\gamma$. Then $D = \{d_\gamma\colon \gamma \in \Gamma_0\}$ is the required relatively closed, discrete subset of S having $D \subset T$.

(b) $\Rightarrow$ *(c)*: Let D be a relatively closed, discrete subset of S having cardinality κ and write D as the union of two disjoint subsets D_1 and D_2 each with cardinality κ.

(c) $\Rightarrow$ *(a)*: Given A and B, disjoint members of $\mathrm{cub}(S)$, the set $\bar{A} \cap \bar{B}$, where the closure is taken in κ, is a member of $\mathrm{cub}(\kappa)$ which is disjoint from S.

(b) $\Rightarrow$ *(d)*: Let $T \in \mathscr{P}(S)$. If $|T| < \kappa$ then $S - T$ contains $S \cap [x, \kappa)$ for some x so that $T \in \mathscr{M}(S)$. And if $|T| = \kappa$ then (b) yields a set $D \in \mathrm{cub}(S)$ with $D \subset T$, so that $T \in \mathscr{M}(S)$.

(d) $\Rightarrow$ *(a)*: If S is stationary in κ, then it follows from Theorem E of the Introduction that there are disjoint subsets T_1 and T_2 of S, each belonging to $\mathscr{S}(\kappa)$. But then neither T_1 nor $S - T_1$ can contain a member of $\mathrm{cub}(S)$, contrary to (d).

(a) $\Rightarrow$ *(e)*: Suppose C is a member of $\mathrm{cub}(\kappa)$ which misses S. Define $f\colon S \to \kappa$ by

$$f(x) = \begin{cases} 0 & \text{if } x \in S \text{ and } x < \min(C) \\ \max\{y \in C\colon y < x\} & \text{if } x \in S \text{ and } x > \min(C). \end{cases}$$

(e) $\Rightarrow$ *(f)*: This implication is obvious.

(f) $\Rightarrow$ *(a)*: Given $g\colon S \to \kappa$ as in (f), let $C_y \in \mathrm{cub}(\kappa)$ have $C_y \cap g^{-1}[\{y\}] = \emptyset$ for each $y \in \kappa$. Define

$$A = \left\{x \in \kappa\colon x \in \bigcap \{C_y\colon y < x\}\right\}$$

Clearly $S \cap A = \emptyset$ and, by (2.4), $A \in \mathrm{cub}(\kappa)$. But then $S \notin \mathscr{S}(\kappa)$, completing the proof.

2.6 COROLLARY. *Let* S *be cofinal in* κ. *Then:*

(a) $\mathscr{M}(S) = \{S \cap A\colon A \in \mathscr{M}(\kappa)\}$;

(b) $\mathscr{M}_+(S) = \{S \cap A\colon A \in \mathscr{M}_+(\kappa)\}$ *if and only if* $S \in \mathscr{S}(\kappa)$.

Proof. Write $\mathcal{N} = \{S \cap A: A \in \mathcal{M}(\kappa)\}$. Suppose $B \in \mathcal{M}(S)$ is disjoint from $C \in \mathrm{cub}(S)$. Then $B \cap \bar{C} = \emptyset$ so that $B \in \mathcal{M}(\kappa)$ and hence $B \in \mathcal{N}$. And if B contains $C \in \mathrm{cub}(S)$ then $A = B \cup \bar{C} \in \mathcal{M}(\kappa)$ so that again $B = S \cap A \in \mathcal{N}$. Thus $\mathcal{M}(S) \subset \mathcal{N}$. Conversely, suppose $B \in \mathcal{N}$. If S is *not* stationary in κ then $\mathcal{M}(S) = \mathcal{P}(S)$ and there is nothing to prove. And if $S \in \mathcal{S}(\kappa)$ then $B \in \mathcal{M}(S)$ because for each $C \in \mathrm{cub}(\kappa)$, $C \cap S \in \mathrm{cub}(S)$. Thus, (a) is proved. Assertion (b) is easily verified.

One important consequence of (2.5) is that the property of being a stationary set is a topological property which does not depend on the particular way in which the set is embedded in κ. We will see in (5.2b) that this is also true for the property of being bistationary. Furthermore, as the reader will see in Section 4, functions which are considerably more general than homeomorphisms preserve stationary sets.

2.7 COROLLARY. *Suppose* $S \in \mathcal{S}(\kappa)$ *and* h *is a homeomorphism from* S *into* κ. *Then* $h[S] \in \mathcal{S}(\kappa)$.

Proof. Since h [S] is cofinal in κ, the result follows from the equivalence of (a) and (c) in (2.5).

3. A GENERALIZATION OF THE BOREL σ-ALGEBRA

In certain cases, it is possible to relate $\mathcal{M}(S)$ to other algebras of subsets of S. As already noted, if S is stationary in ω_1 then $\mathcal{M}(S)$ is precisely $\mathcal{B\ell}(S)$, the usual Borel σ-algebra of subsets of the space S. In case κ is a regular cardinal greater than ω_1, $\mathcal{M}(\kappa) \supsetneqq \mathcal{B\ell}(\kappa)$. (See Example 7.1.) We now introduce a generalization of the Borel σ-algebra.

3.1 *Definition.* Let X be a topological space and let λ be a cardinal number. Then $\mathcal{B}(X,\lambda)$ is the smallest collection of subsets of the space X such that:

(a) each open subset of X belongs to $\mathcal{B}(X,\lambda)$;

(b) if $S \in \mathcal{B}(X,\lambda)$ then $X - S \in \mathcal{B}(X,\lambda)$;

(c) if $\mathcal{C} \subset \mathcal{B}(X,\lambda)$ and if $|\mathcal{C}| < \lambda$ then $\bigcap \mathcal{C} \in \mathcal{B}(X,\lambda)$.

It is clear that for any space X, $\mathcal{B\ell}(X) = \mathcal{B}(X,\omega_1)$.

