A MODEL FOR QUASINILPOTENT OPERATORS

Ciprian Foiaş and Carl Pearcy

Whether every quasinilpotent operator on a separable, infinite-dimensional, complex Hilbert space has a nontrivial invariant (hyperinvariant) subspace is and has long been a stubborn and intractable open question. The purpose of this note is to establish the existence of a model (up to similarity) for such operators (Theorem 1), and to discuss some of the consequences of the existence of this model. It is believed that these results are pertinent to the invariant-subspace problem mentioned above.

We begin by recalling some notation and terminology. Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be separable, infinite-dimensional, complex Hilbert spaces. If X: $\mathscr{H}_1 \to \mathscr{H}_2$ is a bounded linear transformation such that kernel X = kernel X* = $\{0\}$, then X is called a *quasiaffinity*. If A_1 and A_2 are bounded operators on $\mathscr{H}_1$ and $\mathscr{H}_2$, respectively, and there exists a quasiaffinity X: $\mathscr{H}_1 \to \mathscr{H}_2$ such that $XA_1 = A_2 X$, we say that A_1 is a *quasiaffine transform* of A_2, and we write $A_1 \prec A_2$. If A_1 and A_2 are quasiaffine transforms of each other, that is, if there exist quasiaffinities X: $\mathscr{H}_1 \to \mathscr{H}_2$ and Y: $\mathscr{H}_2 \to \mathscr{H}_1$ such that $XA_1 = A_2 X$ and $A_1 Y = YA_2$, then A_1 and A_2 are said to be *quasisimilar*. It is known that if A_1 and A_2 are quasisimilar operators, and A_1 has a nontrivial hyperinvariant subspace, then so does A_2 (see [2], [4], [7]). In the remainder of the paper, $\mathscr{H}$ will always be a separable, infinite-dimensional, complex Hilbert space, and $\mathscr{L}(\mathscr{H})$ will denote the algebra of all bounded linear operators on $\mathscr{H}$. Recall that an operator A is a *part* of an operator B if A is the restriction of B to some invariant subspace of B. The following structure theorem seems interesting and has some noteworthy consequences.

THEOREM 1. *Let* T *be a quasinilpotent operator in* $\mathscr{L}(\mathscr{H})$. *Then there exists a compact* (*quasinilpotent*) *weighted backward shift* K *in* $\mathscr{L}(\mathscr{H})$ *such that* T *is similar to a part of the operator* $K \oplus K \oplus \cdots \oplus K \oplus \cdots$ *acting on the direct sum of countably many copies of* $\mathscr{H}$.

Proof. We treat only the case that T is not nilpotent. The case in which T is nilpotent follows by an obvious modification of our argument below. Consider the (well-defined) sequences

(1) $$\alpha_n = \|T^n\|^{1/2} \quad (n = 0, 1, 2, \cdots),$$

(2) $$\omega_n = \alpha_n/\alpha_{n-1} \quad (n = 1, 2, 3, \cdots),$$

and observe that they satisfy the conditions

(3) $$0 < \alpha_{m+n} \leq \alpha_m \alpha_n,$$

(4) $$\alpha_n^{1/n} \to 0,$$

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(5) $$\sum_{n=0}^{\infty} \alpha_n^2 < +\infty,$$

(6) $$0 < \omega_n \leq \|T\|^{1/2},$$

(7) $$\omega_1 \cdot \omega_2 \cdot \cdots \cdot \omega_n = \alpha_n.$$

We define $\mathscr{H}_\infty$ to be the direct sum of countably many copies of $\mathscr{H}$ indexed by the positive integers, and we define the weighted backward shift W on $\mathscr{H}_\infty$ by the equation

(8) $$W(h_1, h_2, \cdots, h_n, \cdots) = (\omega_1 h_2, \omega_2 h_3, \cdots, \omega_n h_{n+1}, \cdots).$$

It follows from (6) that W is bounded, and (3), (4), and (7) imply that W is quasinilpotent. Consider next the operator X: $\mathscr{H} \to \mathscr{H}_\infty$ defined by the equation

(9) $$Xh = (h, Th/\alpha_1, T^2h/\alpha_2, \cdots, T^{n-1}h/\alpha_{n-1}, \cdots).$$

