q-PLURISUBHARMONIC FUNCTIONS AND A GENERALIZED DIRICHLET PROBLEM

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1. INTRODUCTION

Let Ω be a bounded domain in $\mathbb{C}^n$ and let q be an integer, $0 \leq q \leq n-1$. A C^2 function u defined in Ω is a q-plurisubharmonic function in Ω if its complex Hessian has $(n-q)$ nonnegative eigenvalues at each point of Ω. An obvious question is whether there is a definition for q-plurisubharmonic functions which are not necessarily C^2. Recall that an upper semicontinuous function defined in Ω is plurisubharmonic there if it is essentially subharmonic in every complex direction (see [6]). Thus the definition of plurisubharmonic function is reduced to a 1-complex dimensional definition, and the same is true for plurisuperharmonic functions. We give definitions of q-plurisubharmonic (and q-plurisuperharmonic) functions in Ω, with 0-plurisubharmonic and plurisubharmonic being equivalent. These definitions seem to be very natural for $\mathbb{C}^n$, are invariant under biholomorphic coordinate changes on $\mathbb{C}^n$, and are equivalent to those mentioned above if a function is actually C^2.

Let D be a bounded strictly pseudoconvex domain in $\mathbb{C}^n$, $n > 1$, with C^2 boundary, and suppose that b is a continuous real valued function defined on ∂D. We can solve the Dirichlet problem to find a harmonic function in D which assumes the given boundary values. The problem with this solution is that it is not invariant under biholomorphic coordinate changes on $\mathbb{C}^n$. In order to remedy this, Bremermann [3] considered the class of all continuous plurisubharmonic functions in D which are less than or equal to b on ∂D and applied Perron's method showing that the upper envelope $\bar{u}$ of this class exists and takes on the given boundary values. His solution $\bar{u}$ is plurisubharmonic in D, invariant under biholomorphic coordinate changes on $\mathbb{C}^n$, and if C^2, satisfies the homogeneous complex Monge-Ampere equation

$$[\partial\bar{\partial}u]^n = \underbrace{\partial\bar{\partial}u \wedge \ldots \wedge \partial\bar{\partial}u}_{\text{n times}} = 0$$

in D. Later, Walsh [8] showed that $\bar{u}$ is continuous, and Bedford and Taylor [2] proved that $\bar{u}$ satisfied their distributional definition of the homogeneous complex Monge-Ampere equation.

Let D be a strictly q-pseudoconvex domain in $\mathbb{C}^n$ with C^2 boundary, and let b be a continuous function on ∂D. We prove that the upper envelope of all upper semicontinuous functions on $\bar{D}$ which are q-plurisubharmonic in D and less than

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or equal to b on ∂D exists and takes on the boundary values b(z) on ∂D if we assume that $2q < n$ (for $2q = n$ we give a counterexample). We show that this upper envelope is continuous on $\bar{D}$, is q-plurisubharmonic in D, and is $(n - q - 1)$-plurisuperharmonic in D (if the envelope is C^2, these last two conditions imply that it satisfies the homogeneous complex Monge-Ampere equation).

Given a strictly pseudoconvex domain $D \subset \mathbb{C}^n$ with C^2 boundary, we prove that any continuous plurisubharmonic solution taking on given boundary values b(z) on ∂D, and which is also $(n - 1)$-plurisuperharmonic must actually be Bremermann's solution. Hence this is also the solution to the distributional complex Monge-Ampere equation defined by Bedford and Taylor.

We also examine a problem similar to one discussed by Chern, Levine, and Nirenberg in [4]. Consider an "annulus" with C^2 boundary where the outer boundary is pseudoconvex (not necessarily strictly pseudoconvex) and the inner boundary is weakly restricted. Assign the boundary values 1 on the outer boundary and 0 on the inner boundary. Using the theory developed in this paper we show that there exists a function which is plurisubharmonic and $(n - 1)$-plurisuperharmonic in the annulus, and which takes on the given boundary values. This function is the upper envelope of the plurisubharmonic functions which are less than or equal to 1 on the outer boundary and less than or equal to 0 on the inner boundary. Moreover, this function satisfies the distributional complex Monge-Ampere equation of Bedford and Taylor.

Bremermann also considers his boundary value problem for a pseudoconvex domain D with C^2 boundary. The treatment of general pseudoconvex domains requires the notion of "Šilov boundary" S(D) of the function algebra of holomorphic functions in D which are continuous in $\bar{D}$. He shows that his generalized Dirichlet problem is possible for the upper envelope of the plurisubharmonic functions in $\bar{D}$ that are smaller or equal to the given continuous boundary values b(z) (at those points of ∂D where b(z) is defined) if and only if the boundary values are prescribed on and only on the Šilov boundary S(D) of D. Rossi [7] proved that the Šilov boundary is just the closure of the strictly pseudoconvex boundary points for D.

Suppose we consider our boundary value problem for a q-pseudoconvex domain D, with $2q < n$, which is not necessarily strictly q-pseudoconvex. We prove results that suggest that the boundary values should probably be prescribed only on the closure of the strictly q-pseudoconvex boundary points of D. In doing so we introduce the notion of q-holomorphic functions of Basener [1], and relate properties of q-holomorphic functions to properties of q-plurisubharmonic functions. These q-holomorphic functions essentially play the same role in our problem as the holomorphic functions in Bremermann's problem.

Section 2 of this article contains our definition of q-plurisubharmonic function together with properties of such functions. In section 3 we solve the generalized Dirichlet problem for strictly q-pseudoconvex domains in $\mathbb{C}^n$ with $2q < n$. In section 4 we consider the uniqueness of solutions to the boundary value problem and the "annulus" problem. Section 5 contains a treatment of the problem for q-pseudoconvex domains and a development of the relationship of q-holomorphic functions and q-plurisubharmonic functions.

2. q-PLURISUBHARMONIC FUNCTIONS

As we move toward the consideration of our boundary value problem for q-pseudoconvex domains, we need several definitions and lemmas.

Definition 2.1. A function defined in an open set $\Omega \subset \mathbb{C}$ and with values in $[-\infty, +\infty)$ is called *subharmonic* if (i) u is upper semicontinuous in Ω, and (ii) for every open ball B such that $\bar{B} \subset \Omega$ and every continuous function h on $\bar{B}$ which is harmonic in B and is greater than or equal to u on ∂B we have $u \leq h$ on $\bar{B}$.

Definition 2.2 A function u defined in an open set $\Omega \subset \mathbb{C}^n$ with values in $[-\infty, +\infty)$ is called *plurisubharmonic* if (i) u is upper semicontinuous in Ω, and (ii) u is subharmonic on the intersection of every 1-dimensional complex plane with Ω.

Definitions for *superharmonic* and *plurisuperharmonic* functions can be given by replacing upper semicontinuity by lower semicontinuity in (i) of both definitions and by reversing the inequalities in (ii) of Definition 2.1. Also u is superharmonic (plurisuperharmonic) if $-u$ is subharmonic (plurisubharmonic).

Definition 2.3. A function u defined in an open set $\Omega \subset \mathbb{C}^n$ and with values in $[-\infty, +\infty)$ is called $(n-1)$ *-plurisubharmonic* in Ω if (i) u is upper semicontinuous in Ω, and (ii) for every open ball B with $\bar{B} \subset \Omega$ and every continuous function g in $\bar{B}$ which is plurisuperharmonic in B and greater than or equal to u on ∂B we have $u \leq g$ on $\bar{B}$. Actually, we need only require that g be lower semicontinuous in $\bar{B}$.

The following result shows that we need to consider only C^∞ (or C^2) plurisuperharmonic functions in Definition 2.3

LEMMA 2.4. *An upper semicontinuous function* u *defined in an open set* $\Omega \subset \mathbb{C}^n$ *and with values in* $[-\infty, +\infty)$ *is* $(n-1)$ *-plurisubharmonic if and only if for every open ball* B *such that* $\bar{B} \subset \Omega$ *and every* C^∞ *function* g *on* $\bar{B}$ *which is plurisuperharmonic in* B *and greater than or equal to* u *on* ∂B *we have* $u \leq g$ *on* $\bar{B}$.