3.2 PROPOSITION. *Suppose* S *is a cofinal subset of a successor cardinal* κ. *Let* $\mathcal{A}$ *be the family of subsets of* S *which are either unions of fewer than* κ *closed subsets of* S *or intersections of fewer than* κ *open subsets of* S. *Then*

$$\mathcal{M}(S) = \mathcal{A} = \mathcal{B}(S,\kappa).$$

Proof. Let $\kappa = \lambda^+$. We show that $\mathcal{M}(S) \subset \mathcal{A} \subset \mathcal{B}(S,\kappa) \subset \mathcal{M}(S)$. To prove that $\mathcal{M}(S) \subset \mathcal{A}$, let $B \in \mathcal{M}(S)$. If $|B| < \kappa$, then B is the union of fewer than κ closed sets. If $|B| = \kappa$ and if B is disjoint from some $C \in \mathrm{cub}(S)$ write

$$\kappa - \bar{C} = \bigcup \{I_\gamma : \gamma \in \Gamma\}$$

where the I_γ's are pairwise disjoint convex open sets in κ. Let

$$\Gamma_1 = \{\gamma \in \Gamma : I_\gamma - B \neq \emptyset\}.$$

For each $\gamma \in \Gamma_1$ index $I_\gamma - B$ as $\{b(\gamma,\alpha) : 1 \leq \alpha < \lambda\}$, repetitions being allowed. For $1 \leq \alpha < \lambda$ let $C_\alpha = \mathrm{cl}(\{b(\gamma,\alpha) : \gamma \in \Gamma_1\}$ and let $G_\alpha = S \cap (\kappa - C_\alpha)$ for $1 \leq \alpha < \lambda$. Let $G_0 = (\kappa - \bar{C}) \cap S$. Then $B = \bigcap \{G_\alpha : 0 \leq \alpha < \lambda\}$, showing that $B \in \mathscr{A}$ provided B is disjoint from some member of cub(S). Next suppose $B \in \mathscr{M}(S)$ contains a set $C \in \mathrm{cub}(S)$. Then $S - B \in \mathscr{A}$ by the first part of the proof so that $B \in \mathscr{A}$ since $\mathscr{A}$ is closed under the formation of complements (in S). Therefore $\mathscr{M}(S) \subset \mathscr{A}$. That $\mathscr{A} \subset \mathscr{B}(S,\kappa)$ is automatic. That $\mathscr{B}(S,\kappa) \subset \mathscr{M}(S)$ follows from (2.5d) if S is non-stationary and from (2.3c) in case S is stationary in κ.

3.3 *Remark.* Proposition 3.2 is *not* true if κ is a regular cardinal which is not a successor; *i.e.,* if κ is a regular limit cardinal. See Example 7.3.

4. MEASURABLE FUNCTIONS

Having defined the collections $\mathscr{M}(S)$ and $\mathscr{M}_+(S)$ for each cofinal subset S of κ, we can now introduce the special functions which are appropriate to the study of $\mathscr{S}(\kappa)$.

4.1 *Definition.* Let S and T be cofinal subsets of κ and let $f: S \to T$. Then f is *measurable* if $f^{-1}[B] \in \mathscr{M}(S)$ whenever $B \in \mathscr{M}(T)$ and f is *strongly measurable* if f is measurable and for each $y \in T$ the fiber $f^{-1}[\{y\}]$ of f is non-stationary. Finally, a function $f: S \to T$ is a *measurable isomorphism* if f is a bijection having the property that $A \in \mathscr{M}(S)$ if and only if $f[A] \in \mathscr{M}(T)$.

We remark that the range set T in the definitions of measurable and strongly measurable functions is irrelevant in the light of (2.6a); *i.e.,* $f: S \to T$ is measurable (respectively, strongly measurable) if and only if $f: S \to \kappa$ is measurable (respectively, strongly measurable). We also remark that the definition of strong measurability is topological, because of (2.7).

The relationship between measurable functions and continuous functions is summarized by

4.2 PROPOSITION. *Let* S *be a cofinal subset of* κ *and let* $f: S \to \kappa$.

(a) *If* f *is continuous then* f *is measurable.*

(b) *If* f *is continuous and has* $|f[S]| = \kappa$ *then* f *is strongly measurable.*

Proof. To prove the first assertion, note that if $C \in \mathrm{cub}(\kappa)$ then $f^{-1}[C]$ is closed in S so that $f^{-1}[C] \in \mathscr{M}(S)$.

The second assertion is trivial in case $S \notin \mathscr{S}(\kappa)$, so assume $S \in \mathscr{S}(\kappa)$ and yet for some $y \in \kappa$ the set $f^{-1}[\{y\}]$ is stationary. Let $D = f^{-1}[[y+1,\kappa)]$. Then $\bar{D}$ belongs to $\mathrm{cub}(\kappa)$ and yet is disjoint from the stationary set $f^{-1}[\{y\}]$. But that is impossible because S is stationary.

Our next major result gives a characterization of strongly measurable functions which is crucial for Section 5. We need a preliminary lemma.

4.3 LEMMA. *Let $\mathscr{T}$ be a pairwise disjoint collection of non-stationary subsets of κ and suppose $S \subset \bigcup \mathscr{T}$ where $S \in \mathscr{S}(\kappa)$. Then for some $C \in \mathrm{cub}(S)$, $|C \cap T| \leq 1$ for each $T \in \mathscr{T}$.*

Proof. For each $x \in S$ let $T(x)$ be the unique member of $\mathscr{T}$ which contains x and let $C(x) \in \mathrm{cub}(\kappa)$ have $C(x) \cap T(x) = \emptyset$. According to (2.4) the set

$$C = \{x \in S: x \in C(y) \text{ for each } y \in S \cap [0,x)\}$$

belongs to $\mathrm{cub}(S)$. Suppose $T \in \mathscr{T}$ and yet $C \cap T$ contains two points $x_1 < x_2$. Then $x_2 \in \bigcap \{C(y): y \in S \text{ and } y < x_2\} \subset C(x_1)$ so that $x_2 \in C(x_1) \cap T$. But that is impossible because $x_1 \in T$ forces $T = T(x_1)$ so that $C(x_1) \cap T = \emptyset$.

