ON A PROPERTY OF INDICATORS OF SMOOTH CONVEX BODIES

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1. INTRODUCTION

It is well known that among all convex bodies in $\mathbb{R}^n$ the sphere enjoys several exceptional geometric properties (see [6]). Therefore, it may seem surprising that from the viewpoint of harmonic analysis the sphere also possesses certain unsatisfactory properties that are not shared by all convex bodies. Thus, for instance, it was shown recently by Ch. Fefferman [12] that the sphere is not a good multiplier in the L^p-theory of multiple Fourier series. Another problem in harmonic analysis where the sphere exhibits an unexpected behavior is the question of existence of solutions to convolution equations (see [3], [4], and Problem (B) below). To formulate this problem properly, we need some preliminaries.

We shall use the standard notation of the theory of distributions [17], [18]. In particular, $\mathscr{E}'$ is the convolution algebra of all distributions with compact support in $\mathbb{R}^n$. If $\Phi \in \mathscr{E}'$, we denote by $\hat{\Phi}$ the Fourier transform of Φ; that is, $\hat{\Phi}(\zeta) = \Phi(e^{-i\langle x,\zeta\rangle})$, where

$$x \in \mathbb{R}^n, \qquad \zeta = \xi + i\eta \in \mathbb{C}^n, \qquad \langle x, \zeta \rangle = \sum_{j=1}^{n} x_j \zeta_j .$$

By S^{n-1} we shall denote the unit sphere in $\mathbb{R}^n$, that is, the boundary of the unit ball $\mathbb{B}^n$ in $\mathbb{R}^n$. If K is a subset of $\mathbb{R}^n$, we denote by ch K the convex hull of K. By an *extreme point* of K we shall mean an extreme point of ch K. If A and B are subsets of $\mathbb{R}^n$, the sets $A \pm B$ are defined as $\{z \in \mathbb{R}^n: z = x \pm y,\ x \in A,\ y \in B\}$, with the convention $A \pm \emptyset = \emptyset$. It is easy to see that for each pair of distributions $\Phi, \Psi \in \mathscr{E}'$, the singular support of the convolution $\Phi * \Psi$ satisfies the relation

$$\text{ch sing supp}\,(\Phi * \Psi) \subseteq \text{ch sing supp}\,\Phi + \text{ch sing supp}\,\Psi ;$$

however, the inclusion cannot, in general, be replaced by equality (see [2], [3], [4], [18]). It is therefore natural to say that a distribution $\Phi \in \mathscr{E}'$ *propagates singularities* provided for every $\Psi \in \mathscr{E}'$ it satisfies the condition

$$\text{(1)} \qquad \text{ch sing supp}\,(\Phi * \Psi) = \text{ch sing supp}\,\Phi + \text{ch sing supp}\,\Psi .$$

Every distribution with this property is also *invertible*, that is, it satisfies the weaker condition (see [18])

$$\text{(2)} \qquad \text{ch sing supp}\,\Psi \subseteq \text{ch sing supp}\,(\Phi * \Psi) - \text{ch sing supp}\,\Phi \qquad (\forall \Psi \in \mathscr{E}').$$

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However, the converse is not true, since there exist invertible distributions that do not propagate singularities. The following example shows that the difference between these two concepts is very subtle. Indeed, let T be a compact polyhedron in $\mathbb{R}^n$, and let $\chi_T(x)$ be its characteristic function (= the indicator of T). Then χ_T propagates singularities [3]. However, when $T = \mathbb{B}^n$ $(n \geq 2)$, the indicator $\chi_{\mathbb{B}^n}$, though still invertible, does not propagate singularities [4]. Similar statements are valid when T is the boundary of a polyhedron and when $T = S^{n-1}$, respectively. This curious situation led us to conjecture that from the viewpoint of propagation of singularities, the sphere and the ball may not be exceptional at all; that is, *the indicator of a smooth convex body (or surface) does not propagate singularities.* The objective of the present note is to establish a more general statement from which this conjecture will follow (see the theorem in Section 3). Moreover, it will turn out that the property of propagating singularities does not depend on convexity (see also G. Bengel [1]).

It is not without interest to reformulate the problem in terms of classical analysis. This can be done in two different ways:

(A) A function g will be called *smooth at the point* $p \in \mathbb{R}^n$ provided there exist a neighborhood U of p and a function $g^* \in C^\infty(U)$ such that $g = g^*$ almost everywhere in U. Now the problem can be formulated as follows.