Proof. The necessary statement of this result is trivial so we concern ourselves with only the sufficient statement. Let g' be a continuous function on $\bar{B}$ which is plurisuperharmonic in B and greater than or equal to u on ∂B. Let $0 \leq \varphi \in C_0^\infty(\mathbb{C}^n)$ be equal to 0 when $|z| > 1$, let φ depend only on $|z_1|, \ldots, |z_n|$, and assume that $\int \varphi(z)\, d\lambda(z) = 1$ where $d\lambda$ is the Lebesgue measure. By Theorem 2.6.3 of [6] it follows that

$$g'_\varepsilon(z) = \int g'(z - \varepsilon\zeta)\, \varphi(\zeta)\, d\lambda(\zeta)$$

is plurisuperharmonic, that $g'_\varepsilon(z)$ is C^∞ where the distance from z to the complement of B is greater than ε, and that g'_ε increases monotonically to g' as ε decreases to 0. Given $\delta > 0$ there exists an ε sufficiently small so that $u < g' + \delta/2$ and $u < g'_\varepsilon + \delta$ on ∂B_ε, where B_ε is an open ball concentric with B and with radius equal to the radius of $B - \varepsilon$. By hypothesis $u \leq g'_\varepsilon + \delta$ on B_ε. Letting δ and ε approach 0 we find that $u \leq g'$ on $\bar{B}$.

We can generalize Definition 2.3 to q-plurisubharmonic functions, where $0 \leq q \leq n-1$.

Definition 2.5. A function u defined in an open set $\Omega \subset \mathbb{C}^n$ and with values in $[-\infty, +\infty)$ is called *q-plurisubharmonic* in Ω if (i) u is upper semicontinuous in Ω, and (ii) u is q-plurisubharmonic on the intersection of every $(q+1)$-dimensional complex plane with Ω.

If a function g is continuous in $\bar{B}$ and plurisuperharmonic in B, where B is an open ball whose closure is contained in the intersection of a $(q+1)$-dimensional complex plane with Ω, then g (being independent of $(n-q-1)$ variables) remains plurisuperharmonic under biholomorphic coordinate changes on $\mathbb{C}^n$. Thus the definition of a q-plurisubharmonic function is invariant under such coordinate changes.

Of course similar definitions can be given for $(n-1)$-*plurisuperharmonic* functions and q-*plurisuperharmonic* functions in $\Omega \subset \mathbb{C}^n$.

LEMMA 2.6. *A* C^2 *function* u *defined in an open set* $\Omega \subset \mathbb{C}^n$ *and with values in* $[-\infty, +\infty)$ *is* q-*plurisubharmonic in* Ω *if and only if the complex Hessian of* u *has at least* $(n-q)$ *nonnegative eigenvalues at each point of* Ω.

Proof. From the definition of q-plurisubharmonic function it suffices to prove the following result in $\mathbb{C}^{q+1}$. If $\Omega \subset \mathbb{C}^{q+1}$, a C^2 function u is q-plurisubharmonic in Ω if and only if u has at least 1 nonnegative eigenvalue in its complex Hessian at each point of Ω.

Assume that u has $(q+1)$ negative eigenvalues at some point $z_0 = 0 \in \Omega$. We may write (assuming u, its first derivatives, and certain second derivatives vanish at z_0)

$$u(z) = \sum_{j=1}^{q+1} \lambda_j z_j \bar{z}_j + O(|z|^3),$$

where $\lambda_j < 0$ for $j = 1, 2, \ldots, q+1$. Then there exist a ball B about 0 such that $\bar{B} \subset \Omega$ and an $\varepsilon > 0$ such that $u(z) < -\varepsilon$ on ∂B and $u(0) = 0$, a contradiction.

To prove the other direction, let B be an open ball with $\bar{B} \subset D$, and let g be a C^∞ function on $\bar{B}$ which is plurisuperharmonic in D and greater than or equal to u on ∂B. Then $u - g$ is a C^2 function on $\bar{B}$ whose Hessian has at least one nonnegative eigenvalue. By the maximum principle for such functions (which is easy to prove for C^2 functions), $u \leq g$ on ∂B implies $u \leq g$ on $\bar{B}$. An application of Lemma 2.4 yields the desired result.

The following lemma gives us a maximum principle for q-plurisubharmonic functions. Of course there is an analogous minimum principle for q-plurisuperharmonic functions.

LEMMA 2.7. *Let* K *be a compact set in* $\mathbb{C}^n$ *and suppose that* u *is upper semicontinuous on* K *and* q-*plurisubharmonic in the interior of* K. *Then the maximum of* u *on* K *is attained on* ∂K.

Proof. Since u is upper semicontinuous on K, u does take on its maximum on K. Assume that there is an interior point $z_0 = 0$ of K for which $u(0) > u(z)$ for all $z \in \partial K$ and assume $u(0) = 0$. Intersect K with a complex plane through z_0 of dimension $(q+1)$ and notice that u is q-plurisubharmonic in the interior of this intersection. For this reason we can assume that $n = q + 1$ and $K \subset \mathbb{C}^{q+1}$. Thus we have that $u(0) = 0$ and $u(z) < 0$ for all $z \in \partial K$. Choose $\varepsilon > 0$ so that the function $u_\varepsilon(z) = u(z) + \varepsilon\|z\|^2$, where $\|z\| = (z_1\bar{z}_1 + \dots + z_n\bar{z}_n)^{1/2}$, satisfies $u_\varepsilon(z) < 0$ on ∂K. Since $u_\varepsilon(0) = u(0) = 0$, we have that $u_\varepsilon(z)$ assumes its maximum for K at some interior point p in K. Let B be an open ball of radius $r > 0$ and center p such that $\bar{B}$ is contained in the interior of K. A computation yields

$$u_\varepsilon(z) = u(z) + \varepsilon\|z\|^2 = u(z) + \varepsilon\|z - p\|^2 - \varepsilon\|p\|^2 + 2\varepsilon \operatorname{Re}\langle p,z\rangle,$$

where $\langle .,. \rangle$ is the usual inner product on $\mathbb{C}^n$. Since the function $2\varepsilon \operatorname{Re}\langle p,z\rangle$ is pluriharmonic, it is obvious that the function $u(z) - \varepsilon\|p\|^2 + 2\varepsilon \operatorname{Re}\langle p,z\rangle$ is q-plurisubharmonic in the interior of K. Thus there exists a point $z_1 \in \partial B$ such that

$$\begin{aligned} u(p) + \varepsilon\|p\|^2 &\le u(z_1) - \varepsilon\|p\|^2 + 2\varepsilon \operatorname{Re}\langle p,z_1\rangle \\ &< u(z_1) + \varepsilon r^2 - \varepsilon\|p\|^2 + 2\varepsilon \operatorname{Re}\langle p,z_1\rangle. \end{aligned}$$

Thus $u_\varepsilon(p) < u_\varepsilon(z_1)$, contradicting the fact that u_ε assumes its maximum at p.

Remark. It is clear that in Definition 2.3 we could replace open balls B such that $\bar{B} \subset \Omega$ by compact sets K such that $K \subset \Omega$. Also the definition of q-plurisubharmonic functions is a local definition.

The proof of the next lemma is obvious.

LEMMA 2.8. *Let Ω be an open set in $\mathbb{C}^n$, and let* u_1 *and* u_2 *be* q-*plurisubharmonic in Ω for some* q, $0 \le q \le n - 1$. *Then* $\max(u_1, u_2)$ *is* q-*plurisubharmonic in Ω.*

If Ω is an open set in $\mathbb{C}$, then a continuous function u in Ω which is both subharmonic and superharmonic in Ω is harmonic there. This suggests the following definition for our theory.