4.4 THEOREM. *Suppose $S \in \mathscr{S}(\kappa)$ and suppose $f: S \to \kappa$ has the property that the set $T = f[S]$ is cofinal in κ. Then the following are equivalent:*

(a) *if* $A \in \mathscr{M}_+(T)$ *then* $f^{-1}[A] \in \mathscr{M}_+(S)$;

(b) f *is strongly measurable;*

(c) *if* $A \in \mathscr{S}(S)$ *then* $f[A] \in \mathscr{S}(\kappa)$;

(d) *there is a set* $F \in \mathrm{cub}(S)$ *such that* $f(x) = x$ *for every* $x \in F$.

Proof. We show that (a) ⇒ (b) ⇒ (c) ⇒ (d) ⇒ (a).

(a) ⇒ (b): For each $y \in T$, $T - \{y\}$ contains some $C \in \mathrm{cub}(T)$ so that

$$f^{-1}[T - \{y\}] \in \mathscr{M}_+(S).$$

Since $S \in \mathscr{S}(\kappa)$, it follows that $f^{-1}[\{y\}]$ is non-stationary. Obviously (a) implies that f is measurable.

(b) ⇒ (c): By (4.3) there is a $C \in \mathrm{cub}(S)$ having $|C \cap f^{-1}[\{y\}]| \leq 1$ for each $y \in T$. Let $A \in \mathscr{S}(S)$ and suppose that $f[A] \notin \mathscr{S}(\kappa)$. Then $T_1 = f[A \cap C] \notin \mathscr{S}(\kappa)$ and yet $|T_1| = \kappa$ so that from (2.5d) $\mathscr{M}(T_1) = \mathscr{P}(T_1)$. Let $B \in \mathscr{P}(A \cap C)$. Then $f[B] \in \mathscr{P}(T_1)$ so that by (2.6a) $f[B] = T_1 \cap D$ for some $D \in \mathscr{M}(T)$. But then $f^{-1}[D] \in \mathscr{M}(S)$ and $B = (A \cap C) \cap f^{-1}[D]$ showing that $B \in \mathscr{M}(A \cap C)$, again in the light of (2.6a). Therefore $\mathscr{M}(A \cap C) = \mathscr{P}(A \cap C)$, and that is impossible because from $A \in \mathscr{S}(S)$ and $C \in \mathrm{cub}(S)$ it follows that $A \cap C \in \mathscr{S}(\kappa)$. (See (2.5d).)

(c) ⇒ (d): According to (2.5f) the set $S_1 = \{x \in S: f(x) < x\}$ is not stationary; let $C_1 \in \mathrm{cub}(\kappa)$ have $C_1 \cap S_1 = \emptyset$. Let $S_2 = \{x \in S: f(x) > x\}$ and for each $y \in f[S_2]$ let $m(y) = \min(S_2 \cap f^{-1}[\{y\}])$. Then $m(y) < y$ for each $y \in f[S_2]$ and m is one-to-one. It follows from (2.5f) that $f[S_2]$ is not stationary. By (c), neither is

S_2; let $C_2 \in \text{cub}(\kappa)$ have $S_2 \cap C_2 = \emptyset$. Then $C_1 \cap C_2 \in \text{cub}(\kappa)$ so that, S being stationary, the set $F = S \cap (C_1 \cap C_2)$ is the required member of cub(S).

(d) $\Rightarrow$ *(a)*: Let $A \in \mathscr{M}_+(T)$. Then for some $C \in \text{cub}(T)$, $A \supset C$. Then $\bar{C} \in \text{cub}(\kappa)$ so that $F \cap \bar{C} \in \text{cub}(S)$ because $F \in \text{cub}(S)$ and $S \in \mathscr{S}(\kappa)$. But since $f(x) = x$ for each $x \in F$, $F \cap \bar{C} \subset F \cap A \subset f^{-1}[A]$ so that $f^{-1}[A] \in \mathscr{M}_+(S)$, as required.

Our first corollary significantly sharpens (2.7) because of (4.2).

4.5 COROLLARY. *Suppose that* $S \in \mathscr{S}(\kappa)$ *and that* $f\colon S \to \kappa$ *is strongly measurable. Then* $f[S] \in \mathscr{S}(\kappa)$.

Proof. Write $T = f[S]$. If $|T| < \kappa$ then S is the union of fewer than κ non-stationary sets, namely the sets $f^{-1}[\{y\}]$ for $y \in T$, and that is impossible by (2.3d). Hence T is cofinal in κ so that (4.4c) applies.

Our second corollary summarizes the behavior of measurable and strongly measurable functions with respect to function composition and restriction.

4.6 COROLLARY. *Let* R, S *and* T *be cofinal sets in* κ.

(a) *If* $f\colon R \to S$ *and* $g\colon S \to T$ *are both measurable (respectively, strongly measurable), then so is* $g \circ f$.

(b) *If* $h\colon S \to T$ *is measurable (respectively, strongly measurable) and if* $R \subset S$, *then* $h|R$ *is also measurable (respectively, strongly measurable).*

Proof. Assertion (a) for measurable functions is trivial. If R is non-stationary, then assertion (a) for strongly measurable functions is obvious. If $R \in \mathscr{S}(\kappa)$, use (4.4d).

Assertion (b) for measurable functions is an immediate consequence of (2.6), and assertion (b) for strongly measurable functions is then obvious.

5. MAPPING CLASSIFICATIONS OF STATIONARY SETS

Throughout this section κ denotes a fixed uncountable regular cardinal.

