FOLIATIONS AND LOCALLY FREE TRANSFORMATION GROUPS OF CODIMENSION TWO

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INTRODUCTION

Let M be a manifold with a smooth foliation $\mathscr{F}$ of codimension q. Let E be the bundle of tangents to the leaves, and let $Q = T(M)/E$ be the normal bundle. Imbed Q as $E^{\perp}$ in T(M) once and for all via any Riemannian metric on M.

There is a fairly standard notion (see [1], for example) of a transverse H-structure for $\mathscr{F}$, where H is a Lie subgroup of the group GL_q of $q \times q$ real nonsingular matrices. This is an H-reduction of Q that is invariant under the natural parallelism along leaves.

In the case where H is the group G_k of all matrices of the form

$$\begin{bmatrix} I_k & 0 \\ A & B \end{bmatrix}$$

where $B \in GL_{q-k}$, the existence of a transverse G_k-structure means that some normal k-frame field $(Y_1, \cdots, Y_k)$ is invariant under the linear holonomy of each leaf. Equivalently [1, Corollary 1.5], we require $[Y_i, \Gamma(E)] \subset \Gamma(E)$ for $i = i, \cdots, k$. Letting $V = \mathrm{span}_R \{Y_1, \cdots, Y_k\}$, we say that the G_k-structure is complete if each $Z \in V$ is a complete vector field on M (a condition that is automatic if M is compact).

Definition. $\rho(\mathscr{F})$ is the largest integer k for which $\mathscr{F}$ admits a complete transverse G_k-structure.

In particular, the statement $\rho(\mathscr{F}) = q$ means that $\mathscr{F}$ is a transversally complete e-foliation in the sense of [1], while the statement $\rho(\mathscr{F}) = 0$ means that $\mathscr{F}$ is not invariant under any nonsingular transverse flow on M.

In this paper we investigate the invariant $\rho(\mathscr{F})$ for the case $q = 2$, special applications being made to the situation in which the leaves of $\mathscr{F}$ are the orbits of a locally free Lie transformation group. It will be seen that this amounts to a generalization of results of E. Lima, H. Rosenberg, R. Sacksteder and S. P. Novikov on the rank and file of manifolds (see [5], [7], [8], [9], [11]).

Our basic result is Theorem 1. The term "vanishing cycle" which appears in that theorem, by now standard for foliations of codimension one ([3], [6], [1, Section 6]), is defined for higher codimension in a fairly obvious way in Section 1.

THEOREM 1. *Let* M *be closed and connected,* $\mathrm{codim}(\mathscr{F}) = 2$, *and suppose that* $\mathscr{F}$ *admits no vanishing cycle. Then the condition* $\rho(\mathscr{F}) \geq 1$ *implies that* $\pi_1(M)$ *is infinite, and that if it is abelian it has rank at least* 2.

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It is obvious from the definition (*) in Section 1 that, if a leaf $A \in \mathcal{F}$ supports a vanishing cycle, then the mapping $\pi_1(A) \to \pi_1(M)$ induced by the leaf inclusion has nontrivial kernel. The following corollary is an immediate consequence.

COROLLARY 1. *Let* M *be closed and connected, with* $\operatorname{codim}(\mathcal{F}) = 2$. *If the mapping* $\pi_1(A) \to \pi_1(M)$ *is one-to-one for each leaf* A *of* $\mathcal{F}$ *(a condition that is satisfied, for example, if each leaf is simply connected), and if in addition* $\pi_1(M)$ *is finite or is abelian and has rank at most* 1, *then* $\rho(\mathcal{F}) = 0$.

Before stating the second theorem, we briefly discuss its background. Recall that the *file* of a manifold M is the largest integer r for which there exists a smooth, locally free action of R^r on M, and the *rank* of M is the largest r such that there exists a set of everywhere linearly independent, pairwise commuting vector fields $\{X_1, \cdots, X_r\}$ on M. For compact manifolds, these concepts coincide; but the rank of an open manifold may be greater than its file.

If R^{r+1} has a smooth, locally free action on M, and if $\mathcal{F}$ is the foliation of M by the orbits of $R^r \subset R^{r+1}$, then clearly $\rho(\mathcal{F}) \geq 1$. Thus, if it can be shown that $\rho(\mathcal{F}) = 0$ for every foliation $\mathcal{F}$ by orbits of locally free actions of R^r, we conclude in particular that the file of M is at most r (hence, if M is compact, $\operatorname{rank}(M) \leq r$). Of course, the condition $\rho(\mathcal{F}) = 0$ is stronger because it implies that $\mathcal{F}$ is not invariant under any nonsingular transverse flow.

It is known that $\operatorname{rank}(S^3) = 1$ (Lima [5]), $\operatorname{rank}(S^1 \times S^2) = 1$ (Rosenberg [8]), $\operatorname{rank}(T^{n-2} \times S^2) = n - 2$ (Novikov [7, Theorem 9.5]), $\operatorname{rank}(M^n) \leq n - 2$ if M^n is compact and $\pi_1(M)$ is finite (Sacksteder [14, Theorem 9], see also Rosenberg [9, Theorem 1.7]), and the file of a 3-manifold is one if $\pi_2(M) \neq 0$ (Rosenberg [11]). The following result generalizes all of these facts.