Clearly, X is bounded below by 1 and thus X is one-to-one and has closed range. Furthermore, the inequality

$$\|Xh\|^2 = \sum_{n=0}^{\infty} \|T^n h/\alpha_n\|^2 \leq \|h\|^2 \sum_{n=0}^{\infty} \|T^n\|$$

together with (5) shows that X is bounded. An elementary calculation now shows that XT = WX. If the sequence $\{\omega_n\}$ were convergent, then, because of (4) and (7), its limit would have to be zero, and W would be (unitarily equivalent to) an operator of the form $K \oplus K \oplus \cdots \oplus K \oplus \cdots$ where K is a compact weighted backward shift. But it is easy to construct examples of quasinilpotent operators T such that the sequence $\{\omega_n(T)\}$ does not converge (for example, let T be a weighted backward shift on (ℓ_2) with weight sequence 1, 1/2, 1, 1/4, 1, 1/8, $\cdots$). Thus we want to "replace" the operator W with a weighted backward shift whose weight sequence tends to zero, and to this end we make some definitions. If n is a positive integer, then there exist unique nonnegative integers k_n and m_n such that

(10) $$n = 2^{k_n} + m_n \quad \text{and} \quad 1 \leq m_n \leq 2^{k_n}.$$

Clearly, the sequence $\{k_n\}$ is increasing and converges to $+\infty$. We employ the sequence $\{k_n\}$ to define the sequence

(11) $$\kappa_n = (\alpha_{2^{k_n}}^{1/2^{k_n}})^{1/2}(\alpha_1)^{1/2} \qquad (n = 1, 2, \cdots),$$

where $\{\alpha_n\}$ is the sequence defined in (1). From (3) we know that $\alpha_{2n} \leq \alpha_n^2$ for all n and thus that

(12) $$\alpha_{2^{k+1}}^{1/2^{k+1}} \leq \alpha_{2^k}^{1/2^k} \qquad (k = 1, 2, \cdots).$$

This, together with the fact that the sequence $\{k_n\}$ is increasing, implies that the sequence $\{\kappa_n\}$ is decreasing, and it follows from (4) that $\kappa_n \to 0$. The last sequence we shall need is defined by induction. We set $\sigma_1 = 1$ and in general

$$\sigma_{n+1} = \sigma_n \frac{\omega_n}{\kappa_n} \quad (n = 1, 2, \cdots). \tag{13}$$

Clearly, all σ_n are positive. To show that the sequence $\{\sigma_n\}$ is bounded, we note that by virtue of (7), (13), and the fact that the sequence $\{\kappa_n\}$ is decreasing,

$$0 < \sigma_{n+1} = \frac{\omega_1 \cdot \omega_2 \cdot \cdots \cdot \omega_n}{\kappa_1 \cdot \kappa_2 \cdot \cdots \cdot \kappa_n} \leq \frac{\alpha_n}{\kappa_n^n}. \tag{14}$$

We show that the sequence $\{\sigma_n\}$ is bounded above by 1 by proving that

$$\alpha_n \leq \kappa_n^n \quad (n = 1, 2, \cdots). \tag{15}$$

Clearly, (15) is equivalent to the inequality

$$\alpha_n^{1/n} \leq \kappa_n \quad (n = 1, 2, \cdots), \tag{16}$$

which we verify as follows. Write $n = 2^{k_n} + m_n$, as in (10). Then, by (3),

$$\alpha_n^{1/n} \leq (\alpha_{2^{k_n}} \alpha_{m_n})^{1/n} = (\alpha_{2^{k_n}})^{1/n} (\alpha_{m_n})^{1/n}$$

$$\leq (\alpha_{2^{k_n}}^{1/2^{k_n}})^{2^{k_n}/n} (\alpha_1)^{m_n/n} = \left[\frac{\alpha_{2^{k_n}}^{1/2^{k_n}}}{\alpha_1}\right]^{2^{k_n}/n} (\alpha_1).$$