5.1 THEOREM. *Suppose* S *and* T *are cofinal subsets of* κ. *Then the following statements are equivalent:*

(a) $S - T \notin \mathscr{S}(\kappa)$;

(b) *there is a continuous* $f\colon S \to T$, *not necessarily surjective, such that* $|f[S]| = \kappa$;

(c) *there is a one-to-one measurable map from* S *onto* T;

(d) *there is a strongly measurable mapping from* S *into* T.

Proof. There are two cases, depending upon whether S is stationary in κ.

Suppose $S \notin \mathscr{S}(\kappa)$. We shall show that assertions (a), (b), (c) and (d) are always true in this case. If $S \notin \mathscr{S}(\kappa)$ then surely $S - T \notin \mathscr{S}(\kappa)$. The proof that (a) $\Rightarrow$ (b) in Theorem 2.5 shows that S can be mapped continuously onto a discrete space D having cardinality κ; since such a D is homeomorphic to the set of isolated points of T, we have (b). To obtain (c) and (d), take any bijection $f\colon S \to T$. Since $S \notin \mathscr{S}(\kappa)$, (2.5d) guarantees that f is strongly measurable.

The interesting case in the proof of (5.1) occurs when $S \in \mathscr{S}(\kappa)$. We prove (a) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (a) and then (a) $\Rightarrow$ (b) $\Rightarrow$ (d).

(a) $\Rightarrow$ *(c)*: Since $S - T \notin \mathscr{S}(\kappa)$, some $C \in \mathrm{cub}(\kappa)$ is disjoint from $S - T$. Then $S \cap C \in \mathrm{cub}(S)$ and $S \cap C \subset S \cap T$. Let B be the set of nonisolated points of the subspace $S \cap C$. Then $|(S \cap C) - B| = \kappa = |T - B|$. Let f be any bijection from S onto T having $f(x) = x$ for each $x \in B$. Then f is (strongly) measurable by (4.4d) and (2.3b).

(c) $\Rightarrow$ *(d)*: This follows from the definition of strong measurability.

(d) $\Rightarrow$ *(a)*: In the light of (4.4d), some $F \in \mathrm{cub}(S)$ has $f(x) = x$ for each $x \in F$. Then $\bar{F}$ is a member of $\mathrm{cub}(\kappa)$ which is disjoint from $S - T$, so $S - T \notin \mathscr{S}(\kappa)$.

(a) $\Rightarrow$ *(b)*: Let $C \in \mathrm{cub}(\kappa)$ be disjoint from $S - T$. Then $S \cap C \in \mathrm{cub}(S)$ so that by [2, III, Thm. 8] there is a continuous $f\colon S \to S \cap C$ having $f(x) = x$ for each $x \in S \cap C$. (Indeed, since $S \cap C$ is cofinal, one can define f by

$$f(x) = \min(\{y \in S \cap C\colon x \le y\}).)$$

Since $S \cap C \subset T$, this f is the required continuous function.

(b) $\Rightarrow$ *(d)*: Apply (4.2b).

5.2 *Remarks.* (a) Theorems A and C of the Introduction follow immediately from (5.1).

(b) An easy consequence of (5.1a) is that the property of being a bistationary set is a topological property, independent of the way in which the bistationary set is embedded in κ. (Compare Remark 2.6.)

(c) It also follows from (5.1) that a one-to-one strongly measurable mapping f from a stationary set S onto a stationary set T may *fail* to be a measurable isomorphism in the sense of (4.1): take S to be a bistationary subset of $\omega_1 = T$ and apply (5.1).

We now turn to the proof of an expanded version of Theorem B of the Introduction.

5.3 THEOREM. *There is a collection $\mathscr{B}$ of bistationary subsets of κ such that:*

(a) $|\mathscr{B}| = 2^\kappa$;

(b) *if* $S \neq T$ *are in* $\mathscr{B}$ *then no continuous* $f\colon S \to T$ *has* $|f[S]| = \kappa$;

(c) *if* $S \neq T$ *are in* $\mathscr{B}$ *then there is no one-to-one measurable function from* S *into* T;

(d) *if* $S \neq T$ *are in* $\mathscr{B}$ *then there is no strongly measurable function from* S *into* T.

Proof. In the light of (4.2) and (5.1) it will be enough to find $\mathscr{B}$ satisfying (a) and (d). Using Theorem E of the Introduction, let $\mathscr{D}$ be a family of pairwise disjoint stationary subsets of κ having $|\mathscr{D}| = \kappa$. Write

$$\mathscr{D} = \{S_\alpha : \alpha < \kappa\} \cup \{T_\alpha : \alpha < \kappa\}$$

where $S_\alpha \neq S_\beta$, $T_\alpha \neq T_\beta$ for distinct $\alpha, \beta < \kappa$, and where $S_\alpha \neq T_\beta$ for any $\alpha, \beta < \kappa$.

For each $A \subset \kappa$ define $U(A) = \left(\bigcup \{S_\alpha : \alpha \in A\}\right) \cup \left(\bigcup \{T_\alpha : \alpha \in \kappa - A\}\right)$. Each $U(A)$ contains a member of $\mathscr{D}$ and is disjoint from another member of $\mathscr{D}$; hence each $U(A)$ is bistationary. Further, if A and B are distinct subsets of κ, then $U(A) - U(B)$ contains a member of $\mathscr{D}$ so that $U(A) - U(B) \in \mathscr{S}(\kappa)$. It follows from (5.1) that there is no strongly measurable mapping from $U(A)$ into $U(B)$.

We conclude this section by establishing the relevant generalization of the Introduction's Theorem D.

5.4 THEOREM. *Let* S *and* T *be cofinal subsets of* κ. *The following are equivalent:*

(a) $S \Delta T \notin \mathscr{S}(\kappa)$;

(b) *there is a measurable isomorphism* g *from* S *onto* T;

(c) *there are strongly measurable functions* $g: S \to T$ *and* $h: T \to S$.