THEOREM 2. *Let* M *be a connected* n-*manifold, and let* G *be a connected Lie group of dimension* $n - 2$ *having a smooth, locally free action on* M. *Let* $\mathcal{F}$ *be the foliation of* M *by the* G-*orbits, and suppose* $\rho(\mathcal{F}) \geq 1$. *Then the following statements hold.*

(1) $\pi_2(M) = 0$.

(2) *If* M *is compact, then* $\pi_1(M)$ *is infinite, and if it is abelian, it has rank at least* 2.

(3) *If* G *is contractible*, M *is compact, and* $\pi_1(M)$ *is abelian, then* $\operatorname{rank}(\pi_1(M)) \geq n - 1$.

While it is not a direct corollary of Theorem 1, this result is readily deduced from the considerations which yield that theorem (see Section 3). Crucial for this is Proposition 2, which generalizes a result of Novikov [7, Lemma 9.3] and asserts that a foliation by orbits of a locally free Lie transformation group admits no vanishing cycles. Indeed, a unified understanding of many of the known results about rank and file centers on the phenomenon of the vanishing cycle; this is clear in Novikov's work [7, pp. 302-304] and implicit in Rosenberg's paper [11] (compare the deformations in [11] with [7, pp. 287-288]).

In the following corollaries to Theorem 2, we fix the hypothesis that M is a compact connected 3-manifold and $\mathcal{F}$ is a foliation of M by curves, with $\rho(\mathcal{F}) \geq 1$.

COROLLARY 2. *The universal cover* $\hat{M}$ *is contractible, hence* $\pi_1(M)$ *is infinite and contains no element of finite order.*

COROLLARY 3. *If the commutator* $[\pi_1(M), \pi_1(M)]$ *is finitely generated and of infinite index in* $\pi_1(M)$, *then* M *is a fiber bundle over* S^1 *whose fiber is a closed* 2-*manifold of genus at least* 1.

COROLLARY 4. *If* $\pi_1(M)$ *is abelian, then* $M \cong T^3$.

By reason of [13], one might guess that a closed connected 3-manifold admitting a foliation $\mathcal{F}$ by curves with $\rho(\mathcal{F}) \geq 1$ must be an orientable T^2-bundle over S^1. This is false. Indeed, the tangent circle bundle to any closed orientable surface of genus ≥ 2 has the form $SL(2, R)/\Gamma$, where Γ is a discrete subgroup, hence supports a pair of everywhere linearly independent vector fields X, Y with $[X, Y] = Y$.

It would be interesting to know whether some odd-dimensional sphere supports a foliation by curves with $\rho(\mathcal{F}) \geq 1$.

Throughout this paper, everything in sight is assumed to be smooth of class C^k for $k \geq 2$.

1. VANISHING CYCLES

The notion discussed here is due, in codimension one, to Novikov [7], although the terminology "vanishing cycle" is due to other authors (see [3] and [6] for example). The following definition generalizes the concept to higher codimensions.

(*) *Definition.* Suppose that $\mathcal{F}$ is a foliation of M of codimension $q \geq 1$, that $L_0 \in \mathcal{F}$, and that $\sigma_0: S^1 \to L_0$ is smooth. The loop σ_0 is called a *vanishing cycle* if there exists a smooth mapping $F: S^1 \times [0, 1] \to M$ such that

(1) $F(\theta, 0) = \sigma_0(\theta)$ for each $\theta \in S^1$,

(2) for each $\theta \in S^1$, the curve $s_\theta(\lambda) = F(\theta, \lambda)$ is transverse to $\mathcal{F}$,

(3) for each λ in $[0, 1]$, the curve $\sigma_\lambda(\theta) = F(\theta, \lambda)$ defines a loop σ_λ on a leaf $L_\lambda \in \mathcal{F}$,

(4) σ_0 is not nullhomotopic on L_0,

(5) $\sigma_\lambda \sim 0$ on L_λ for $0 < \lambda \leq 1$.

LEMMA 1. *The property of being a vanishing cycle is invariant under base-point-preserving homotopy; hence it is a property of* $[\sigma_0] \in \pi_1(L_0, x_0)$.

LEMMA 2. *Let* σ_0 *and* F *satisfy the conditions in* (*). *Let* $\overline{F}: S^1 \times [0, 1] \to M$ *be a smooth mapping such that* (1) *and* (2) *of* (*) *hold, and let* $\overline{F}(\theta, \lambda)$ *lie on the same local leaf of* $\mathcal{F}$ *as* $F(\theta, \lambda)$, *for all* θ, λ. *Let* $\bar{\sigma}_\lambda(\theta) = \overline{F}(\theta, \lambda)$. *Then there exists an* $\varepsilon > 0$ *such that* $\bar{\sigma}_\lambda \sim 0$ *on* L_λ $(0 < \lambda \leq \varepsilon)$.