Definition 2.9. If Ω is an open set in $\mathbb{C}^n$ and if q satisfies $0 \le q \le n - 1$, then a continuous function u in Ω is called q-*complex Monge-Ampere* (q-CMA) in Ω if it is both q-plurisubharmonic and $(n - q - 1)$-plurisuperharmonic.

LEMMA 2.10. *If* u *is a* C^2 *function in an open set* $\Omega \subset \mathbb{C}^n$, *and if* u *is* q-CMA *in Ω, then* u *has at least one zero eigenvalue in its complex Hessian at each point of Ω.*

Proof. We have that u is q-plurisubharmonic in Ω for some q, $0 \le q \le n - 1$, and also $(n - q - 1)$-plurisuperharmonic in Ω. By Lemma 2.6 u has at least $(n - q)$ nonnegative eigenvalues in its complex Hessian at every point. Of course a corresponding lemma for $(n - q - 1)$-plurisuperharmonic functions would show that this Hessian has at least $(q + 1)$ nonpositive eigenvalues in Ω. Hence the complex Hessian of u has at least 1 zero eigenvalue at each point of Ω.

Thus if u is C^2 and q-CMA in $\Omega \subset \mathbb{C}^n$, then u satisfies the homogeneous complex Monge-Ampere equation

$$[\partial\bar{\partial}u]^n = \underbrace{\partial\bar{\partial}u \wedge \dots \wedge \partial\bar{\partial}u}_{\text{n times}} = 0$$

in Ω.

3. A GENERALIZED DIRICHLET PROBLEM

We want to consider a boundary value problem for particular domains in $\mathbb{C}^n$, $n > 1$. Let D be a domain in $\mathbb{C}^n$ with a C^2 boundary. Let $p \in \partial D$ and U be an open neighborhood of p in $\mathbb{C}^n$. A C^2 real-valued function ρ on U satisfying

$$D \cap U = \{z \in U: \rho(z) < 0\}$$

and $d\rho \neq 0$ on $\partial D \cap U$ is a defining function on U for D. We let $L_p(\rho)$ denote the Levi form of ρ at p. Let λ_P, λ_Z, and λ_N denote the number of positive, zero, and negative eigenvalues of $L_p(\rho)$ respectively.

Definition 3.1. The domain D is q-*pseudoconvex* at the boundary point p if $\lambda_N \leq q$ and is *strictly* q-*pseudoconvex* at p if $\lambda_N + \lambda_Z \leq q$. Also D is (strictly) q-*pseudoconvex* if it is (strictly) q-pseudoconvex at each boundary point.

The boundary value problem that we are interested in is the following one. Let D be a bounded strictly q-pseudoconvex domain in $\mathbb{C}^n$, $n > 1$, of the form $D = \{z \in \mathbb{C}^n: \rho(z) < 0\}$, where $\rho(z)$ is a C^2 defining function for $\bar{D}$ which is strictly q-plurisubharmonic in a neighborhood of $\bar{D}$. Let b(z) be a continuous function given on ∂D. We consider the set of all q-plurisubharmonic functions in D which are upper semicontinuous in $\bar{D}$ and less than or equal to b(z) on ∂D. As a consequence of our next result we have that if $2q < n$, then the upper envelope of this class of functions exists and takes on the prescribed boundary values b(z).

First we give a counter example to show that this may not be true if $2q \geq n$. Let $D \subset \mathbb{C}^2$ be defined as $D_1 - \bar{D}_2$, where

$$D_1 = \{z \in \mathbb{C}^2: (z_1\bar{z}_1 + z_2\bar{z}_2)^{1/2} < e\} \quad \text{and}$$
$$D_2 = \{z \in \mathbb{C}^2: (z_1\bar{z}_1 + z_2\bar{z}_2)^{1/2} < 1\}.$$

Then $\partial D = \partial D_1 \cup \partial D_2$, and D is 1-pseudoconvex in $\mathbb{C}^2$. Let

$$b(z) = \begin{cases} 1 & \text{on } \partial D_1 \\ 0 & \text{on } \partial D_2 \end{cases}.$$

Then we can define a C^∞ 1-plurisubharmonic function u in $\bar{D}$ such that on each 1-dimensional complex plane through the origin (which we assume is the z_1-plane for convenience) we have u is a function of $|z_1|^2$ such that $u(1) = 0$ and $u(e^2) = 1$ and $-1 \leq u \leq 1$ on $\bar{D}_1$ with $0 \leq u \leq 1$ on $\bar{D}$. Given $\varepsilon > 0$ and $\delta > 0$ we can choose such a u so that $u(z) \geq 1 - \varepsilon$ for $(z_1\bar{z}_1 + z_2\bar{z}_2)^{1/2} \geq 1 + \delta$. Thus the upper envelope of such functions takes on the boundary value 1 on ∂D_2, and our problem does not have a solution.

THEOREM 3.2. *Let* D *be a domain in* $\mathbb{C}^n$ *given by* $D = \{z \in \mathbb{C}^n: \rho(z) < 0\}$, *where* $\rho(z)$ *is* C^2 *in a neighborhood of* $\bar{D}$. *Let* b(z) *be a continuous real valued function defined on* ∂D. *Then the upper envelope* $\tilde{u}(z)$ *of the class of all functions which are* q-*plurisubharmonic in* D, *upper semicontinuous on* $\bar{D}$, *and less than or equal to* b(z) *on* ∂D *exists. If* ∂D *is strictly* q-*pseudoconvex in a neighborhood of* $p \in \partial D$, *then we have*

$$\liminf_{z \to p} \bar{u}(z) \geq b(p).$$

If D *is a strictly* q*-pseudoconvex domain in* $\mathbb{C}^n$ *and* $2q < n$, *we also have*

$$\limsup_{z \to p} \bar{u}(z) \leq b(p)$$

for every boundary point p *of* D.

Proof. The family of q-plurisubharmonic functions in D which are upper semicontinuous on $\bar{D}$ and less than or equal to b(z) on ∂D is denoted by P_q (b (z), D). The fact that the upper envelope $\bar{u}(z)$ of functions in P_q (b (z), D) exists follows from the maximum principle of Lemma 2.7.

The first inequality if proved using the arguments of Bremermann [3] with his strictly plurisubharmonic functions replaced by the strictly q-plurisubharmonic functions.

If D is a strictly q-pseudoconvex domain, we define $U_\varepsilon(z) = \rho(z) - \varepsilon\|z - p\|^2$, where p is an arbitrary boundary point, and $\varepsilon > 0$ is chosen sufficiently small so that $U_\varepsilon(z)$ is strictly q-plurisubharmonic in $\bar{D}$. Exactly as in Bremermann's proof, we prove that $\bar{u}(p) \leq b(p)$ by constructing a "barrier" at the point p, the barrier function being $W_p(z) = -U_\varepsilon(z)$. The only difference here is that we must show that $u(z) - CW_p(z)$ satisfies the maximum principle in $\bar{D}$, where $C > 0$ is a constant.

From the proof of Lemma 2.7, it suffices to show that $u(z) - CW_p(z) + \varepsilon\|z\|^2$ satisfies the maximum principle in $\bar{D}$ for $\varepsilon > 0$. Assume there exists a point $z' \in D$ such that

$$u(z) - CW_p(z) + \varepsilon\|z\|^2 < u(z') - CW_p(z') + \varepsilon\|z'\|^2 \qquad \text{for all } z \in \partial D$$

and the maximum of the function on $\bar{D}$ is assumed at z'. Suppose we can prove that $u(z) - CW_p(z)$ is q-plurisubharmonic on the intersection of a $(q + 1)$-dimensional complex plane H' through z' with an open ball B with center at z' and of small radius. Then the proof of Lemma 2.7 will show that there exists a point $z \in \partial B \cap H'$ such that

$$u(z) - CW_p(z) + \varepsilon\|z\|^2 > u(z') - CW_p(z') + \varepsilon\|z'\|^2,$$

a contradiction.