Proof. If neither S nor T is stationary in κ, then (a), (b) and (c) are true, where g is any bijection from S onto T and $h = g^{-1}$ (see 2.5d). We consider only the case where S is stationary in κ, the other case being analogous. Suppose (a) holds. Let $C \in \text{cub}(\kappa)$ have $C \cap (S \Delta T) = \emptyset$. Then $C \cap S = C \cap (S \cap T) = C \cap T$ so that since $C \cap S \in \text{cub}(S) \subset \mathscr{S}(\kappa)$, $T \in \mathscr{S}(\kappa)$ and hence $C \cap T \in \text{cub}(T)$. Let D be the set of non-isolated points in the space $C \cap (S \cap T)$. Then $D \in \text{cub}(S) \cap \text{cub}(T)$ and $|S - D| = \kappa = |T - D|$. Hence there is a bijection $g: S \to T$ having $g(x) = x$ for each $x \in D$; then g is the function required by (b).

Clearly (b) implies (c) and the implication (c) $\Rightarrow$ (a) follows immediately from (5.1a) and (2.3d).

6. STATIONARY SETS AND METRIZABLE SPACES

Stationary sets can also be characterized in terms of the ways in which they can be mapped into metrizable spaces. While the results of this section do not contribute to the classification of stationary sets, they do provide new ways for recognizing stationary sets and show that the notion of a measurable mapping has wider applications. The technique used to prove (a) $\Rightarrow$ (b) in Theorem (6.2) below is the same as the technique used to prove (b) $\Rightarrow$ (c) in Theorem 4.4. We begin by giving a slight generalization of the notion of a measurable mapping.

6.1 *Definition.* Let X be any topological space and let S be a cofinal subset of κ. A function $f: S \to X$ is *measurable* if $f^{-1}[U] \in \mathscr{M}(S)$ for each open set U in X.

It is clear that $f: S \to X$ is measurable if and only if $f^{-1}[C] \in \mathscr{M}(S)$ for each closed set C in X. Therefore, this wider definition of measurable mappings coincides with our earlier definition in case $X = T$ is a cofinal subset of κ.

Recall that a space X is *subparacompact* provided each open cover of X has a σ-discrete closed refinement [B].

6.2 THEOREM. *If* S *is a cofinal subset of* κ, *the following are equivalent:*

(a) S *is stationary;*

(b) *if* X *is any subparacompact space in which each point is the intersection of fewer that* κ *open sets and if* $f: S \to X$ *is a measurable surjection, then* $f^{-1}[\{x\}] \in \mathscr{M}_+(S)$ *for some* $x \in X$;

(c) *if* $f: S \to X$ *is a continuous mapping from* S *into a metrizable space* X, *then* $f^{-1}[\{x\}] \in \mathscr{M}_+(S)$ *for some* $x \in X$;

(d) *if* $f: S \to X$ *is continuous and if* X *is metrizable then* $|f[S]| < \kappa$;

(e) *if* $f: S \to X$ *is continuous and if* X *is metrizable, then* $|f^{-1}[\{x\}]| = \kappa$ *for some* $x \in X$.

Proof. Since (b) $\Rightarrow$ (c) and (d) $\Rightarrow$ (e) are trivial, it suffices to prove (a) $\Rightarrow$ (b), (c) $\Rightarrow$ (d), and (e) $\Rightarrow$ (a).

(a) $\Rightarrow$ *b)*: Suppose X and f are as in (b) and that $f^{-1}[\{x\}] \notin \mathscr{M}_+(S)$ for each $x \in X$. By (2.3c), $\bigcap \mathscr{A} \in \mathscr{M}_+(S)$ whenever $\mathscr{A} \subset \mathscr{M}_+(S)$ has $|\mathscr{A}| < \kappa$; hence each $x \in X$ has a neighborhood U_x such that $f^{-1}[U_x] \notin \mathscr{M}_+(S)$. Since X is subparacompact there is a refinement $\mathscr{D} = \cup\{\mathscr{D}(n): n \in \omega\}$ of $\{U_x: x \in X\}$ where each $\mathscr{D}(n)$ is a discrete collection of closed subsets of X. Then $f^{-1}[D] \notin \mathscr{M}_+(S)$ and yet $f^{-1}[D] \in \mathscr{M}(S)$ for each $D \in \mathscr{D}$; hence $f^{-1}[D] \notin \mathscr{S}(S)$ for each $D \in \mathscr{D}$.

By (2.3d) there is a natural number n such that the set $T = f^{-1}[\bigcup \mathscr{D}(n)]$ is stationary. Then $\{f^{-1}[D]: D \in \mathscr{D}(n)\}$ is a disjoint cover of T by non-stationary sets so that by (4.3) there is a $C \in \text{cub}(T)$ such that $|C \cap f^{-1}[D]| \le 1$ for each $D \in \mathscr{D}(n)$. Then (*) $f|C$ is a one-to-one map and $f[C]$ is closed and discrete. By Theorem E of the Introduction, since $C \in \mathscr{S}(\kappa)$, there are disjoint sets A_0 and A_1 in $\mathscr{S}(C)$. It follows from (*) that $f[A_0]$ is closed in X so that $f^{-1}[f[A_0]]$ is measurable. But that is impossible because $A_0 \subset f^{-1}[f[A_0]]$ and $A_1 \cap f^{-1}[f[A_0]] = \emptyset$, again by (*).

(c) $\Rightarrow$ *(d)*: Suppose $f: S \to X$ is continuous where X is metrizable. According to c), some set $f^{-1}[\{x\}]$ belongs to $\mathscr{M}_+(S)$. Since $X - \{x\}$ is an F_σ-subset of X, the set $T = f^{-1}[X - \{x\}]$ is an F_σ-subset of S. But then, since $T \cap f^{-1}[\{x\}] = \emptyset$, there is some $s \in S$ having $T \subset [0, s)$. Therefore $|f[S]| < \kappa$.