Indeed, if ε is positive and sufficiently small, then for $0 < \lambda \leq \varepsilon$ the path $\bar{\sigma}_\lambda$ approximates σ_λ uniformly, in the topology of L_λ. Hence we see that $\bar{\sigma}_\lambda \sim \sigma_\lambda \sim 0$ on L_λ. Lemma 1 is conveniently deduced from Lemma 2. Here, take σ_0 and F as in (*), let $U \subset L_0$ be an open, relatively compact neighborhood of $\sigma_0(S^1)$, let N be a normal neighborhood of U in M with fibers whose radius is bounded away from zero, and use the normal fibers ν_θ through $\sigma_0(\theta)$ to define the point $\overline{F}(\theta, \lambda)$ on ν_θ in the same local leaf as $F(\theta, \lambda)$, for sufficiently small values of λ. Then the curve $\bar{s}_\theta(\lambda) = \overline{F}(\theta, \lambda)$ lies along the normal fibers ν_θ, for each $\theta \in S^1$. By a suitable change of the parameter λ, we see from Lemma 2 that $\bar{\sigma}_\lambda \sim 0$ on L_λ when $0 < \lambda \leq 1$. But now a whole base-point-preserving homotopy of σ_0 on L_0 can be lifted along the normal fibers out to each L_λ, where $U \subset L_0$ is rechosen, if necessary, to contain the entire homotopy.

The following is completely obvious.

LEMMA 3. *If* $\sigma_0\colon S^1 \to L_0 \in \mathcal{F}$ *is a vanishing cycle, then* $[\sigma_0] \in \pi_1(L_0, x_0)$ *is a nontrivial element of the kernel of the homomorphism* $\pi_1(L_0, x_0) \to \pi_1(M, x_0)$.

In codimension one, useful criteria for the existence of vanishing cycles are given by a theorem of Novikov.

THEOREM 3 (see [7, Theorem 6.1]). *Under each of the following conditions, a codimension-one foliation* $\mathcal{F}$ *of a connected manifold* M *admits a vanishing cycle.*

(a) M *is compact and* $\pi_1(M)$ *is finite.*

(b) $\pi_1(A) \to \pi_1(M)$ *has nontrivial kernel for some* $A \in \mathcal{F}$.

(c) $\pi_2(M) \neq 0$ *and* $\pi_2(A) = 0$ *for each* $A \in \mathcal{F}$.

(d) *There exists a closed nullhomotopic transversal to* $\mathcal{F}$.

The proof of (c) in [7] seems particularly obscure, but the basic deformations needed there are described quite well in [11] (although for a slightly different purpose). It is not difficult to tidy up the proofs of the other parts.

Suppose that $\rho(\mathcal{F}) \geq 1$, and let $Z \in \Gamma(Q)$ be nowhere zero and invariant under the linear holonomy of each leaf. Suppose further that Z is a complete vector field on M (this is always the case if M is compact). Then, as in [1, Proposition 2.1], there exists a foliation $\mathcal{F}_Z$ of codimension $q - 1$ that has tangent bundle generated by E and Z. Each leaf $L \in \mathcal{F}_Z$ is foliated in codimension one by leaves of $\mathcal{F}$, this foliation being denoted by $\mathcal{F} \mid L$. Therefore $\mathcal{F} \mid L$ is a transversally complete e-foliation in the sense of [1] (thus $\rho(\mathcal{F} \mid L) = \operatorname{codim}(\mathcal{F} \mid L) = 1$), the e-structure being given by $Z \mid L$. By [1, Theorem 5.5], we can make the following assertions. Here the notation $\hat{X}$ denotes the universal cover of X.

LEMMA 4. *If* $L \in \mathcal{F}_Z$ *and* $A \in \mathcal{F} \mid L$, *then* $\hat{L} \cong \hat{A} \times R$, *the foliation of* $\hat{L}$ *by the leaves* $\hat{A} \times \{t\}$ *being the lift of the foliation* $\mathcal{F} \mid L$. *Furthermore, all leaves of* $\mathcal{F} \mid L$ *are mutually diffeomorphic, and either all or none are closed in the manifold topology of* L.

LEMMA 5. *If* $L \in \mathcal{F}_Z$, $A \in \mathcal{F} \mid L$, *then the inclusion map induces a monomorphism* $\pi_1(A) \to \pi_1(L)$.

Suppose that $L \in \mathcal{F}_Z$ and $x \in L$, and let A denote the leaf of $\mathcal{F} \mid L$ containing x. Identify $\pi_1(A, x)$ with its monomorphic image in $\pi_1(L, x)$, and let $\pi_T(L, x)$ denote the subset of elements $[\sigma] \in \pi_1(L, x)$ that can be represented by a closed transversal σ to $\mathcal{F} \mid L$. Here we do not count the trivial loop as a closed transversal.

LEMMA 6. $\pi_1(A, x) \cap \pi_T(L, x) = \emptyset$ *and* $\pi_1(L, x) = \pi_1(A, x) \cup \pi_T(L, x)$.