Since $CW_p(z)$ is a C^2 strictly q-plurisuperharmonic function in D, there exists an open ball B, of sufficiently small radius and center z', and an $(n - q)$-dimensional complex plane H through z' such that $CW_p(z)$ is C^2 plurisuperharmonic in $B \cap H$. Because $2q < n$, we have $q + 1 \leq n - q$, and there exists a $(q + 1)$-dimensional complex plane H' through z' such that $CW_p(z)$ is plurisuperharmonic in $B \cap H'$. Let B' be an open ball with $\bar{B}' \subset B \cap H'$, and let g(z) be a C^∞ plurisuperharmonic function in B' which is continuous on $\bar{B}'$ and satisfies $u(z) - CW_p(z) \leq g(z)$ on $\partial B'$. Since u is q-plurisubharmonic in D, and $CW_p(z)$ is plurisuperharmonic in $B \cap H'$, we have that $u(z) - CW_p(z) \leq g(z)$ on $\bar{B}'$. Thus $u - CW_p(z)$ is q-plurisubharmonic on $B \cap H'$.

Our next result concerns properties of the solution $\bar{u}(z)$ given to us by Theorem 3.2.

THEOREM 3.3. *Let* D *be a strictly* q*-pseudoconvex domain in* $\mathbb{C}^n$ *and assume* $2q < n$. *Let* $b(z)$ *be a continuous real-valued function defined on* ∂D. *Then the upper envelope* $\bar{u}(z)$ *of* $P_q(b(z),D)$ *is*

(i) q*-plurisubharmonic in* D,

(ii) *continuous in* $\bar{D}$, *and*

(iii) q-CMA *in* D.

Proof. First we shall prove (i). Let B be an open ball with closure contained in the intersection of a $(q + 1)$ -dimensional complex plane with D. Suppose g is C^∞ plurisuperharmonic in B, continuous on $\bar{B}$, and satisfies $\bar{u}(z) \leq g(z)$ for all $z \in \partial B$. Since $u(z) \leq g(z)$ for all $z \in \partial B$ and all $u \in P_q(b(z), D)$, we have $u(z) \leq g(z)$ for all $z \in B$ and all $u \in P_q(b(z), D)$. Then $\bar{u}(z) \leq g(z)$ for all $z \in B$, and $\bar{u}$ is q-plurisubharmonic in D if it is upper semicontinuous there. However, this will follow from part (ii) of our result, which is proved using the technique of Walsh [8] with q-plurisubharmonic functions instead of plurisubharmonic functions.

It remains to prove that $\bar{u}$ is q-CMA in D. To do this we must show that $\bar{u}$ is $(n - q - 1)$ -plurisuperharmonic in D. Since $\bar{u}$ is continuous in $\bar{D}$, we must show only that $\bar{u}$ satisfies the appropriate minimum principle. Let B be an open ball whose closure is contained in the intersection of a $(n - q)$ -dimensional complex plane H with D. Suppose f is a C^∞ plurisubharmonic function on B which is continuous on $\bar{B}$ and satisfies $f(z) < \bar{u}(z)$ on ∂B. Assume that the complex plane H is defined by $z_1 = 0, z_2 = 0, \ldots, z_q = 0$, so $f(z)$ is actually $f(z_{q+1}, \ldots, z_n)$. Let $f'(z_1, z_2, \ldots, z_n)$ be defined by $f'(z_1, \ldots, z_n) = f(z_{q+1}, \ldots, z_n) - K(z_1\bar{z}_1 + \ldots + z_q\bar{z}_q)$, where $K > 0$ is chosen sufficiently large so that $f'(z) \leq \bar{u}(z)$ on $\partial D'$, where D' is a still to be chosen strictly pseudoconvex domain with C^∞ boundary and with closure in D whose intersection with H is B. Since $f'(z)$ has $(n - q)$ nonnegative eigenvalues in its complex Hessian, $f'(z)$ is q-plurisubharmonic on D' if D' is chosen appropriately (for example an ellipsoid whose intersection with any $(n - q)$ -dimensional complex plane parallel to H results in a smaller ball than B, the centers of both balls being contained in the same q-dimensional complex plane normal to H). Since $f'(z) \leq \bar{u}(z)$ on $\partial D'$, we find $f'(z) \in P_q(\bar{u}(z), D')$. In Lemma 3.4, we will show that if D' is a strictly pseudoconvex domain in D, the function $\hat{u}$ defined as

$$\hat{u}(z) = \begin{cases} \bar{u}(z), & z \in D - D' \\ \bar{u}_{D'}(z) & z \in D', \end{cases}$$

where $\bar{u}_{D'}(z)$ is the upper envelope of all q-plurisubharmonic functions in D' which are upper semicontinuous in $\bar{D}$, and less than or equal to $\bar{u}(z)$ on $\partial D'$, is actually equal to $\bar{u}(z)$ on D.

Now $f'(z) \in P_q(\bar{u}(z), D')$, implying that $f'(z) \leq \bar{u}_{D'}(z)$ for all $z \in D'$, and hence $f'(z) \leq \bar{u}(z)$ for all $z \in D'$. Thus $f(z) \leq \bar{u}(z)$ on B, and $u(z)$ is $(n - q - 1)$ -plurisuperharmonic in D.

Remark. Theorems 3.2 and 3.3 are valid if q-plurisubharmonic is replaced everywhere by q-plurisuperharmonic, $P_q(b(z), D)$ is replaced by $S_q(b(z), D)$, the set of q-plurisuperharmonic functions in D which are greater than or equal to $b(z)$ on ∂D, and the upper envelope is replaced by the lower envelope.

We now consider the lemma which is mentioned in the proof of Theorem 3.3.

LEMMA 3.4. *Let* D *be a* q*-pseudoconvex domain in* $\mathbb{C}^n$ *with* C^2 *boundary where* $2q < n$, *and let* $b(z)$ *be a continuous function defined on* ∂D. *Denote by* $\bar{u}(z)$ *the upper envelope of the class* $P_q(b(z),D)$. *If* D' *is a strictly pseudoconvex domain with* C^2 *boundary such that* $\bar{D}' \subset D$, *then the function* $\hat{u}(z)$, *defined by*

$$\hat{u}(z) = \begin{cases} \bar{u}(z), & z \in D - D' \\ \bar{u}_{D'}(z), & z \in D', \end{cases}$$

where $\bar{u}_{D'}(z)$ *is the upper envelope of* $P_q(\bar{u}(z), D')$, *is equal to* $\bar{u}(z)$ *in* D.

Proof. By Theorems 3.2 and 3.3 we have $\bar{u}(z) = b(z)$ on ∂D, $\bar{u}(z)$ is continuous in D, $\bar{u}_{D'}(z) = \bar{u}(z)$ on $\partial D'$, and $\hat{u}(z)$ is continuous in D. Also $\bar{u}(z)$ is q-plurisubharmonic in D, and $\hat{u}(z)$ is q-plurisubharmonic in $\{D - \bar{D}'\} \cup D'$. It is obvious that $\bar{u}_{D'}(z) \geq \bar{u}(z)$ on $\bar{D}$, since $\bar{u}(z) \in P_q(\bar{u}(z),D')$. If we can show that $\hat{u}(z)$ is q-plurisubharmonic in D, then $\hat{u}(z) \in P_q(b(z),D)$ and $\hat{u}(z) \leq \bar{u}(z)$ on D. Thus $\hat{u}(z) = \bar{u}(z)$ on D, and our result is proved.