Let T be a convex body in $\mathbb{R}^n$, and let $f \in L^1(\mathbb{R}^n)$ be a function with compact support. Set

$$F(z) = \int_T f(z - y)\, dy\,. \tag{3}$$

Let z_0 be an extreme point of $T + \operatorname{supp} f$. Then, by the Titchmarsh-Lions theorem, $z_0 \in \operatorname{supp} F$, and z_0 has a unique decomposition $z_0 = x_0 + y_0$, with $x_0 \in \operatorname{supp} f$ and $y_0 \in T$. Is it then true that for all f and z_0 as above, the smoothness of F at z_0 implies that of f at x_0?

The second formulation, essentially equivalent, of our problem reads as follows:

(B) Let Ω be an open convex set in $\mathbb{R}^n$, and let T be as above. Let u be a function (or distribution) with compact support contained in $\Omega' = \Omega + T$, and consider the integral equation

$$\int_T f(z - y)\, dy = u(z)\,. \tag{4}$$

Under what conditions on T is this equation solvable for all Ω and u? [For an exact and more general formulation of (B) and its relationship to (A), the reader is referred to [3]. As mentioned below, these problems can be considered for a convolution more general than that in (3), (4).]

Finally, it should be mentioned that as an immediate corollary of the asymptotic formulas (15) and (19) derived in the next section one obtains the following interesting fact about indicators of smooth convex bodies: While $\hat{\chi}_K(\zeta)$ is an entire function of asymptotic (or completely regular) growth in the sense of A. Pfluger and B. Ja. Levin (see [13], [19], [23]), $\hat{\chi}_K(\zeta)$ is not a function of asymptotic growth in the sense of [7]. (Indeed, growth of the latter type is a sufficient condition for propagation of singularities (see [7]), which, as we know, does not hold for $\hat{\chi}_K(\zeta)$.) In a more geometric language, this could also be expressed as follows: the entire function $\hat{\chi}_K(\zeta)$

grows regularly along (almost all) rays $\zeta = r\cdot z$ ($z \in \mathbb{C}^n$, z fixed, $r \to +\infty$), but does not grow regularly when ζ tends to infinity while remaining on a variety of logarithmic shape. Since the exact definitions are too technical to be stated here, we refer the reader to [7], [13]. — An analogous statement holds for indicators of smooth, compact, convex surfaces.

Remark. The solution of our conjecture on smooth convex bodies was announced in our note [5]. Recently, Bengel [1] showed by a different method that the characteristic function of an *arbitrary* (that is, not necessarily convex) C^∞-body T does not propagate singularities. The theorem established in Section 3 below represents a further generalization of this result.

2. AN ASYMPTOTIC FORMULA

The proof of the theorem in the next section is based on the knowledge of the asymptotic behavior of the function $|\hat{\chi}_S(\zeta)|$ for *complex* ζ, as $|\zeta| \to +\infty$. While the corresponding problem for real values $\zeta = \xi$ was studied by several authors [11], [14], [15], [16], [20], [21], [24], for complex ζ the problem was treated only for special surfaces S. Thus, for instance, when S is a sphere, the answer is given by the well-known asymptotic formulas for the Bessel functions $J_{n/2}(z)$ ($z \in \mathbb{C}^1$) (see [4], [25]). In the present section we shall consider the case of a general closed convex C^∞-surface S with positive Gaussian curvature. Moreover, the same result can be obtained for surfaces of class $C^{(k)}$, where $k = \left[\frac{n+1}{2}\right] + 3$ (see the remark at the end of this section).

In deriving the asymptotic formula for $|\hat{\chi}_S(\zeta)|$ for $\zeta \in \mathbb{C}^n$ ($|\zeta| \to +\infty$), we shall combine a multi-dimensional version of the method of stationary phase (see M. V. Fedoryuk [11], Theorem 2.2) with the approach employed by W. Littman [21], [22] in his study of the asymptotic behavior of the Fourier integral

$$\hat{\alpha}(\xi) = \int_S \alpha(x)\, e^{-i\langle x,\xi\rangle}\, d\sigma(x),$$

where ξ is a real vector and $d\sigma(x)$ denotes the surface element. However, it should be mentioned that the same result can also be obtained by other methods (see for example [14], [15]). Littman's approach has the advantage of also being applicable to the study of a convex neighborhood of an extreme point p on a *nonconvex* surface S, at which some of the principal curvatures may vanish (see [20, p. 769]). We shall therefore limit ourselves to closed convex surfaces with positive Gaussian curvature. Furthermore, without loss of generality, we may assume that the origin lies in the interior of the surface S, in other words, that if

$$h_S(\theta) = \max_{x\in S} \langle x, \theta\rangle \qquad (\theta \in \mathbb{R}^n) \tag{5}$$

denotes the supporting function of S, then $h_S(\theta) > 0$ for all $\theta \neq 0$. Let T be the convex body bounded by S.