Since the number inside the brackets is dominated by 1 by virtue of (3) or (12), a larger number may be obtained by replacement of the exponent $2^{k_n}/n$ by a smaller exponent. By (10), $1/2 \leq 2^{k_n}/n$, and thus

$$\alpha_n^{1/n} \leq \left[\frac{\alpha_{2^{k_n}}^{1/2^{k_n}}}{\alpha_1}\right]^{1/2} (\alpha_1) = \kappa_n,$$

which establishes (16) and completes the proof that the sequence $\{\sigma_n\}$ is bounded.
We now define the operators Y and J in $\mathscr{L}(\mathscr{H}_\infty)$ as follows:

$$\begin{cases} Y(h_1, h_2, \cdots, h_n, \cdots) = (\sigma_1 h_1, \sigma_2 h_2, \cdots, \sigma_n h_n, \cdots), \\ J(h_1, h_2, \cdots, h_n, \cdots) = (\kappa_1 h_2, \kappa_2 h_3, \cdots, \kappa_n h_{n+1}, \cdots). \end{cases} \tag{17}$$

Clearly, Y is a positive quasiaffinity and J is a weighted backward shift of infinite multiplicity. Moreover, by virtue of (13), $YW = JY$, and thus

$$YXT = YWX = JYX. \tag{18}$$

Note also that the product $YX\colon \mathscr{H} \to \mathscr{H}_\infty$ is bounded below by 1 (since $\sigma_1 = 1$), and hence has closed range. Choose now an orthonormal basis $\{e_n\}_{n=1}^\infty$ for $\mathscr{H}$, and let K denote the operator in $\mathscr{L}(\mathscr{H})$ satisfying the equations $Ke_1 = 0$ and $Ke_{n+1} = \kappa_n e_n$ for $n \geq 1$. Since the weight sequence $\{\kappa_n\}$ converges to 0, K is a compact weighted backward shift, and thus is necessarily quasinilpotent. Furthermore, it is obvious that the operator J is unitarily equivalent to the ampliation

$$\tilde{K} = K \oplus K \oplus \cdots \oplus K \oplus \cdots$$

acting on $\mathcal{H}_\infty$, and thus there exists a unitary operator U in $\mathcal{L}(\mathcal{H}_\infty)$ satisfying the equation $UJ = \tilde{K}U$. Hence it follows from (18) that

$$UYXT = UJYX = \tilde{K}UYX. \tag{19}$$

Moreover it is clear that the operator $UYX\colon \mathcal{H} \to \mathcal{H}_\infty$ is bounded below by 1 and thus has closed range $\mathcal{R} \subset \mathcal{H}_\infty$. Let $S\colon \mathcal{H} \to \mathcal{R}$ be the invertible operator defined by the equation $Sh = UYXh$ ($h \in \mathcal{H}$), and note that it follows from (19) that $\tilde{K}\mathcal{R} \subset \mathcal{R}$. Thus (19) can be rewritten as $ST = (\tilde{K} \mid \mathcal{R})S$, or, equivalently, as

$$STS^{-1} = \tilde{K} \mid \mathcal{R}. \tag{20}$$

Thus T is similar to a part of the operator $\tilde{K}$, and the proof is complete.

A first consequence of Theorem 1 is the following.

THEOREM 2. *Let* T *be a quasinilpotent operator in* $\mathcal{L}(\mathcal{H})$, *and let* $\tilde{K}$ *be the operator constructed in Theorem* 1 *and satisfying* (20). *Then* $\tilde{K}$ *is quasisimilar to a compact operator* L *in* $\mathcal{L}(\mathcal{H}_\infty)$; *that is, there exist quasiaffinities* X *and* Y *in* $\mathcal{L}(\mathcal{H}_\infty)$ *satisfying the equations*

$$\tilde{K}X = XL, \qquad Y\tilde{K} = LY. \tag{21}$$

Proof. By construction, $\tilde{K} = K \oplus \cdots \oplus K \oplus \cdots$, where K is a quasinilpotent compact operator in $\mathcal{L}(\mathcal{H})$. It follows from a theorem of G.-C. Rota [3, p. 77] that for every positive integer n, there exists an invertible operator Q_n on $\mathcal{H}$ such that the compact operator $L_n = Q_n K Q_n^{-1}$ satisfies the inequality $\|L_n\| < 1/n$. We define the operators L, X, and Y on $\mathcal{H}_\infty$ by the equations