It suffices to show that if B is an open ball with closure contained in the intersection of a $(q + 1)$ -dimensional complex plane with D, if g is a C^∞ plurisuperharmonic function in D which is continuous on $\bar{D}$ and greater than or equal to $\hat{u}(z)$ on ∂B, and if B contains a boundary point p of $\partial D'$, then $g(z) \geq \hat{u}(z)$ on $\bar{B}$. If $\hat{u}(z_0) > g(z_0)$ for some point $z_0 \in B \cap \{D - \bar{D}'\}$, then our maximum principle (Lemma 2.7) implies there exists a point z' in $\{\partial B \cup \partial D'\} \cap \{D - D'\}$ such that $\hat{u}(z') > g(z')$, a contradiction if $z' \in \partial B$. If $z_0 \in B \cap D'$, then there exists a point $z' \in \{B \cup \partial D'\} \cap \{\bar{D}'\}$ such that $\hat{u}(z') > g(z')$, a contradiction if $z' \in \partial B$. Thus we need only consider the case where $\hat{u}(z) - g(z)$ has a positive maximum for $\bar{B}$ at $z_0 \in \partial D' \cap B$. We know that $\bar{u}(z) \leq \hat{u}(z)$ in D' and $u(z) = \hat{u}(z)$ in $D - D'$, implying that $\bar{u}(z) - g(z)$ has a positive maximum for $\bar{B}$ at z_0, a contradiction since $\bar{u}$ is q-plurisubharmonic in D.

Remark. It is interesting that in the statement of Theorem 3.2 we have that if ∂D is strictly q-pseudoconvex in a neighborhood of $p \in \partial D$, then

$$\liminf_{z \to p} \bar{u}(z) \geq b(p).$$

However, we need that D is a strictly q-pseudoconvex domain in $\mathbb{C}^n$ to prove

$$\limsup_{z \to p} \bar{u}(z) \leq b(p).$$

Thus one direction is local and the other is global. If we consider the same boundary value problem as in Theorem 3.2, but with $P_q(b(z),D)$ replaced by $P_q^2(b(z),D)$, the class of C^2 plurisubharmonic functions on D which are less than or equal to $b(z)$

on ∂D, the authors have a proof that the upper envelope exists and takes on the given boundary values. However, we need only that ∂D is strictly q-pseudoconvex near $p \in \partial D$ to show that

$$\limsup_{z \to p} \bar{u}(z) \leq b(p).$$

for $\bar{u}$ the upper envelope of $P_q{}^2(b(z),D)$. Thus if we could prove that the C^2 q-plurisubharmonic functions are "dense" in the q-plurisubharmonic functions, then our boundary value problem is a local one. This is of course true in the case $q = 0$.

4. UNIQUENESS AND THE "ANNULUS" PROBLEM

Suppose we consider our boundary value problem of Theorem 3.2 in the case where D is an open ball in $\mathbb{C}^3$. Then there exists a plurisubharmonic solution $\bar{u}_0$ and a 1-plurisubharmonic solution $\bar{u}_1$ with the same boundary values b(z), since D is both 0 and 1-pseudoconvex. It is easy to give examples for which $\bar{u}_0$ and $\bar{u}_1$ are not equal in D. However, one may ask if $\bar{u}_0$ is the only continuous plurisubharmonic solution which is also 2-plurisuperharmonic, or if $\bar{u}_1$ is the only continuous 1-plurisubharmonic solution which is also 1-plurisuperharmonic. In this direction we state the following conjecture.

Conjecture 4.1. Let D be a strictly q-pseudoconvex domain in $\mathbb{C}^n$, $n > 1$ and $2q < n$, such that $D = \{z \in \mathbb{C}^n: \rho(z) < 0\}$, where ρ is C^2 and strictly q-plurisubharmonic in a neighborhood of $\bar{D}$. Let $\bar{u}(z)$ be the upper envelope (of $P_q(b(z),D)$) solution which is q-plurisubharmonic in D, continuous in $\bar{D}$, q-CMA in D in the sense that $\bar{u}$ is also $(n - q - 1)$-plurisuperharmonic in D, and takes on prescribed continuous boundary values b(z) on ∂D. Then $\bar{u}$ is the unique solution to this boundary value problem with these properties.

Suppose $\tilde{u}$ is another solution with the same properties. Obviously, $\tilde{u} \leq \bar{u}$ on D and $\tilde{u} = \bar{u}$ on ∂D. If we could show that $\bar{u} - \tilde{u}$ is $(n - 1)$-plurisubharmonic in D, then $\bar{u} - \tilde{u} \leq 0$ in D by the maximum principle, and $\tilde{u} = \bar{u}$ in D. If $\tilde{u}$ and $\bar{u}$ are C^2 in D (it is shown in [2] that $\bar{u}$ is not necessarily C^2 in D), then it is easy to see that $\tilde{u} - \bar{u}$ is $(n - 1)$ -plurisubharmonic in D.

It is interesting that in case D is strictly pseudoconvex and $\bar{u}$ and $\tilde{u}$ are both plurisubharmonic and $(n - 1)$ plurisuperharmonic, that we have $\tilde{u} = \bar{u}$ on ∂D implies $\tilde{u} \leq \bar{u}$ on D since $\bar{u}$ is $(n - 1)$ -plurisuperharmonic and $\tilde{u}$ is plurisubharmonic on D. Thus we have proved the following result.

THEOREM 4.2. *Let* D *be a strictly pseudoconvex domain in* $\mathbb{C}^n$ *such that* $D = \{z \in \mathbb{C}^n: \rho(z) < 0\}$, *where* $\rho(z)$ *is* C^2 *and strictly plurisubharmonic in a neighborhood of* $\bar{D}$. *If a continuous function* b(z) *is given on* ∂D, *then there exists one and only one solution* (*the upper envelope solution* $\bar{u}$) *which is continuous on* $\bar{D}$, *plurisubharmonic and* $(n - 1)$ *-plurisuperharmonic in* D, *and equal to* b(z) *on* ∂D.

Remark. In [2] Bedford and Taylor give a distributional definition for the complex Monge-Ampere equation which can be applied to continuous plurisubharmonic functions in D. For a strictly pseudoconvex domain $D \subset \mathbb{C}^n$, they find a solution to their homogeneous complex Monge-Ampere equation which takes on the given boundary values $b(z)$. Then by using their minimum principle, they show that their solution is actually Bremermann's upper envelope solution $\bar{u}$. In this case, their solution satisfying the homogeneous complex Monge-Ampere equation is equivalent to our continuous solution being in the intersection class of plurisubharmonic and $(n-1)$-plurisuperharmonic functions on D.

Now we turn our attention to an "annulus" problem. Let D_1 and D_2 be bounded domains in $\mathbb{C}^n$ such that $\bar{D}_2 \subset \bar{D}_1$. Let $D = D_1 - \bar{D}_2$ and suppose that there exists a function $\rho_1(z)$ which is C^2 and plurisubharmonic in a neighborhood of ∂D_1 so that $\partial D_1 = \{z \in \mathbb{C}^n: \rho_1(z) = 0\}$, $D_1 = \{z \in \mathbb{C}^n: \rho_1(z) < 0\}$, and $d\rho_1 \neq 0$ on ∂D_1. Also assume there exists another function $\rho_2(z)$ which is C^2 in a neighborhood ∂D_2 and such that $\partial D_2 = \{z \in \mathbb{C}^n: \rho_2(z) = 0\}$, $D_2 = \{z \in \mathbb{C}^n: \rho_2(z) < 0\}$, and $d\rho_2 \neq 0$ on ∂D_2.

THEOREM 4.3. *Let* D *be defined as above. Then there exists a continuous plurisubharmonic function* $w(z)$ *in* D *such that* $w(z) = 1$ *on* ∂D_1, $w(z) = 0$ *on* ∂D_2, *and* $w(z)$ *is* 0-CMA *in* D *in the sense that* $w(z)$ *is* $(n-1)$*-plurisuperharmonic in* D. *This solution* $w(z)$ *solves the complex Monge-Ampere equation* (*as in* [2]) *in* D.