(e) $\Rightarrow$ *(a)*: Suppose S is not stationary. Then $S \subset \kappa - C$ for some $C \in \text{cub}(\kappa)$. Write $\kappa - C = \bigcup \mathscr{I}$ where $\mathscr{I}$ is a disjoint collection of open, convex subsets of κ. Since no member of $\mathscr{I}$ is cofinal in κ, $|\mathscr{I}| = \kappa$. Index $\mathscr{I}$ without repetitions as $\mathscr{I} = \{I_\gamma: \gamma < \kappa\}$ and let $D = \{d_\gamma: \gamma < \kappa\}$ be a discrete topological space having $|D| = \kappa$. Define $F: \cup \mathscr{I} \to D$ by the rule that $F(x) = d_\gamma$ if $x \in I_\gamma$. Then F is continuous. Let $f = F|S$. Then $f: S \to D$ is continuous and yet for each $d \in D$, $|f^{-1}[\{d\}] < \kappa$, contrary to (e).

6.3 *Remarks.* (a) In Theorem 6.2, the only metric spaces which need to be considered are discrete spaces.

(b) If $f: S \to X$ is continuous, where S is stationary in κ and where X is metrizable, then $|S - f^{-1}[\{x\}]| < \kappa$ for some $x \in X$.

7. EXAMPLES

Our first example and the subsequent proposition elucidate the relationship between the σ-algebras $\mathscr{M}(S)$ and $\mathscr{B\ell}(S)$ (see Section 2).

7.1 EXAMPLE. *If κ is any regular cardinal with $\kappa > \omega_1$, then $\mathscr{M}(\kappa) \supsetneqq \mathscr{B\ell}(\kappa)$.*

Proof. Otherwise, since $\omega_1 < \kappa$, $\mathscr{P}([0,\omega_1)) \subset \mathscr{M}(\kappa) = \mathscr{B\ell}(\kappa)$ so that

$$\mathscr{P}([0,\omega_1)) = \mathscr{B\ell}([0,\omega_1))$$

which is false.

When one considers stationary subsets of κ instead of all of κ, a more interesting result is available.

7.1a PROPOSITION. *Let S be a stationary subset of κ. Then $\mathscr{M}(S) = \mathscr{B\ell}(S)$ if and only if, for every ordinal $\lambda < \kappa$ having $\mathrm{cf}(\lambda) \geq \omega_1$, there is some closed cofinal subset of $[0,\lambda)$ which is disjoint from $S \cap [0,\lambda)$.*

Before proving (7.1a) we pause to comment on the existence of the kind of stationary set described in the hypothesis of the proposition. Suppose $\kappa = \mu^+$ for some regular cardinal μ and consider the set $S = \{\alpha < \kappa: \mathrm{cf}(\alpha) = \mu\}$. Then S is stationary in κ. Suppose $\lambda < \kappa$ is a limit ordinal. Then there is a closed cofinal C in $[0,\lambda)$ such that $|C| = \mathrm{cf}(\lambda)$ and $\{\alpha \in C: \alpha \text{ is isolated in } \kappa\}$ is dense in C. Then $\mathrm{cf}(x) < \mathrm{cf}(\lambda) \leq \mu$ for each $x \in C$ so that $S \cap C = \emptyset$. Consequently S satisfies the hypotheses in (7.1a).

Our proof of (7.1a) requires a lemma.

7.1b LEMMA. *Let $\{I_\gamma: \gamma \in \Gamma\}$ be a pairwise disjoint collection of open convex subsets of κ and for each $\gamma \in \Gamma$ let $B_\gamma \subset I_\gamma$ be a Borel subset of κ. Then $B = \bigcup \{B_\gamma : \gamma \in \Gamma\}$ is a Borel subset of κ.*

Proof. According to a result of Mauldin [6], each B_γ can be written as

$$B_\gamma = \bigcup \{O(n,\gamma) \cap K(n,\gamma): n \geq 1\}$$

where each $O(n,\gamma)$ is an open subset of I_γ and each $K(n,\gamma)$ is a closed subset of κ. Let $O(n) = \bigcup \{O(n,\gamma): \gamma \in \Gamma\}$ and let $K(n) = \mathrm{cl}\left(\bigcup \{K(n,\gamma): \gamma \in \Gamma\}\right)$.

Then $O(n) \cap K(n) = \bigcup \{O(n,\gamma) \cap K(n,\gamma): \gamma \in \Gamma\}$ so that

$$\bigcup \{O(n) \cap K(n) : n \geq 1\} = \bigcup \left\{ \bigcup \{O(n,\gamma) \cap K(n,\gamma) : n \geq 1\} : \gamma \in \Gamma \right\}$$

$$= \bigcup \{B_\gamma : \gamma \in \Gamma\}$$

as required to show that B is a Borel subset of κ.

Proof of (7.1a). Suppose $\mathscr{BL}(S) = \mathscr{M}(S)$, and let $\lambda < \kappa$ have $\mathrm{cf}(\lambda) \geq \omega_1$. Then $\mathscr{P}(S \cap [0,\lambda)) = \mathscr{BL}(S \cap [0,\lambda)) \subset \mathscr{M}(S \cap [0,\lambda))$. But then a trivial modification of the proof that (a) $\Rightarrow$ (b) in (2.5) shows that some closed cofinal subset of $[0,\lambda)$ is disjoint from S. Conversely, suppose that whenever $\lambda < \kappa$ has $\mathrm{cf}(\lambda) \geq \omega_1$, some closed cofinal subset of $[0,\lambda)$ is disjoint from S. We assert that

$$\mathscr{P}(S \cap [0,\lambda)) \subset \mathscr{BL}(S)$$

for each $\lambda < \kappa$ having $\mathrm{cf}(\lambda) \geq \omega_1$. For if not, let λ be the first ordinal satisfying:

(a) $\lambda < \kappa$;

(b) some $T \subset S \cap [0,\lambda)$ is not in $\mathscr{BL}(S)$.