$$L = L_1 \oplus L_2 \oplus \cdots \oplus L_n \oplus \cdots,$$

$$X = \gamma_1 Q_1^{-1} \oplus \gamma_2 Q_2^{-1} \oplus \cdots \oplus \gamma_n Q_n^{-1} \oplus \cdots,$$

$$Y = \delta_1 Q_1 \oplus \delta_2 Q_2 \oplus \cdots \oplus \delta_n Q_n \oplus \cdots,$$

where $\{\gamma_n\}$ and $\{\delta_n\}$ are sequences of positive numbers chosen so that $\sup_n \|\gamma_n Q_n^{-1}\|$ and $\sup_n \|\delta_n Q_n\|$ are bounded. It is clear that L is compact and that X and Y are quasiaffinities. Moreover, since $\delta_n Q_n K = \delta_n L_n Q_n$ and $\gamma_n Q_n^{-1} L_n = \gamma_n K Q_n^{-1}$ for all n, an elementary computation shows that (21) is valid, and the proof is complete.

The following consequence of Theorem 2 seems remarkable.

THEOREM 3. *Let* T *be a nonzero quasinilpotent operator in* $\mathcal{L}(\mathcal{H})$. *Then there exist nonzero compact operators* K_1 *and* K_2 *such that* T *is a quasiaffine transform of* K_2 *and such that* K_1 *is a quasiaffine transform of* T, *that is,* $K_1 \prec T \prec K_2$.

Proof. It suffices to show that T is the quasiaffine transform of a compact operator K_2, since the same argument applied to T^* shows that T^* is the quasiaffine transform of a compact operator K_1^*, and thus that K_1 is the quasiaffine transform of T. Let $\tilde{K}$ and $\mathcal{R}$ be as in Theorem 1, so that $\mathcal{R}$ is an invariant subspace of $\tilde{K}$ and T is similar to $\tilde{K} \mid \mathscr{R}$. It follows easily that it suffices to show that $\tilde{K} \mid \mathscr{R}$ is the quasiaffine transform of a compact operator. The argument for that goes as follows.

By virtue of Theorem 2, we know that there exist a compact operator L and a quasiaffinity Y in $\mathscr{L}(\mathscr{H}_\infty)$ such that $Y\tilde{K} = LY$. Let $\mathscr{L}_1$ denote the closure of the linear manifold $Y\mathscr{R}$, and note that $L\mathscr{L}_1 \subset \mathscr{L}_1$. Let L_1 denote the compact operator $L \mid \mathscr{L}_1$, and let $Y_1: \mathscr{R} \to \mathscr{L}_1$ denote the quasiaffinity defined by $Y_1 r = Yr$ for every r in $\mathscr{R}$. Then $Y_1(\tilde{K} \mid \mathscr{R}) = L_1 Y_1$, and the theorem is proved.

Theorems 2 and 3 give rise to many interesting questions. For example: is every quasinilpotent operator quasisimilar to a compact operator? The next theorem indicates that the answer might be affirmative.

THEOREM 4. *Every nilpotent operator in* $\mathscr{L}(\mathscr{H})$ *is quasisimilar to a compact operator.*

Proof. An operator J in $\mathscr{L}(\mathscr{H})$ is called a *Jordan operator* [1] if there exists a decomposition of $\mathscr{H}$ as an orthogonal direct sum $\mathscr{H} = \sum_{n=1}^{\infty} \oplus \mathscr{H}_{k_n}$, where for each n, the space $\mathscr{H}_{k_n}$ is a finite-dimensional reducing subspace of J of dimension k_n and $J_n = J \mid \mathscr{H}_{k_n}$ has a matrix relative to an orthonormal basis for $\mathscr{H}_{k_n}$ that is a single Jordan block. (We assume that the operator 0 on a one-dimensional space is a single Jordan block.) According to [1, Theorem 1], every nilpotent operator T in $\mathscr{L}(\mathscr{H})$ is quasisimilar to a Jordan operator J in $\mathscr{L}(\mathscr{H})$. And, as we indicated above, $J = \sum_{n=1}^{\infty} \oplus J_n$, where each J_n acts on a finite-dimensional space and is either the operator 0 on a one-dimensional space or has a matrix consisting of a single Jordan block. In either case, it is trivial to show that J_n is similar to the operator $(1/n)J_n$, and it follows easily (as in the proof of Theorem 2) that J, and thus T, is quasisimilar to the compact operator $\sum_{n=1}^{\infty} \oplus (1/n)J_n$. Hence the proof is complete.