Proof. First we show that there exists a continuous plurisubharmonic function $f(z)$ in D such that $f(z) = 1$ on ∂D_1 and $f(z) = 0$ on ∂D_2. Given $\delta > 0$ sufficiently small, there exists a domain D_3 with $\bar{D}_2 \subset D_3 \subset \bar{D}_3 \subset D_1$ such that $\rho_1(z) < 0$ in $D_1 - D_3$ and $\rho_1(z) < -\delta$ on ∂D_3. Now we can multiply ρ_1 by a sufficiently large positive constant K such that $K\rho_1(z)$ is less than -1 on ∂D_3. Then the function $f(z) = f_1(z) + 1$, where $f_1(z)$ is defined as the maximum of $K\rho_1(z)$ and -1 on $\bar{D} - D_3$ and as -1 on D_3, is the desired continuous plurisubharmonic function.

By the maximum principle for plurisubharmonic functions, we know that the upper envelope $w(z)$ of all plurisubharmonic functions in D which are less than or equal to 1 on ∂D_1 and less than or equal to 0 on ∂D_2 exists. Since $f(z)$ is in this class of functions, we have that $w(z) \geq 1$ on ∂D_1 and $w(z) \geq 0$ on ∂D_2.

Now consider the function $e^{K'\rho_2(z)} - 1$, where K' is a positive constant still to be chosen. Now for $i, j = 1, \ldots n$ we have

$$\frac{\partial^2(e^{K'\rho_2} - 1)}{\partial z_i \partial \bar{z}_j} = K' \frac{\partial^2 \rho_2}{\partial z_i \partial \bar{z}_j} e^{K'\rho_2} + (K')^2 \frac{\partial \rho_2}{\partial z_i} \frac{\partial \rho_2}{\partial \bar{z}_j} e^{K'\rho_2}.$$

Since $d\rho_2 \neq 0$ on ∂D_2, we can choose K' sufficiently large so that $e^{K'\rho_2} - 1$ is $(n-1)$-plurisubharmonic in a neighborhood of ∂D_2. This function $e^{K'\rho_2(z)} - 1$ is a new defining function for D_2, which for the sake of simplicity we rename $\rho_2(z)$. The function $-\rho_2(z)$ is $(n-1)$-plurisuperharmonic in a neighborhood of ∂D_2 and is positive on the intersection of this neighborhood with D. Given a $\delta > 0$ sufficiently small, there exists a domain D_3 with $\bar{D}_2 \subset D_3 \subset \bar{D}_3 \subset D_1$ so that $-\rho_2(z) > \delta$ on ∂D_3. By multiplying $-\rho_2(z)$ by a sufficiently large positive constant L, we have that $-L\rho_2(z) > 1$ on ∂D_3. Then the function $g(z)$ defined as the minimum of $-L\rho_2(z)$ and 1 on $\bar{D} - D_3$ and as 1 on D_3 is a continuous $(n-1)$-plurisuperharmonic function on D.

Let $u \in P_0(b(z),D)$, where $b(z) = 1$ on ∂D_1 and $b(z) = 0$ on ∂D_2. By the maximum principle for the function $u(z) - g(z)$ on D, we have that $u(z) - g(z) \leq 0$ on $\bar{D}$, implying that $w(z) \leq g(z)$ on $\bar{D}$. Thus $w(z) = 1$ on ∂D_1 and $w(z) = 0$ on ∂D_2.

The solution w(z) also solves the complex Monge-Ampere equation in D in the sense of Bedford and Taylor [2]. Let D′ be a strictly pseudoconvex domain contained in D with a C^∞ boundary. Define the function

$$\hat{w}(z) = \begin{cases} w(z), & z \in D - D' \\ \bar{w}_{D'}(z), & z \in D' \end{cases}$$

analogously to the way we defined $\hat{u}(z)$ in the proof of Lemma 3.4. As in that proof, it is true that $w(z) = \hat{w}(z)$ in D. But $\hat{w}(z)$ solves the complex Monge-Ampere equation in the Bedford-Taylor sense in D′, and so w(z) solves this equation in D.

Remark. The last part of this proof shows that in the case of continuous plurisubharmonic functions our concept of a 0-CMA function is equivalent to a solution of the homogeneous complex Monge-Ampere equation of Bedford and Taylor.

Remark. The technique used in the proof of Theorem 4.3 may be useful in finding a solution of the boundary value problem involving the nonhomogenerous complex Monge-Ampere equation with Dirac δ right hand side and continuous boundary values. This problem for the unit ball in $\mathbb{C}^2$ is considered in [2].

Remark. Theorem 4.3 and the method developed in its proof may also give an approach to defining the "capacity" of a Stein manifold.

5. THE NONLINEAR CAUCHY-RIEMANN EQUATIONS

As mentioned in the introduction, Bremermann points out that for an arbitrary domain of holomorphy $D \subset \mathbb{C}^n$ the boundary values cannot be prescribed on the whole boundary of D, but only on the Šilov boundary S(D) of the function algebra of holomorphic functions in D which are continuous in $\bar{D}$. Also Rossi showed that if ∂D is C^2, then the set S(D) is just the closure of the strictly pseudoconvex boundary points.

We consider a q-pseudoconvex domain $D \subset \mathbb{C}^n$ with a C^2 boundary. We prove results which indicate that the boundary values b(z) for our generalized Dirichlet problem (*i.e.,* taking the upper envelope of all continuous functions in $\bar{D}$ which are q-plurisubharmonic in D and less than or equal to b(z) on ∂D) should be described only on the closure of the strictly q-pseudoconvex boundary points of D for $2q < n$. In doing this we make use of the q-holomorphic functions of Basener [1], and we prove results which indicate the relationships of the real part and log of the modulus of such functions to the q-plurisubharmonic functions and the homogeneous complex Monge-Ampere equation. In addition, we prove results concerning complex fiberings in the sense of Rossi [7] and Freeman [5] for certain boundary points of D which are not strictly q-pseudoconvex.

For the remainder of this section q will be an integer between and including 0 and $n - 1$ except when we discuss our generalized Dirichlet problem, in which case we assume that $2q < n$.

Let Ω be an open subset of $\mathbb{C}^n$ (or more generally Ω can be a complex manifold). The following definition is found in [1]. This definition is given for C^∞ functions, but it and the results found in [1] are certainly true for C^2 functions.

Definition 5.1. Let f be a C^∞ function defined on Ω and q a nonnegative integer. Then f is a q-*holomorphic function* on Ω if f satisfies

$$\bar{\partial} f \wedge (\partial\bar{\partial} f)^q = \bar{\partial} f \wedge \underbrace{(\partial\bar{\partial} f \wedge \dots \wedge \partial\bar{\partial} f)}_{q \text{ times}} = 0$$

on Ω. Thus f satisfies a system of nonlinear partial differential equations (for $q > 0$) which can be regarded as a generalization of the usual Cauchy-Riemann equations ($q = 0$).

Let $z = (z_1, \dots, z_n)$ be coordinates for $\mathbb{C}^n$. If f is C^∞ in Ω and $z_0 \in \Omega$, we define

$$M^z_{z_0}(f) = \begin{bmatrix} \dfrac{\partial f}{\partial \bar{z}_1}(z_0) & \cdots & \dfrac{\partial f}{\partial \bar{z}_n}(z_0) \\ \dfrac{\partial^2 f}{\partial z_1 \partial \bar{z}_1}(z_0) & \cdots & \dfrac{\partial^2 f}{\partial z_1 \partial \bar{z}_n}(z_0) \\ \vdots & & \vdots \\ \dfrac{\partial^2 f}{\partial z_n \partial \bar{z}_1}(z_0) & \cdots & \dfrac{\partial^2 f}{\partial z_n \partial \bar{z}_n}(z_0) \end{bmatrix}$$

Then Basener has proved the following result.