Proof. Let $\hat{\sigma}$ be a lift of σ to $\hat{L} = \hat{A} \times R$. Either $\hat{\sigma}$ begins and ends on the same leaf $\hat{A} \times \{t\}$ or it does not, and this depends only on the class $[\sigma] \in \pi_1(L, x)$. In the first case, $[\sigma] \in \pi_1(A, x)$, and in the second case, $[\sigma] \in \pi_T(L, x)$. ■

PROPOSITION 1. *If* $\mathcal{F}_Z$ *admits a vanishing cycle, so does* $\mathcal{F}$.

Proof. Let $\sigma_0\colon S^1 \to L_0$ be a vanishing cycle for $\mathcal{F}_Z$. If $[\sigma_0] \in \pi_T(L_0, x_0)$, then nearby displacements $\sigma_\lambda\colon S^1 \to L_\lambda$ have the property that $[\sigma_\lambda] \in \pi_T(L_\lambda, x_\lambda)$, hence σ_λ cannot be nullhomotopic on L_λ for small $\lambda > 0$. Thus $[\sigma_0] \in \pi_1(A_0, x_0)$, where $A_0 \in \mathcal{F} \mid L_0$. By Lemma 1, we lose no generality in assuming $\sigma_0\colon S^1 \to A_0$. Let the mapping $F\colon S^1 \times [0, 1] \to M$ satisfy the condition in (*) relative to the foliation $\mathcal{F}_Z$. Observe that local-product coordinate neighborhoods for the foliation $\mathcal{F}_Z$ can also be chosen so that they are simultaneously local-product neighborhoods for

$\mathcal{F}$. Let $U_0, U_1, \cdots, U_r = U_0$ be a sequence of such neighborhoods chosen so that $\sigma_0[t_i, t_{i+1}] \subset U_i$ $(0 \leq i \leq r - 1)$, where

$$[1, t_1] = [t_0, t_1], [t_1, t_2], \cdots, [t_{r-1}, t_r] = [t_{r-1}, 1]$$

partitions S^1 into consecutive closed subarcs. One can assume that

$$F([t_i, t_{i+1}] \times [0, 1]) \subset U_i \quad (0 \leq i \leq r - 1),$$

by reparametrizing along the second coordinate if necessary. Using the local-product structure in each U_i consecutively, one replaces $F \mid ([1, t_{r-1}] \times [0, 1])$ by a mapping $\tilde{F}: [1, t_{r-1}] \times [0, 1] \to M$ such that

(1) $\tilde{F}(\theta, 0) = \sigma_0(\theta)$ for each $\theta \in [1, t_{r-1}]$,

(2) for each $\theta \in [1, t_{r-1}]$, the formula $\tilde{s}_\theta(\lambda) = \tilde{F}(\theta, \lambda)$ $(0 \leq \lambda \leq 1)$ defines a path $\tilde{s}_\theta$ transverse to $\mathcal{F}$, and $\tilde{s}_1 = s_1$,

(3) $\tilde{\sigma}_\lambda(\theta) = \tilde{F}(\theta, \lambda) \in A_\lambda \in \mathcal{F} \mid L_\lambda$ for each $\lambda \in [0, 1]$.

Since the $\mathcal{F}_Z$-holonomy of σ_0 leaves fixed the points of R^{q-1} corresponding to the points $\tilde{s}_1(\lambda) = s_1(\lambda)$, when λ is sufficiently small, we see that, for some $\varepsilon > 0$, the point $\tilde{\sigma}_\lambda(t_{r-1})$ lies on the same local leaf of $\mathcal{F}_Z$ in $U_r = U_0$ as $\tilde{\sigma}_\lambda(1)$ $(0 \leq \lambda \leq \varepsilon)$. Again we can suppose $\varepsilon = 1$. If, in fact, $\tilde{\sigma}_\lambda(t_{r-1})$ lies on the same local leaf of $\mathcal{F}$ in $U_r = U_0$ as $\tilde{\sigma}_\lambda(1)$, for each $\lambda \in [0, 1]$, then $\tilde{F}$ can be completed to a mapping $\bar{F}: S^1 \times [0, 1] \to M$ such that properties (1) to (4) of (*) hold relative to the foliation $\mathcal{F}$. In this case Lemma 2 implies that $\bar{\sigma}_\lambda \sim 0$ on $L_\lambda \in \mathcal{F}_Z$, for all sufficiently small $\lambda > 0$. Therefore Lemma 5 implies that $\tilde{\sigma}_\lambda \sim 0$ on $A_\lambda \in \mathcal{F}$ for all sufficiently small $\lambda > 0$. Thus σ_0 will be a vanishing cycle for $\mathcal{F}$. The problem, then, is to show that $\tilde{\sigma}_\lambda(t_{r-1})$ and $\tilde{\sigma}_\lambda(1)$ lie on the same local leaf of $\mathcal{F}$ in $U_r = U_0$ for all sufficiently small $\lambda > 0$. If not, $\tilde{F}$ can still be completed to $\check{F}: S^1 \times [0, 1] \to M$ so that conditions (1) to (4) of (*) hold relative to the foliation $\mathcal{F}_Z$, and, by Lemma 2, one can also assume condition (5). For some $\lambda > 0$, we can assert that $\check{\sigma}_\lambda \sim 0$ on L_λ, that $\check{\sigma}_\lambda[1, t_{r-1}] \subset A_\lambda \in \mathcal{F} \mid L_\lambda$, and that $\check{\sigma}_\lambda(t_{r-1})$ and $\check{\sigma}_\lambda(1)$ are on different local leaves of $\mathcal{F}$; indeed, we can assume that $\check{\sigma}_\lambda \mid [t_{r-1}, 1]$ is transverse to $\mathcal{F} \mid L_\lambda$. Arguing exactly as in [3, Lemme 1], we deform $\check{\sigma}_\lambda$ to a closed transversal to $\mathcal{F} \mid L_\lambda$. This transversal, being nullhomotopic on L_λ, cannot exist (see Lemma 6). ■