The following proposition shows, however, that the answer to the question posed before Theorem 4 is negative.

THEOREM 5. *If* T *is an operator that is quasisimilar to a nonzero compact operator, then* T *commutes with a nonzero compact operator. Furthermore, there exists a quasinilpotent operator in* $\mathscr{L}(\mathscr{H})$ *that does not commute with any nonzero compact operator, and hence is not quasisimilar to any compact operator.*

Proof. Suppose that $TX = XK$ and $YT = KY$, where X and Y are quasiaffinities and K is a nonzero compact operator. Then $TXKY = XK^2Y = XKYT$; thus T commutes with the nonzero compact operator XKY. To prove the second statement of the proposition, let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis for $\mathscr{H}$, and let T be the weighted backward shift on $\mathscr{H}$ defined by $Te_1 = 0$ and $Te_{n+1} = \tau_n e_n$ $(n = 1, 2, \cdots)$, where the weight sequence $\{\tau_n\}_{n=1}^{\infty}$ is defined as follows. If $n \equiv 1 \pmod 2$, set $\tau_n = 1/2$. If $n \equiv 2 \pmod 4$, set $\tau_n = 1/2^4$. In general, if $n \equiv 2^k \pmod{2^{k+1}}$, set $\tau_n = 1/2^{4^k}$. An easy computation shows that T is quasinilpotent, and one knows from [6] that if T' is a nonzero operator commuting with T, then T' is a formal power series in T. Since, by inspection, the matrix $M = (\mu_{i,j})$ of T' relative to the orthonormal basis $\{e_n\}$ has the property that every diagonal $\{\mu_{i,i+k}\}$ containing at least one nonzero entry contains infinitely many equal nonzero entries, T' cannot be compact. Thus the proof is complete.

Nevertheless, if T is a quasinilpotent operator and if the compact operators K_1 and K_2 associated with T appearing in Theorem 3 are sufficiently well-behaved, it is possible to conclude that T has a nontrivial hyperinvariant subspace.

THEOREM 6. *Let* T *be a nonzero quasinilpotent operator in* $\mathscr{L}(\mathscr{H})$, *and suppose that* K_1 *and* K_2 *are (nonzero) compact operators such that* $K_1 \prec T \prec K_2$. *Furthermore, suppose that* $\{Z_n\}_{n=1}^{\infty}$ *is a bounded sequence of nonzero operators satisfying the equation* $K_1 Z_n = Z_{n+1} K_2$ *for all* n. *Then* T *has a nontrivial hyperinvariant subspace.*

Proof. We are given quasiaffinities X and Y such that $TX = XK_1$ and $YT = K_2 Y$. If K_1 has a nontrivial kernel, then so does T, and the kernel of T is a nontrivial hyperinvariant subspace for T. Thus we may suppose that K_1 has trivial kernel. It follows that the sequence $\{S_n\} = \{XK_1 Z_n Y\}$ is a bounded sequence of nonzero compact operators satisfying the relation $TS_n = S_{n+1} T$. The result now follows from [5, Theorem 4].

We close by remarking that the separability of the Hilbert space $\mathscr{H}$ plays no role in Theorem 1, and the Hilbert space structure itself plays only a minor role. Thus it is possible to prove an analogue of Theorem 1 for quasinilpotent operators on more general spaces. We shall return to this subject in a later note, where we shall also show that Theorem 3 is valid for a larger class of operators than the class of quasinilpotent operators.

Added in proof. The special case of Theorem 3 in which T and T^* have cyclic vectors was obtained by E. Gerlach in his paper *Generalized invariant subspaces for linear operators*, Studia Math. 42 (1972), 87-90. Peter Rosenthal, in his paper *Commutants of reductive operator algebras*, Duke Math. J. 41 (1974), 829-834, proved a modest generalization of the first statement in Theorem 5.

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Institute of Mathematics, Bucharest, Roumania
and
The University of Michigan, Ann Arbor, Michigan 48104