PROPOSITION 5.2 [1]. *If* $f \in C^\infty(\Omega)$, *then* $\bar{\partial} f \wedge (\partial\bar{\partial} f)^q = 0$ *on* Ω *if and only if*

$$\operatorname{rank} M^z_{z_0}(f) \leq q \qquad \text{for all } z_0 \in \Omega.$$

Basener [1] shows that the q-holomorphic functions define a notion of convexity which is related (at least locally) to q-pseudoconvexity. Also the collection of q-holomorphic functions on Ω does not form a function algebra if $q > 0$.

It is well known that the real part of a holomorphic function in an open set $\Omega \subset \mathbb{C}^n$ has a zero complex Hessian at every point of Ω. The function $z_1\bar{z}_2$ is 1-holomorphic in any open set in $\mathbb{C}^2$, but its real part does not even satisfy the homogeneous complex Monge-Ampere equation $[\partial\bar{\partial}(\operatorname{Re} z_1\bar{z}_2)]^2 = 0$. However, $\operatorname{Re}(z_1\bar{z}_2)$ is a 1-plurisubharmonic function in $\mathbb{C}^2$. This leads us to our first result in this direction.

THEOREM 5.3. *Let* Ω *be an open subset of* $\mathbb{C}^n$. *If* $f \in C^\infty(\Omega)$ *is a* q-*holomorphic function in* Ω, *then* Re f *is a* q-*plurisubharmonic function in* Ω.

Proof. We must show that if f satisfies $\bar\partial f \wedge (\partial\bar\partial f)^q = 0$ on Ω, then

$$\left[\frac{\partial^2 \mathrm{Re}\, f}{\partial z_i \partial \bar z_j}\right], \qquad i, j = 1, \ldots, n,$$

has at least $(n-q)$ nonnegative eigenvalues on Ω. It suffices to prove this for $n = q+1$, since for $n > q+1$ we can then proceed as follows. Let $z_0 \in \Omega$ and diagonalize

$$\left[\frac{\partial^2 \mathrm{Re}\, f}{\partial z_i \partial \bar z_j}\right], \qquad i, j = 1, \ldots, n,$$

at z_0. Suppose fewer than $(n-q)$ eigenvalues are nonnegative, so that at least $(q+1)$ are negative. Pull back to the $(q+1)$-dimensional subspace E of $\mathbb{C}^n$ through z_0 spanned by the eigenvectors belonging to these eigenvalues. Here we have

$$\bar\partial f|_E \wedge (\partial\bar\partial f|_E)^q = 0,$$

but

$$\left[\frac{\partial^2 \mathrm{Re}\, f}{\partial z_i \partial \bar z_j}\right]$$

restricted to E has no nonnegative eigenvalues at z_0.

Thus we want to prove if Ω is an open set in C^{q+1} and $f \in C^\infty(\Omega)$ satisfies $\bar\partial f \wedge (\partial\bar\partial f)^q = 0$, then

$$\left[\frac{\partial^2 \mathrm{Re}\, f}{\partial z_i \partial \bar z_j}\right], \qquad i, j = 1, \ldots, q+1,$$

has at least 1 nonnegative eigenvalue at an arbitrary point $z_0 \in \Omega$. By Proposition 5.2 we know that the matrix

$$\left[\frac{\partial^2 f}{\partial z_i \partial \bar z_j}\right], \qquad i, j = 1, \ldots, q+1,$$

is singular at z_0. Hence there exists a nonzero $(q, 0)$-form α such that $\alpha \wedge \partial\bar\partial f = 0$ when evaluated at z_0. Then

$$\begin{aligned}
\alpha \wedge \partial\bar\partial \mathrm{Re}\, f \wedge \bar\alpha &= \alpha \wedge \partial\bar\partial f \wedge \bar\alpha + \alpha \wedge \partial\, \overline{\partial f} \wedge \bar\alpha \\
&= \alpha \wedge \overline{(\bar\partial\partial f \wedge \alpha)} = -\alpha \wedge \overline{(\partial\bar\partial f \wedge \alpha)} \\
&= -\alpha \wedge \overline{(\alpha \wedge \partial\bar\partial f)} = 0
\end{aligned}$$

at z_0. Hence the matrix

$$\left[\frac{\partial^2 \mathrm{Re}\, f(z_0)}{\partial z_i \partial \bar{z}_j}\right], \qquad i, j = 1, \ldots, q+1,$$

cannot be definite, and Re f is q-plurisubharmonic in Ω.

The following corollary to Theorem 5.3 is a result of the relation

$$\log|f|^2 = 2\mathrm{Re}(\log f),$$

for a q-holomorphic function f.

COROLLARY 5.4. *Let Ω be an open set in $\mathbb{C}^n$. If* $f \in C^\infty(\Omega)$ *is a* q-*holomorphic function in* Ω, *then* $\log|f|$ *is a* C^∞q-*plurisubharmonic function in* Ω *except at the zeros of* f.

Earlier we gave an example of a 1-holomorphic function f in $\mathbb{C}^2$ whose real part failed to satisfy the homogeneous complex Monge-Ampere equation

$$[\partial\bar{\partial}(\mathrm{Re}\, f)]^2 = 0.$$

However, we find that the following general result is true.

THEOREM 5.5. *Let Ω be an open set in $\mathbb{C}^n$. If* $f \in C^\infty(\Omega)$ *is a* q-*holomorphic function in* Ω *and if* $2q < n$, *then we have* $[\partial\bar{\partial}(\mathrm{Re}\, f)]^{2q+1} = 0$ *in* Ω. *Moreover,* $\log|f|$ *satisfies* $[\partial\bar{\partial}(\log|f|)]^{2q+1} = 0$ *in* Ω *except at the zeros of* f.

Proof. Since f is q-holomorphic,

$$\left[\frac{\partial^2 \mathrm{Re}\, f}{\partial z_i \partial \bar{z}_j}\right], \qquad i, j = 1, \ldots, n,$$

has at least $(n-q)$ nonnegative eigenvalues at each point of Ω by Theorem 3.3. This also applies to $-f$ so that

$$\left[\frac{\partial^2 \mathrm{Re}\, f}{\partial z_i \partial \bar{z}_j}\right]$$

also has at least $(n-q)$ nonpositive eigenvalues at each point. Suppose that

$$\left[\frac{\partial^2 \mathrm{Re}\, f}{\partial z_i \partial \bar{z}_j}\right]$$

has k zero eigenvalues at a point $p \in \Omega$. This implies that

$$[\partial\bar{\partial}\mathrm{Re}\, f]^r = 0 \qquad \text{for } r > (n-k) \text{ at p.}$$

Counting eigenvalues we have at least $n-q-k$ positive, at least $n-q-k$ negative, and k zero eigenvalues at p. Thus $(n-q-k)+(n-q-k)+k \le n$ and $(n-k) \le 2q < 2q+1$. Thus $[\partial\bar{\partial}\mathrm{Re}\, f]^{2q+1} = 0$. The statement concerning $\log|f|$ follows from the equation preceding Corollary 5.4.

Alternatively, we can argue that

$$[\partial\bar{\partial}f]^{q+1} = 0 \quad \text{and} \quad [\partial\overline{\partial f}]^{q+1} = 0$$

imply

$$\begin{aligned}[\partial\bar{\partial}(f+\bar{f})]^{2q+1} &= (\partial\bar{\partial}f + \partial\overline{\partial f})^{2q+1} \\ &= \sum_{j=0}^{2q+1}\binom{2q+1}{j}(\partial\bar{\partial}f)^j \wedge (\partial\overline{\partial f})^{2q+1-j} = 0.\end{aligned}$$

Let D be our bounded q-pseudoconvex domain in $\mathbb{C}^n$ with C^2 boundary. Recall that a q-plurisubharmonic function in D which is upper semicontinuous on $\bar{D}$ assumes its maximum for $\bar{D}$ on ∂D. The next result gives the relationship between the strictly q-pseudoconvex boundary points of D and the points on ∂D where q-plurisubharmonic functions in $\bar{D}$ can assume their peak values. A point $p \in \partial D$ is a *peak point* for the family of q-plurisubharmonic functions in D which are upper semicontinuous on $\bar{D}$ if there is such a function with $u(p) = 1$ and $u(z) < 1$ for all points $z \in \bar{D}$ with $z \neq p$. The value $u(p)$ is the *peak value* of u on $\bar{D}$.