2. PROOF OF THEOREM 1

We suppose that M is compact and connected, $\operatorname{codim}(\mathcal{F}) = 2$, and $\rho(\mathcal{F}) \geq 1$. We also suppose that the vector field $Z \in \Gamma(Q)$ and the foliation $\mathcal{F}_Z$ satisfy the conditions preceding Lemma 4.

If $\mathcal{F}$ does not admit a vanishing cycle, then, by Proposition 1, neither does $\mathcal{F}_Z$. By Theorem 3, which applies since $\operatorname{codim}(\mathcal{F}_Z) = 1$, the group $\pi_1(M)$ is infinite. Note that Theorem 3 also implies, for each $L \in \mathcal{F}_Z$, that the homomorphism $\pi_1(L) \to \pi_1(M)$ is one-to-one.

Continuing to suppose that $\mathcal{F}$ admits no vanishing cycle, we further assume that $\pi_1(M)$ is abelian. No generality will be lost in assuming transverse orientability of $\mathcal{F}$, since this can be brought about by passing to a 2-fold covering. We must show that the abelian group $\pi_1(M)$ has rank at least 2. To this end we consider two cases, according as $\mathscr{F}_Z$ does or does not admit a compact leaf.

First we suppose that no leaf of $\mathscr{F}_Z$ is compact. Then, by a theorem of Moussu and Roussarie [6, Théorème 2], the facts that $\pi_1(M)$ is abelian and that $\mathscr{F}_Z$ admits no vanishing cycle imply that $\mathscr{F}_Z$ admits no limit cycle. Then a theorem of Novikov [7, Theorem 5.1] (see also [1, Theorem 5.5], [9]) implies that each leaf $L \in \mathscr{F}_Z$ has the property that $\pi_1(M)/\pi_1(L) = \mathbb{Z}^k$, where $k \geq 1$, with $k = 1$ if and only if L is closed in M. Thus, in our case, $k \geq 2$ and therefore $\operatorname{rank}(\pi_1(M)) \geq 2$.

Suppose that $\mathscr{F}_Z$ admits a compact leaf. If all leaves are compact, then the holonomy of each leaf is finite. But transverse orientability for $\mathscr{F}$ implies the same for $\mathscr{F}_Z$, hence the holonomy of each $L \in \mathscr{F}_Z$ is trivial. Thus $\mathscr{F}_Z$ is the foliation of M by the fibers of a bundle $M \to S^1$. Each $L \in \mathscr{F}_Z$ is compact with $\hat{L} = \hat{A} \times R$, hence the abelian group $\pi_1(L)$ has rank at least 1. The exact sequence

$$0 \to \pi_1(L) \to \pi_1(M) \to \pi_1(S^1) \to 0$$

then shows that $\operatorname{rank}(\pi_1(M)) \geq 2$. We may assume, therefore, that $\mathscr{F}_Z$ admits both compact and noncompact leaves.

Suppose that for some $L \in \mathscr{F}_Z$, the leaves of $\mathscr{F} \mid L$ are not closed in L (if one fails to be closed, then by Lemma 4 they all do). If L is compact, then [1, Theorem 5.5] again applies and shows that $\operatorname{rank}(\pi_1(L)) \geq 2$. Since the homomorphism $\pi_1(L) \to \pi_1(M)$ is one-to-one, we see that $\operatorname{rank}(\pi_1(M)) \geq 2$. If, on the other hand, L is noncompact, let $\tau: S^1 \to L$ be a closed transversal to $\mathscr{F} \mid L$ (constructed as in [3, Lemme 1]), and note that τ fails to be nullhomotopic because of Lemma 4, and that similarly $\mathscr{F}_Z$ admits a closed transversal $\sigma: S^1 \to M$. By Theorem 3 and the fact that $\mathscr{F}_Z$ admits no vanishing cycles, σ also fails to be nullhomotopic in M. Likewise, all iterates σ^r and τ^r $(r \neq 0)$ fail to be nullhomotopic, and, in addition, σ^r is not base-point homotopic to a loop on L. In order to verify the last assertion, we suppose it is false and lift σ^r to a transversal s on $\hat{M}$, s necessarily beginning and ending on the same covering leaf $\hat{L}$ of L. We then construct a closed nullhomotopic transversal on $\hat{M}$, hence on M, and this again contradicts the absence of vanishing cycles for $\mathscr{F}_Z$. It follows that $[\sigma]$ and $[\tau]$ generate a free abelian subgroup of rank 2 in $\pi_1(M)$.