(Notice that $\mathrm{cf}(\lambda) \geq \omega_1$; this follows from minimality of λ plus property (b).) Let C be a closed cofinal subset of $[0,\lambda)$ which is disjoint from S. Write

$$[0,\lambda) - C = \bigcup \{I_\gamma : \gamma \in \Gamma\}$$

as the union of its convex components. Then each I_γ has the form $I_\gamma = (a_\gamma, b_\gamma)$ where $b_\gamma < \lambda$. Then each set $T_\gamma = T \cap I_\gamma$ belongs to $\mathscr{BL}(S)$ so that, by (7.1b), $T \in \mathscr{BL}(S)$, contrary to the choice of T.

Now, to show that $\mathscr{M}(S) = \mathscr{BL}(S)$, let $R \in \mathscr{M}(S)$. It is enough to consider the case where R is cofinal in κ and is disjoint from some $C \in \mathrm{cub}(\kappa)$. Write $\kappa - C = \bigcup \{J_\delta : \delta \in \Delta\}$, the J_δ's being the convex components of $\kappa - C$. Then each J_δ has the form $J_\delta = (c_\delta, d_\delta)$ where $d_\delta < \kappa$. But then the set $R_\delta = R \cap J_\delta$ belongs to $\mathscr{P}(S \cap [0,d_\delta)) \subset \mathscr{BL}(S)$ so that, by (7.1b), $R = \bigcup \{R_\delta : \delta \in \Delta\} \in \mathscr{BL}(S)$. Thus $\mathscr{M}(S) \subset \mathscr{BL}(S)$. Since $\mathscr{BL}(S) \subset \mathscr{M}(S)$ is always true, the proof is complete.

Our second example shows that the class of measurable mappings cannot profitably be used to study stationary sets. Instead one must consider measurable mappings with nonstationary fibers; *i.e.*, the class of strongly measurable mappings (see Section 4).

7.2 EXAMPLE. *Let* S *be a stationary subset of the regular cardinal* κ *and let* T *be any cofinal subset of* κ. *Then there is a measurable function* f *from* S *onto* T *such that exactly one fiber of* f *has more than one point.*

Proof. Let C be the set of non-isolated points of S. Then $|S - C| = \kappa = |T|$. Let t_0 be the first element of T and let g be any bijection from $S - C$ onto $T - \{t_0\}$. Define $f: S \rightarrow T$ by the rule

$$f(s) = \begin{cases} g(s) & \text{if } s \in S - C \\ t_0 & \text{if } s \in C. \end{cases}$$

Our next example shows that Proposition (3.2) is valid only for successor cardinals.

7.3 EXAMPLE. *Let κ be an uncountable regular limit cardinal. Then there is a non-stationary set* B *which is not in $\mathscr{B}(\kappa,\kappa)$ (see* (3.1)) *even though* $B \in \mathscr{M}(\kappa)$.

Proof. Let C be the set of uncountable regular cardinals λ with $\lambda < \kappa$. One easily constructs a collection $\{A_\alpha : \alpha \in C\}$ of pairwise disjoint open subsets of κ such that

(a) A_α is homeomorphic to $[0,\alpha)$ for each $\alpha \in C$

(b) $\kappa - \bigcup \{A_\alpha : \alpha \in C\}$ is cofinal in κ and hence belongs to $\mathrm{cub}(\kappa)$.

For each $\alpha \in C$ choose a subset $S_\alpha \subset A_\alpha$ which is a homeomorph of a bistationary subset of $[0,\alpha)$, and let $B = \bigcup \{S_\alpha : \alpha \in C\}$. Since $B \cap A_\alpha = S_\alpha \notin \mathscr{B}(A_\alpha, \alpha)$ for each $\alpha \in C$, $B \notin \mathscr{B}(\kappa,\alpha)$ for each $\alpha \in C$. Since C is cofinal in κ, $B \notin \mathscr{B}(\kappa,\kappa)$. Since B misses $\kappa - \bigcup \{A_\alpha : \alpha \in C\} \in \mathrm{cub}(\kappa)$, B is the required set.

We now present an easy example showing that the condition "$S - T \notin \mathscr{S}(\kappa)$" is a necessary but *not* sufficient condition for there to exist a continuous mapping from S onto T. (See Theorem 5.1.)

7.4 EXAMPLE. *There exist stationary subsets* S *and* T *of* ω_1 *such that* $S \Delta T \notin \mathscr{S}(\omega_1)$ *and yet there is no continuous mapping from* S *onto* T.

Proof. Let $S = \omega_1$ and let $T = \omega_1 - \{\omega_0\}$.

Even though Theorem (5.3) yields a rich supply of bistationary subsets of ω_1 which are topologically distinct, it is difficult to describe a simple topological property which one, but not every, bistationary set in ω_1 possesses. (We remark that the existence of a countably compact cofinal subspace distinguishes any superset of a member of $\mathrm{cub}(\omega_1)$ from any bistationary set in ω_1.) The following example is a simplification of an example communicated to the authors by W. Fleissner. It yields a stronger result than does (7.4): the sets in (7.4) are not bistationary.

7.5 EXAMPLE (Fleissner). *For any stationary set* S *in* ω_1 *there are stationary sets* S_1 *and* S_2 *such that* $S \Delta S_i \notin \mathscr{S}(\omega_1)$ *for* $i = 1,2$ *and yet* S_1 *and* S_2 *are not homeomorphic.*

Proof. For any space X let I(X) be the set of isolated points of X and let $X^d = X - I(X)$. Define $X^{(n)}$ recursively by $X^{(0)} = X$ and $X^{(n+1)} = [X^{(n)}]^d$. The sets

S_1 and S_2 which we will construct are not homeomorphic since all points of $I(S_1^{(2)})$ have compact neighborhoods in S_1, while not all points of $I(S_2^{(2)})$ have compact neighborhoods in S_2.