Fix $\omega^0 \in S^{n-1}$ and consider $\zeta = r\omega^0 + i\eta$ ($r > 0$). Let x^0 be the point defined by the equation $x^0 = x(\omega^0)$, where we set

$$x(\omega) = \operatorname{grad} h_S(\omega); \tag{6}$$

x^0 is then the only point where the tangent plane to S with the exterior normal ω^0 touches the surface S (see [6], Section 26). The surface S can be covered by finitely many open pieces S^j $(0 \leq j \leq \ell)$, so that for each j there is an integer $k = k_j$ $(1 \leq k \leq n)$ such that the surface S^j has the parametric representation $x_k = g_k(\tilde{x}^k)$, where

$$\tilde{x}^k \overset{\text{def}}{=} (x_1, \cdots, x_{k-1}, x_{k+1}, \cdots, x_n) \in X_k$$

and X_k is an open subset in $\mathbb{R}^{n-1}$. We may assume that $x^0 \in S^0$ and $k_0 = n$. Let $\iota_0, \iota_1, \cdots, \iota_\ell$ be a partition of unity on S, subordinate to the covering $\{S^j\}$, such that $\iota_0 \equiv 1$ in a neighborhood U of x^0, supp $\iota_0 \subseteq S^0$. Rotating the axes, if necessary, we may assume that $\omega^0 = (0, \cdots, 0, 1)$. Set

$$S(z) = \{x \in S: x_n \leq z\}, \qquad S_z = \partial S(z), \qquad \alpha_j(x; \eta) = \iota_j(x) e^{\langle x, \eta \rangle}.$$

Then

$$\hat{\chi}_S(\zeta) = I_0 + \sum_{j=1}^{\ell} I_j = I_0 + J \qquad (\zeta = r\omega^0 + i\eta), \tag{7}$$

where

$$I_j(\zeta) = \int_S e^{-i\langle x, r\omega^0 \rangle} \alpha_j(x; \eta)\, d\sigma(x) \qquad (0 \leq j \leq \ell). \tag{8}$$

Furthermore, set

$$A = \sum_{j=1}^{\ell} \alpha_j, \qquad a(z; \eta) = \int_{S(z)} A(x; \eta)\, d\sigma(x). \tag{9}$$

By Fubini's theorem,

$$\begin{cases} a(z; \eta) = \displaystyle\int_{-\infty}^{z} dw \int_{S_w} A(\tilde{x}^n, w; \eta)\, d\sigma(\tilde{x}^n), \\ J(\zeta) = \displaystyle\int_{-\infty}^{\infty} \left(e^{-irz} \int_{S_z} A(\tilde{x}^n, z; \eta)\, d\sigma(\tilde{x}^n) \right) dz, \end{cases} \tag{10}$$

which yields the formula

$$J(\zeta) = \int_{-\infty}^{\infty} e^{-irz} \frac{\partial a(z; \eta)}{\partial z}\, dz. \tag{11}$$

Set

$$T_0 = \operatorname{ch}(S \setminus U) \tag{12}$$

and $c = h_S(-\omega^0)$, $d = h_{T_0}(\omega^0)$ (see (5), (12)). Then $\{x \in S: x_n > d\} \subset S^0$. Hence, for every η, $\operatorname{supp}_{x_n} A \subset [c, d]$ ($\operatorname{supp}_s f(s, t)$ denotes the support of f in the variable s for fixed t); thus we see that $\mathrm{supp}_z\left(\frac{\partial a}{\partial z}\right) \subset [c, d]$ for all η. Moreover, we claim that $\frac{\partial a}{\partial z} \in C^\infty[c, d]$ (as a function of z). Indeed, if we set $a(z; \eta) = \sum_{j=1}^{\ell} a_j(z; \eta)$, where

$$a_j(z; \eta) = \int_{S(z)} \alpha_j(x; \eta)\, d\sigma(x) \qquad (1 \le j \le \ell), \tag{13}$$

then $\mathrm{supp}_x\, \alpha_j \subset S^k$, where $k = k_j$, as above. Hence, by (9), (10), and (13),

$$\frac{\partial a_j}{\partial z}(z; \eta) = \int_{S_z} \alpha_j(\tilde{x}^n, z; \eta)\, d\sigma(\tilde{x}^n).$$