We are now reduced to assuming that $\mathscr{F}_Z$ admits both compact and noncompact leaves and that the leaves of $\mathscr{F} \mid L$ are closed in L, for all $L \in \mathscr{F}_Z$. If $L \in \mathscr{F}_Z$ is a compact leaf, then $\mathscr{F} \mid L$ is an e-foliation by closed leaves, hence L is a bundle $L \to S^1$, and $\operatorname{rank}(\pi_1(L)) \geq 1$. Since $\pi_1(L) \to \pi_1(M)$ is one-to-one, $\operatorname{rank}(\pi_1(M)) \geq 1$.

We shall suppose that $\operatorname{rank}(\pi_1(M)) = 1$ and produce a contradiction. Since M is compact, we see that $\pi_1(M) = \mathbb{Z} \oplus T$, where T is a finite group.

If $L \in \mathscr{F}_Z$ is compact and $A \in \mathscr{F} \mid L$, then $\pi_1(A)$ is finite. Indeed, by the theorem of Novikov already cited [7, Theorem 5.1], $\pi_1(L)/\pi_1(A) \cong \mathbb{Z}$ while $\operatorname{rank}(\pi_1(L)) \leq 1$; therefore the exact sequence

$$0 \to \pi_1(A) \to \pi_1(L) \to \pi_1(L)/\pi_1(A) \to 0$$

shows that $\operatorname{rank}(\pi_1(L)) = 1$ and $\pi_1(A)$ is finite. By Reeb stability [2, Paragraph 2.6] there exists a neighborhood U of A in M that is a union of compact leaves of $\mathscr{F}$ with finite fundamental group.

Suppose $A' \in \mathscr{F}$ and $A' \subset L' \in \mathscr{F}_Z$. Since $\mathscr{F}_Z$ does not admit a vanishing cycle and $\pi_1(M)$ is abelian, [6, Théorème 1] implies that $\mathscr{F}_Z$ has no exceptional minimal set in M, so that the closure of each leaf of $\mathcal{F}_Z$ is either all of M or contains a compact leaf. Thus, in any case, $\overline{L}'$ contains a compact leaf of $\mathcal{F}_Z$, say L of the previous paragraph. For some $A \in \mathcal{F} \mid L$ and U as above, $L' \cap U \neq \emptyset$, hence a leaf $A_0 \in \mathcal{F} \mid L'$ meets U, hence A_0 is compact with finite fundamental group. Since $\mathcal{F} \mid L'$ is a transversally complete e-foliation, every leaf of $\mathcal{F} \mid L'$ is compact with finite fundamental group. Thus our arbitrary A' (hence every leaf of $\mathcal{F}$) is compact with finite fundamental group.

Let $\hat{M} \to M$ be the universal cover. Dividing out the infinite cyclic component of $\pi_1(M) = \mathbb{Z} \oplus T$, we obtain a covering space $\tilde{M} \to M$ with T as group of covering transformations, and therefore $\tilde{M}$ is compact. Let $\tilde{\mathcal{F}}$ be the lift of $\mathcal{F}$ to $\tilde{M}$. We claim that each $\tilde{A} \in \tilde{\mathcal{F}}$ is simply connected. Indeed, $\tilde{A}$ is covered by $\hat{A} \subset \hat{M}$ and $\hat{A}$ is simply connected, since the homomorphism $\pi_1(A) \to \pi_1(M)$ is one-to-one. If σ is a loop on $\tilde{A}$, its lift $\hat{\sigma}$ to $\hat{A}$ must be closed, since otherwise σ determines an element of infinite order in $\pi_1(\tilde{M})$, hence in $\pi_1(\tilde{A})$, hence in $\pi_1(A)$, and this is a contradiction. Thus $\pi_1(\tilde{A}) = 0$, as desired. It follows that the leaves $\tilde{A} \in \tilde{\mathcal{F}}$ have trivial holonomy; hence, these leaves being compact, $\tilde{\mathcal{F}}$ is a foliation by leaves of a fiber bundle $\tilde{M} \to N$, where N is a compact 2-manifold. Since the fiber is simply connected, we obtain the relation $\pi_1(N) = \pi_1(\tilde{M}) = \mathbb{Z}$, which is impossible for any compact 2-manifold. This contradiction completes the proof of Theorem 1.

By reason of Lemma 3, Corollary 1 is an immediate consequence of Theorem 1.

3. PROOF OF THEOREM 2 AND ITS COROLLARIES

The following adapts an argument of Novikov [7, Lemma 9.3], and it is the main step in the proof of Theorem 2. We emphasize that there is no assumption of compactness on M.

PROPOSITION 2. *Let* $\mathcal{F}$ *be the foliation of* M *by orbits of a locally free smooth action of a connected Lie group* G. *Then* $\mathcal{F}$ *does not admit a vanishing cycle.*

Proof. We suppose there is a vanishing cycle, and we obtain a contradiction.