Let $A = I(\omega_1) \cup I(\omega_1^{(1)}) \cup I(\omega_1^{(2)})$. Clearly $A \notin \mathscr{S}(\omega_1)$ and it is easy to see that the set $S_1 = S \cup A$ has the required property.

The first point in $I(\omega_1^{(2)})$ is ω_0^2 and ω_0^2 is the limit of the sequence $\langle \omega_0 \cdot k : k \geq 1 \rangle$. It is easily seen that the set $S_2 = S_1 - \{\omega_0 \cdot (2k) : k \geq 1\}$ has the required properties. (We leave it to the reader to construct S_2 in such a way that no point of $I(S_2^{(2)})$ has a compact neighborhood in S_2.) A trivial modification of the above construction yields disjoint stationary sets T_1 and T_2 having analogous properties.

8. QUESTIONS AND DIRECTIONS FOR FURTHER RESEARCH

8.1 Give necessary and sufficient conditions on two stationary subsets S and T of κ (or even of ω_1) so that T is a continuous image of S (a homeomorph of S).

8.2 Let D be κ with the discrete topology. For each $A \in \mathscr{S}(\kappa)$ let

$$\tilde{A} = \bigcap \{\mathrm{Cl}_{\beta D}(A \cap C) : C \in \mathrm{cub}(\kappa)\}.$$

Then $\tilde{A}$ is a nonvoid compact Hausdorff space contained in $\beta D - D$. Using standard techniques, one can show that for $A, B \in \mathscr{S}(\kappa)$, the spaces $\tilde{A}$ and $\tilde{B}$ are homeomorphic if and only if there is an almost isomorphism $h: \mathscr{S}(A) \to \mathscr{S}(B)$, where we make the following definitions:

(a) $S \subset^{\cdot} T$ if and only if $S - T \notin \mathscr{S}(\kappa)$ (where $S, T \in \mathscr{S}(\kappa)$);

(b) $S =^{\cdot} T$ if and only if $S \,\Delta\, T \notin \mathscr{S}(\kappa)$ (again where $S, T \in \mathscr{S}(\kappa)$);

(c) $f: S(A) \to S(B)$ is an *almost isomorphism* if

(i) for $S_1, S_2 \in \mathscr{S}(A)$, $f(S_1) \subset^{\cdot} f(S_2)$ if and only if $S_1 \subset^{\cdot} S_2$

(ii) for each $T \in \mathscr{S}(B)$ some $S \in \mathscr{S}(A)$ has $f(S) =^{\cdot} T$.

If $A, B \in \mathscr{S}(\kappa)$ and $A =^{\cdot} B$ then it follows from (5.4) that there is a bijection $f: A \to B$ having the property that $S \in \mathscr{S}(A)$ if and only if $f[S] \in \mathscr{S}(B)$. Defining $h: \mathscr{S}(A) \to \mathscr{S}(B)$ by $h(S) = f[S]$, one obtains an almost isomorphism and therefore a homeomorphism from $\tilde{A}$ onto $\tilde{B}$. The question posed here asks about the converse of that last assertion; *i.e.*, given that $\tilde{A}$ and $\tilde{B}$ are homeomorphic does it follow that $A =^{\cdot} B$?

The referee pointed out a reformulation of this question in terms of Boolean algebra. If we let $\mathscr{B}(A) = \mathscr{P}(A)/\mathscr{M}^+(A)$, then $\tilde{A}$ is just the Stone space of $\mathscr{B}(A)$ and $\tilde{A}$ is homeomorphic to $\tilde{B}$ if and only if $\mathscr{B}(B)$ is isomorphic to $\mathscr{B}(A)$. Thus the question posed here reduces to "if $\mathscr{B}(B)$ and $\mathscr{B}(A)$ are isomorphic, does it follow that $A =^{\cdot} B$?"

8.3 In this question, we are asking for topological properties which rather strongly distinguish different types of bistationary subsets of ω_1. To explain what we have in mind, we first consider ω_2.

Consider the following properties of topological spaces.

$\mathscr{C}$: every non-isolated point has a countable neighborhood base;

$\mathscr{N}$: no non-isolated point has a countable neighborhood base.

Define $B \subset \omega_2$ by $B = \{x < \omega_2 : \mathrm{cf}(x) < \omega_1\}$. Then B is bistationary, B has property $\mathscr{C}$ and $\omega_2 - B$ has property $\mathscr{N}$. Conversely, if S,T $\subset \omega_2$ are such that S has property $\mathscr{C}$ and T has property $\mathscr{N}$, then $S \cap T$ cannot be stationary since $S \cap T$ is discrete, even if both S and T are stationary.

This suggests the problem of finding "nice" topological properties $\mathscr{A}$ and $\mathscr{B}$ such that

(1) there is a bistationary set $A \subset \omega_1$ such that A has $\mathscr{A}$ and $\omega_1 - A$ has $\mathscr{B}$;

(2) given any two stationary sets S and T in ω_1, if S has $\mathscr{A}$ and T has $\mathscr{B}$, then $S \cap T$ is not stationary.

The relevance of this question is this: the set $B \subset \omega_2$ defined above is a "constructive" example of a bistationary set in ω_2. If one could find $A \subset \omega_1$, $\mathscr{A}$ and $\mathscr{B}$ satisfying (1) and (2), that would give a bistationary set in ω_1 which is "more constructive" than the ones obtained using either Theorem E of the Introduction or the more *ad hoc* constructions such as that described in [10].

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