THEOREM 5.6. *If* $p \in \partial D$ *is a strictly* q*-pseudoconvex boundary point of* D, *then there exists a* q*-plurisubharmonic function* u *in* D *which is continuous on* $\bar{D}$ *such that* u *assumes its peak value on* $\bar{D}$ *at* p. *Conversely, if* u *is a* C^2 q*-plurisubharmonic function in* $\bar{D}$ *which peaks at some point* $p \in \partial D$, *then* p *must be contained in the closure of the strictly* q*-pseudoconvex boundary points of* D.

Proof. Suppose $p \in \partial D$ is an arbitrary strictly q-pseudoconvex boundary point of D. Basener proves in [1] that there exists an open neighborhood U of p in $\mathbb{C}^n$ and a q-holomorphic function f in U such that $f(p) = 1$ and $0 < |f| < 1$ in $U \cap \bar{D} - \{p\}$. Then by Corollary 5.4., $\log|f|$ is a C^2 q-plurisubharmonic function in U which peaks locally at p. Suppose $-\infty < \log|f| < \alpha < 1$ for all $z \in U \cap \bar{D}$, where $\alpha > 0$. Then the maximum u of the functions $\log|f|$ and $\alpha/2$ is a q-plurisubharmonic function on D which is continuous on $\bar{D}$ and has its peak value at p.

The converse statement is proved by applying the arguments of Rossi [7] with plurisubharmonic and strictly pseudoconvex replaced by q-plurisubharmonic and strictly q-pseudoconvex respectively.

It is interesting that Theorem 5.6 is also true for the modulus of a q-holomorphic function (with the exception that one direction is only local). The first conclusion in the following theorem is proved by Basener in [1]. The converse statement is proved by replacing u in the statement of Theorem 3.6 by $\log|f|$ and applying Corollary 5.4.

THEOREM 5.7. *If* $p \in \partial D$ *is a strictly* q*-pseudoconvex boundary point of* D, *then there exists a neighborhood* U *of* p *in* $\mathbb{C}^n$ *and a* C^2 q*-holomorphic function* f *in* U *such that the peak value of* $|f|$ *on* $\bar{D} \cap U$ *is assumed at* p. *Conversely, if* f *is a* C^2 q*-holomorphic function in a neighborhood of* p *in* $\mathbb{C}^n$ *such that* $|f|$ *peaks at some point* $p \in \partial D$, *then* p *must be contained in the closure of the strictly* q*-pseudoconvex boundary points of* D.

Theorems 5.6 and 5.7 indicate that the boundary values for our generalized Dirichlet problem should be given at most on the closure of the strictly q-pseudoconvex boundary points for our q-pseudoconvex domain D. The following result, the proof of which is a direct analogue of one due to Bremermann, shows that we should not fail to give boundary values on an open neighborhood of a peak point of a function which is q-holomorphic in a neighborhood of $\bar{D}$.

THEOREM 5.8. *Let* D *be a* q*-pseudoconvex bounded domain in* $\mathbb{C}^n$. *Suppose* $p \in \partial D$ *is a peak point for the modulus of some function which is* q*-holomorphic in a neighborhood of* $\bar{D}$. *If the boundary values* b(z) *for our generalized Dirichlet problem are prescribed in such a way that a neighborhood of the point* p *is omitted, then the upper envelope of the class of* q*-plurisubharmonic functions in* D, *which are upper semicontinuous on* $\bar{D}$ *and less than or equal to* b(z) *on that part of* ∂D *where* b(z) *is given, does not exist.*

As conjectured by Rossi (and proved for $n = 2$) in [7], it is probably true that in the case that D is a domain of holomorphy in $\mathbb{C}^n$ with C^2 boundary, a point $p \in \partial D$ which is not in the closure of the strictly pseudoconvex boundary points of D must be contained in a complex variety of complex dimension at least one and contained in ∂D.

THEOREM 5.9. *Let* D *be a* q*-pseudoconvex domain in* $\mathbb{C}^2$ with C^2 *boundary. Let* $p \in \partial D$ *and suppose that the Levi form has exactly* s *negative eigenvalues and* t *zero eigenvalues for all points in an open neighborhood of* p *in* ∂D. *Then for some sufficiently small neighborhood of* p *in* $\mathbb{C}^n$ *there exist a complex variety* M *of complex dimension* s *through* p *and contained in* $\bar{D}$ *(actually intersecting* ∂D *only at* p) *and a complex variety* N *of complex dimension* t *through* p *and contained in* ∂D. *Moreover,* M *and* N *intersect transversally at* p.

Proof. If $t > 0$ the work of Freeman in [5] assures us that there exists a complex variety N of dimension t and an open neighborhood U of p in $\mathbb{C}^n$ such that $N \cap U \subset \partial D \cap U$ and $p \in N$. Choose coordinates for $\mathbb{C}^n$ and ∂D at p so that $p = 0$ and N is the $z_1, \dots z_t$ plane. Fixing the coordinates $z_1, \dots z_t$ at 0, we have a hypersurface (∂D restricted by $z_1 = 0, \dots, z_t = 0$) in $\mathbb{C}^{n-t}$ such that the Levi form on this hypersurface at p has s negative and no zero eigenvalues (*i.e.,* this hypersurface is strictly s-pseudoconvex at p in $\mathbb{C}^{n-t}$). By Proposition 6 in [1] there exists a complex variety M of dimension s and an open neighborhood V of p in $\mathbb{C}^{n-t}$ such that $M \cap V \subset \bar{D} \cap V$ and $\partial D \cap (M \cap V) = \{p\}$. Assume that we have chosen coordinates in $\mathbb{C}^n$ so that M is the $z_{t+1}, \dots, z_{s+t}$ plane. It is obvious that M and N intersect transversally at p.

Remark. The following question seems to be a difficult one to answer. Given the hypothesis of Theorem 5.9, does there exist a complex variety of dimension $(s + t)$ through p and contained in $\bar{D}$? If so and if $(s + t) \geq q + 1$, then by the maximum principle for q-holomorphic functions in [1], p cannot be a peak point for such functions.

REFERENCES

1. R. Basener, *Nonlinear Cauchy-Riemann equations and* q*-pseudoconvexity.* Duke Math J. 43(1976), 203–213.

2. E. Bedford and B. A. Taylor, *The Dirichlet problem for a complex Monge-Ampère equation.* Invent. Math. 37(1976), 1–44.

3. H. J. Bremermann, *On a generalized Dirichlet problem for plurisubharmonic functions and pseudo-convex domains. Characterization of Šilov boundaries.* Trans. Amer. Math. Soc. 91(1959) 246–276.

4. S. S. Chern, H. I. Levine, and L. Nirenberg, *Intrinsic norms on a complex manifold,* Global Analysis (Papers in honor of K. Kodaira), pp. 119–139. Univ. Tokyo Press, Tokyo, 1969.

5. M. Freeman, *Local complex foliations of real submanifolds.* Math. Ann. 209 (1974), 1–30.

6. L. Hörmander, *An introduction to complex analysis in several variables.* D. Van Nostrand Co., Inc., Princeton, N.J., 1966.

7. H. Rossi, *Holomorphically convex sets in several complex variables.* Ann. of Math.(2) 74(1961), 470–493.

8. J. B. Walsh, *Continuity of envelopes of plurisubharmonic functions.* J. Math. Mech. 18 (1968/69), 143–148.

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