Without loss of generality, assume that G is simply connected. Suppose that $x_0 \in M$ and $L_0 = G \cdot x_0$, and let σ_0 and F be as in (*) with σ_0 based at x_0. For $0 \leq \lambda \leq 1$, let $x_\lambda = F(1, \lambda)$, and let $p_\lambda\colon G \to L_\lambda = G \cdot x_\lambda$ be the covering map defined by $p_\lambda(g) = g \cdot x_\lambda$. Since G is the universal cover of L_0, we see that $\sigma_0 = p_0 \circ \hat{\sigma}_0$, where the path $\hat{\sigma}_0\colon [0, 1] \to G$ satisfies the condition $\hat{\sigma}_0(0) = e$, where e denotes the identity of G, and $\hat{\sigma}_0(1) \neq e$. The mapping $G \times [0, 1] \to M$ defined by the formula $(g, \lambda) \mapsto g \cdot x_\lambda$ extends to a smooth mapping $G \times R \to M$ with Jacobian of maximal rank at $(\hat{\sigma}_0(1), 0)$; hence this is an imbedding near that point, and it follows that there exists a smooth curve $(\tau(\lambda), t(\lambda))$ in $G \times R$, defined for all λ in some interval $[0, \varepsilon]$, and satisfying the conditions $\tau(0) = \hat{\sigma}_0(1)$, $t(0) = 0$, and $\tau(\lambda) \cdot x_{t(\lambda)} = x_\lambda$ for all λ. We show that $t(\lambda) \equiv \lambda$. Indeed, $\hat{\sigma}_0(t) \cdot x_\lambda$ is in the same local leaf as $F(t, \lambda)$ for small λ and all t; hence $\hat{\sigma}_0(1) \cdot x_\lambda$ is in the same local leaf as $F(1, \lambda) = x_\lambda$. Thus the mapping $G \times R \to M$ takes (g, λ) to the local leaf of x_λ, for all g near $\hat{\sigma}_0(1)$ and all small $\lambda \geq 0$. In particular, $t(\lambda) \equiv \lambda$, and it follows that $p_\lambda(\tau(\lambda)) = x_\lambda$ and $\tau(\lambda) \neq e$ if $0 \leq \lambda \leq \varepsilon$. Without loss of generality we assume that $\varepsilon = 1$. Let $\hat{\sigma}_\lambda$ be the piecewise smooth curve obtained by following $\hat{\sigma}_0$ by $\tau \mid [0, \lambda]$ and suitably reparametrizing over [0, 1]. A modification of $\hat{\sigma}_0$ will smooth the corner at $\hat{\sigma}_0(1)$

without changing the homotopy class of σ_0; therefore Lemma 1 allows us to assume that the mapping

$$\overline{F}: S^1 \times [0, 1] \to M,$$

$$\overline{F}(\theta, \lambda) = p_\lambda(\hat{\sigma}_\lambda(\theta)) = \overline{\sigma}_\lambda(\theta)$$

is smooth (it is well-defined, since $p_\lambda(\hat{\sigma}_\lambda(1)) = x_\lambda = p_\lambda(\hat{\sigma}_\lambda(0))$, for each λ). This satisfies conditions (1) and (2) of (*), and for small λ the fact that $\overline{F}(1, \lambda) = x_\lambda = F(1, \lambda)$ will imply that all $\overline{F}(\theta, \lambda)$ lie on the same local leaf of $\mathscr{F}$ as $F(\theta, \lambda)$. By Lemma 2, each $\overline{\sigma}_\lambda \sim 0$ on L_λ, if λ is small, and this contradicts the relation $\hat{\sigma}_\lambda(1) \neq e = \hat{\sigma}_\lambda(0)$. ■

We prove Theorem 2. Here $\mathscr{F}$ is a codimension-two foliation of a connected manifold M by the orbits of a locally free action of G. If $\rho(\mathscr{F}) \geq 1$, let $\mathscr{F}_Z$ be as usual. It is well known that $\pi_2(G) = 0$; therefore, if $L \in \mathscr{F}_Z$, then $\pi_2(L) = 0$ by Lemma 4. By Proposition 2 and Proposition 1, $\mathscr{F}_Z$ admits no vanishing cycle; therefore, Theorem 3 implies that $\pi_2(M) = 0$. Furthermore, Theorem 1 and Proposition 2 imply that if M is compact, then $\pi_1(M)$ is infinite, and that if it is abelian, it has rank at least 2.

It remains to establish part (3) of Theorem 2. Suppose, then, that G is contractible, M is compact, and $\pi_1(M)$ is abelian. The foliation $\mathscr{F}_Z$ has no vanishing cycles; therefore Theorem 3 implies that the homomorphism $\pi_1(L) \to \pi_1(M)$ is one-to-one, for each $L \in \mathscr{F}_Z$. We consider two cases.