Expressing the last (n - 2)-dimensional surface integral in the local parameter $x_k = g_k(\tilde{x}^k)$, we find that $\frac{\partial a_j}{\partial z}$ is a C^∞-function. Hence, by repeated integration by parts in (11), we get the formula

$$J(\omega^0, r, \eta) = (ir)^{-\nu} \int_{-\infty}^{\infty} e^{-irz}\left(\frac{\partial}{\partial z}\right)^{\nu+1} a(z; \eta)\, dz. \tag{14}$$

Taking into account the special form of $a(z; \eta)$, we can easily obtain from (14) the estimate

$$|J(\omega^0, r, \eta)| \le \frac{c_\nu}{r^\nu}(1 + |\eta|)^{\nu+1} \exp h_{T_0}(\eta), \tag{15}$$

where the constant c_ν also depends on ω^0.

Moreover, it is easy to see that if we take a small neighborhood U_0 of x^0 ($U_0 \subset U$), the estimate (15) will hold with a larger c_ν for an entire cone G_0 of points ω consisting of all ω for which $x = x(\omega) \in U_0$ (see (6)). If U_0 is sufficiently small, the estimate (15) will hold with some T_0 depending only on U_0 and not on ω^0. Since S is compact, we finally obtain a finite collection of estimates (15) such that one of them applies to each point of the surface S.

Next we have to consider the integral I_0 (see (7), (8)), which represents the main contribution to $\hat{\chi}_S$. Let us write $t = \tilde{x}^n$, $t^0 = (x_1^0, \cdots, x_{n-1}^0)$, $g = g_n$, and choose $\delta > 0$ so that

$$\mathrm{supp}\, \iota_0(t, g(t)) \subset \Delta_\delta = \{t \in \mathbb{R}^{n-1}: |t - t^0| < \delta\}.$$

Then

$$I_0(\zeta) = \int_{\Delta_\delta} \exp(-irg(t)) f(t; \eta)\, dt, \tag{16}$$

where

$$f(t; \eta) = \iota_0(t, g(t)) \exp(\langle t, \tilde{\eta}^n\rangle + g(t)\eta_n)[1 + |\mathrm{grad}\, g(t)|^2]^{1/2}.$$

Let $H_g(t)$ be the Hessian matrix of g at t; that is, let

$$H_g(t) = \left(\frac{\partial^2 g(t)}{\partial t_i\, \partial t_j}\right)_{i,j}.$$

Since grad $g(t^0) = 0$ and $|\det H_g(t^0)| = K_S(x^0) > 0$, we see that t^0 is a nondegenerate critical point of g. Hence to the integral (16) we can apply the following multi-dimensional version of the stationary-phase method as it appears in the paper of Fedoryuk [11, Theorem 2.2] {this theorem is obtained from the one-dimensional method of stationary phase [10] combined with a lemma of M. Morse [22]; see [11]}.

THEOREM. *Let Ω be a bounded domain in $\mathbb{R}^m$, and let $f \in C_0^\infty(\Omega)$. Furthermore, let g be a real-valued C^∞-function in Ω whose only critical point t^0 in Ω is nondegenerate. Then there exists a positive integer N such that the integral*

$$I(r) = \int_\Omega f(t) \exp(irg(t))\, dt$$

can be written in the form

$$I(r) = \left(\frac{2\pi}{r}\right)^{m/2} |\det H_g(t^0)|^{-1/2} f(t^0) \exp\left(irt^0 + \frac{i\pi}{4}\sigma_g(t^0)\right) + R(r), \tag{17}$$

where $\sigma_g(t)$ denotes the signature of $H_g(t)$ and the remainder $R(r)$ satisfies the estimate

$$|R(r)| \leq c(g)\, \|f\|_{C^{(N)}(\Omega)}\, r^{-m/2-1} \qquad (c(g) = \text{const}). \tag{18}$$

In our case, we set $m = n - 1$, replace g by $-g$ (hence $\sigma_g(t^0) = 1 - n$), and take into account the role of η. After a simple calculation we obtain from (17) and (18) the equation

$$\begin{aligned} I_0(\zeta) &= \left(\frac{2\pi}{r}\right)^{(n-1)/2} (K_S(x^0))^{-1/2} \exp\left(\frac{i\pi}{4}(1-n) - i\langle x^0, \zeta\rangle\right) \\ &\quad\times \left[1 + \frac{C_1}{r}(1+|\eta|)^N\right], \end{aligned} \tag{19}$$

where C_1 is bounded. This relation, together with estimate (15), describes the asymptotic behavior of $|\hat{\chi}_S(\zeta)|$ for $\zeta \in \mathbb{C}^n$ ($|\zeta|$ large).

Remark. Let $\phi \in C^\infty(S)$ be such that $\phi(x^0) \neq 0$. Then the proof above (that is, the proof for $\phi \equiv 1$) carries over to this more general case. Moreover, it is not necessary to limit ourselves to measures with C^∞-densities carried by C^∞-surfaces. A careful analysis (see [15], [20], for example) would show that it suffices to take S of class $C^{(\nu)}$, where $\nu = [(n+1)/2] + 3$, and $\phi \in C^{(\nu-1)}(S)$. However, whether similar results hold for smaller values of ν (for example, $\nu = 1, 2$) seems to be an open question. Finally, the assumption about the Gaussian curvature is not essential either. If at least one of the principal curvatures at x^0 does not vanish, an estimate analogous to (15) and (19) still holds. (For real values $\zeta = \xi$, this is shown in [20].) Since the proofs of these statements are similar to the proof of (15) and (19), but technically more complicated, they will be omitted.

3. THE THEOREM

Now we are able to state and prove the theorem mentioned in Section 1:

THEOREM. *Let* S *be a closed surface in* $\mathbb{R}^n$, *of class* $C^{(\nu)}$, *where* $\nu = [(n+1)/2] + 3$. *Let* ϕ *be a distribution on* S *for which there exists an extreme point* x^0 *on* S *such that* $K_S(x^0) \neq 0$, $\phi(x^0) \neq 0$, *and* ϕ *is of class* $C^{(\nu-1)}$ *in some neighborhood* U *of* x^0. *Then* ϕ *does not propagate singularities. A similar statement holds for distributions supported by the compact set* T *with boundary* $\partial T = S$.

The proof of this theorem is based on Hörmander's description (see [18]) of the set ch sing supp Φ ($\Phi \in \mathscr{E}'$), which we shall recall briefly.

Given $\Phi \in \mathscr{E}'$ and $\xi \in \mathbb{R}^n$, consider the plurisubharmonic function $v_\Phi(z;\xi)$ ($z \in \mathbb{C}^n$) defined by the formula

$$v_\Phi(z;\xi) = \frac{\log|\hat{\Phi}(\xi + z\log|\xi|)|}{\log|\xi|}. \tag{20}$$

Every sequence $\{\xi_j\}$ ($|\xi_j| \to +\infty$) contains a subsequence $\{\xi_{j_k}\}$ such that $\lim_{k\to\infty} v_\Phi(z;\xi_{j_k}) = v(z)$ is a plurisubharmonic function (see [18, p. 293]). It turns out that the function

$$h_v(y) = \lim_{t\to+\infty} \frac{\sup\{v(z):\ \Im z = ty\}}{t} \tag{21}$$

is the supporting function of some compact convex set R_v [18, p. 288]. Let $\mathscr{H}(\Phi)$ denote the family of all functions h_v resulting from the distribution Φ in this manner. The family $\mathscr{H}(\Phi)$ completely describes the set ch sing supp Φ, for it can be shown (see [18, p. 293]) that the supporting function $H(y)$ of the set ch sing supp Φ is given by $H(y) = \sup\{h(y):\ h \in \mathscr{H}(\Phi)\}$. Using a construction of L. Ehrenpreis [9], Hörmander proves the following remarkable fact (see [18, Theorem 5.3]):

(H) *Let* $\Phi \in \mathscr{E}'$ *and* $h_v \in \mathscr{H}(\Phi)$. *Then there exists* $\Psi \in \mathscr{E}'$ *with* sing supp $f = \{0\}$ *such that the supporting function of the set* ch sing supp$(\Phi * \Psi)$ *is the function* h_v.