For the first case, suppose all leaves of $\mathscr{F}_Z$ are noncompact. As usual, the result [6, Théorème 2] implies that $\mathscr{F}_Z$ admits no limit cycles, hence $\hat{M} = \hat{L} \times R$. But $\mathscr{F} \mid L$ is a transversally complete e-foliation, so that $\hat{M} \cong \hat{L} \times R \cong G \times R^2$ is contractible. If the abelian group $\pi_1(M)$ has an element of finite order, there exists a prime p with $\mathbb{Z}_p \subset \pi_1(M)$, hence $\tilde{M} = \hat{M}/\mathbb{Z}_p$ is a $K(\mathbb{Z}_p, 1)$, a well-known impossibility for every finite-dimensional manifold $\tilde{M}$. Thus $\pi_1(M) = \mathbb{Z}^k$, for some k, and $M = K(\mathbb{Z}^k, 1)$ has the homotopy type of the torus T^k. Since M is a compact n-manifold, $k = n$ and $\operatorname{rank}(\pi_1(M)) = n$.

For the second case, suppose some $L \in \mathscr{F}_Z$ is compact. Then $\hat{L} \cong G \times R$ is contractible, and arguing as above, we conclude that $\operatorname{rank}(\pi_1(L)) = n - 1$. Since the homomorphism $\pi_1(L) \to \pi_1(M)$ is one-to-one, this shows that $\operatorname{rank}(\pi_1(M)) \geq n - 1$, and the proof of Theorem 2 is complete.

We remark that Proposition 2 also enables us to prove the following.

THEOREM 4. *Let* M *be a connected* n-*manifold,* G *a Lie group of dimension* n - 1 *having contractible universal cover* (*for example, a solvable Lie group*) *and having locally free action on* M. *Then* $\pi_2(M) = 0$. *If in addition* M *is compact and* $\pi_1(M)$ *is abelian, then* $\operatorname{rank}(\pi_1(M)) \geq n - 1$.

Indeed, the action defines a codimension-one foliation $\mathscr{F}$ without vanishing cycles, and each $L \in \mathscr{F}$ has the property $\pi_2(L) = \pi_2(G) = 0$, hence by Theorem 3, part (c), $\pi_2(M) = 0$. To obtain the second conclusion, we apply the arguments about $\mathscr{F}_Z$ in the proof of part (3) of Theorem 2 to our present foliation $\mathscr{F}$, considering again the two cases in which $\mathscr{F}$ does or does not admit a compact leaf.

For the Corollaries 2 through 4, let $\mathscr{F}$ be a foliation by curves of a compact, connected 3-manifold M. Passing to a finite cover, if necessary, we may assume both that M is orientable and that $\mathscr{F}$ is transversally orientable; hence we construct a tangent field to $\mathscr{F}$ that is nowhere zero. Thus $\mathscr{F}$ is the foliation by orbits of a locally free action of R, and, by Theorem 2, $\rho(\mathscr{F}) \geq 1$ implies that $\pi_2(M) = 0$ and $\hat{M}$ is noncompact. Thus $\pi_1(\hat{M}) = 0 = \pi_2(\hat{M})$, so that $H_1(\hat{M}) = 0 = H_2(\hat{M})$ (by the Hurewicz theorem [4]). Also, $H_3(\hat{M}) = 0$, since $\hat{M}$ is noncompact. It follows that $H_i(\hat{M}) = 0$ for all $i > 0$, hence that $\pi_i(\hat{M}) = 0$ for all $i > 0$ (again by Hurewicz), hence that $\hat{M}$ is contractible.

If $\pi_1(M)$ contains an element of finite order, there exists a prime p such that $\mathbb{Z}_p$ is a subgroup of $\pi_1(M)$. Let $\tilde{M}$ be the 3-manifold obtained from $\hat{M}$ by dividing out $\mathbb{Z}_p$. Then $\tilde{M} = K(\mathbb{Z}_p, 1)$ and $H_i(\tilde{M}; \mathbb{Z}_p) = \mathbb{Z}_p$ for each $i \geq 0$. This contradicts the fact that $\tilde{M}$ is a finite-dimensional manifold.

The lift of $\mathscr{F}_Z$ foliates $\hat{M}$ by planes R^2, hence, by [12, Section 4], $\hat{M}$ is irreducible (that is, every tamely imbedded 2-sphere bounds a 3-ball). As in [10, Lemma 1] it follows that M is irreducible.

If $[\pi_1(M), \pi_1(M)]$ is finitely generated and of infinite index in $\pi_1(M)$, it follows that the Hurewicz surjection $\pi_1(M) \to H_1(M)$ has finitely generated kernel and that $H_1(M)$ is finitely generated and has rank at least 1. Consequently, there exists a group surjection $\pi_1(M) \to \mathbb{Z}$, the kernel K is finitely generated, and $K \neq \mathbb{Z}_2$, since $\pi_1(M)$ contains no elements of finite order.

By a theorem of Stallings [15], we conclude from the previous two paragraphs that M is fibered over S^1 and that the fiber is a closed 2-manifold T. Evidently, T cannot have S^2 as universal cover, hence genus$(T) \geq 1$.

If $\pi_1(M)$ is abelian, we conclude exactly as in [10, pp. 131-132] that the bundle above is trivial and has fiber T^2. That is, $M = T^3$.

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