Proof of the theorem. First we shall establish the theorem in the special case when S is a convex C^∞-surface, $x^0 \in S$, $K_S(x^0) > 0$, and ω^0 is the exterior normal vector to the surface S at x^0. Then, as was shown in Section 2, there exist sets T_0 and G_0, with $\omega^0 \in G_0 \subset S^{n-1}$, such that if η is a nonzero vector in $\mathbb{R}^n$ and $\omega \in G_0$, then $h_{T_0}(\eta) < \langle x(\omega), \eta\rangle$ and inequality (15) holds.

Set $\nu = [(n+1)/2]$, $z = w + ity$ ($t > 0$, $y \in G_0$), and

$$\omega(w, r) = \frac{r\omega^0 + w\log r}{|r\omega^0 + w\log r|} \qquad (w \in \mathbb{R}^n,\ r > 0). \tag{22}$$

If r is sufficiently large, then $\omega(w,r) \in G_0$. Hence, combining estimate (15) with the asymptotic formula (19), we obtain the estimate

$$\frac{\log|\hat{\chi}_S(r\omega^0 + w\log r + ity\log r)|}{\log r} = t\langle x(\omega), y\rangle + \frac{1-n}{2} + o(1), \tag{23}$$

where $\omega = \omega(w, r)$ is given by (22). However, as $r \to +\infty$, $x(\omega) \to x(\omega^0) = x^0$. Thus the limit of the quotient in (23) is $t \langle x^0, y \rangle + [(1-n)/2]$. Hence, by (20) and (21), each function h_v obtained from $\hat{\chi}_S$ by means of some sequence $\xi_n = r_n \omega^0$ $(r_n \nearrow +\infty)$ is such that

$$h_v(y) = \langle x^0, y \rangle \qquad (y \in G_0). \tag{24}$$

By (H), there is a $\Psi \in \mathscr{E}'$ such that ch sing supp$(\Phi * \Psi) = h_v$. Hence, if (1) were satisfied for this Ψ and $\Phi = \chi_S$, then by (24) it would also imply

$$h_S(y) = \langle x^0, y \rangle \qquad (y \in G_0);$$

in other words, the smooth surface S would have a "corner" at the point x^0, which is impossible.

To prove the theorem in full generality, it suffices to show how to reduce the case of a nonconvex surface to the convex case, and then apply the Remark at the end of Section 2.

Let S be a closed nonconvex surface of class $C^{(2)}$, and let x^0 be an extreme point with $K_S(x^0) \neq 0$. Then $x^0 \in S$ and $K_S(x) > 0$ in some neighborhood V of x^0 on S. It is not difficult to show (for details, see [8]) that some neighborhood W of x^0 in S $(W \subseteq V)$ actually lies on the convex surface $S^* = \partial(\text{ch } S)$. Hence the asymptotic formula (19) holds for S^*. Because of its local character and in view of the inclusion $x^0 \in W \subset S \cap S^*$, formula (19) also holds for S. On the other hand, estimate (15) cannot be immediately generalized to S^*, because even when S is a C^∞-surface, its convex hull may not be of class $C^{(2)}$ (see [8]). However, since we never used the convexity of the set $S \setminus U$ in the derivation of inequality (15), we still may apply (15). In view of the Remark of Section 2, this concludes the proof of the first part of the theorem. The corresponding statement for convex bodies can be obtained either directly by the same method as above, or by reduction to the previous case by means of the divergence theorem (see [15], [16]).

Remarks. (i) As we noted above, it is not clear whether the theorem holds for surfaces S that are not smooth enough, for example, for surfaces of class $C^{(\nu)}$ $(\nu \leq n/2)$. It seems that while the result may still hold for $\nu \geq 2$, it could be false for some $C^{(1)}$-surface S that is not of class $C^{(2)}$. Indeed, the geometric structure of $C^{(1)}$-surfaces is more complicated; thus, for instance, it is not difficult to construct a closed convex $C^{(1)}$-surface S that is nowhere strictly convex, in other words, such that each open subset of S contains a straight-line segment (see [8]).

(ii) Consider again the indicator χ_S of a closed convex C^∞-surface S with positive Gaussian curvature. We know that the function χ_S does not propagate singularities. However, it is not difficult to verify that χ_S is invertible, that is, satisfies (2). The proof proceeds along the same lines as the proof of the same statement for the sphere (see [4]). One has only to use the asymptotic formulas for $|\hat{\chi}_S(\xi)|$ $(\xi \in \mathbb{R}^n,\ |\xi| \to \infty)$, as given in [14], [16], [18].

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