ISOLATED SUBGROUPS

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A subgroup H of a group G is *isolated* provided its conjugates are strictly disjoint; explicitly, H is isolated provided $xHx^{-1} \cap H = \{1\}$ whenever $x \in G$ and $x \notin H$. Isolated subgroups seem to have been used only in Frobenius's theorem for finite groups and its developments [4]. Our aim is to consider the effect on the structure of a (possibly infinite) group G of its supply of isolated subgroups. At one extreme, G has no isolated subgroups except $\{1\}$ and G (G is I-*simple*); at the other extreme, G admits a nontrivial partition by isolated subgroups (G is *multic*). Most well-known classes of groups are *monic*, that is, nonmultic (Sections 1, 2); however, we obtain several noteworthy classes of multic groups. Our interest in these questions arose from geometry, and in Section 5 we show that the isolated subgroups of the fundamental group of a Riemannian manifold M are closely related to the curvature of M. Finally, in Section 6 we discuss finite and infinite Frobenius groups.

Our late colleague Theodore Motzkin participated in the beginning investigations of this paper. We consider him a coauthor, even though the completed paper could not have his customary meticulous scrutiny.

1. TOTAL GROUPS

1.1. LEMMA. (1) *The intersection of an arbitrary collection of isolated subgroups of* G *is isolated.*

(2) *If* A *is an isolated subgroup of* B, *and* B *is an isolated subgroup of* C, *then* A *is isolated in* C.

(3) *If* I *is an isolated subgroup of* G *and* H *is a subgroup of* G, *then* $I \cap H$ *is isolated in* H.

(4) *If* I *is isolated in* G *and* $x \in G$, *then* $x^n \in I \setminus \{1\}$ *implies* $x \in I$.

(5) *No proper isolated subgroup of* G *contains a nontrivial normal subgroup of* G.

By the first of these properties, if S is a subset of a group G, we may define I_S to be the smallest isolated subgroup of G containing S. In particular, for each $x \in G$ we have the isolated subgroup I_x. An element $x \in G$ is *total* if $I_x = G$.

We now distinguish some classes of groups that have successively richer supplies of isolated subgroups.

1.2. *Definition.* (1) G is I-*simple* if $\{1\}$ and G are the only isolated subgroups of G.

(2) G is *total* if it contains a total element.

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(3) G is *monic* if there exists no nontrivial partition (see Section 2) of G into isolated subgroups. Otherwise, G is *multic*.

Obviously, I-simplicity implies totality, and totality implies monicity. We shall see that the four (mutually exclusive) types implied by the definition actually occur: I-simple, total but not I-simple, monic but not total, multic.

Remark. If elements $x \neq 1$ and $y \neq 1$ of G commute, then $I_x = I_y$. In fact, since $xI_yx^{-1} \cap I_y \neq \{1\}$ we see that $x \in I_y$, hence $I_x \subset I_y$. Symmetrically, $I_y \subset I_x$.

1.3. LEMMA. (1) *If* G *has a nontrivial center, then* G *is* I-*simple.*

(2) *A direct product* $A \times B$ *of nontrivial groups is* I-*simple.*

Proof. (1) Let $z \in Z(G) \setminus \{1\}$ and $x \in G \setminus \{1\}$. For each $g \in G \setminus \{1\}$, the element z commutes with both x and g; hence, by the preceding remark, $g \in I_g = I_z = I_x$. Thus $I_x = G$.

(2) First we show that each nontrivial element of $S = (A \times \{1\}) \cup (\{1\} \times B)$ is total. Suppose $a \in A \setminus \{1\}$ and $b \in B \setminus \{1\}$. Since (a, 1) and (1, b) commute, it follows that $I_{(a,1)} = I_{(1,b)}$. Hence S is contained in this subgroup, so that (a, 1) and (1, b) are total. Now $(a, b) \neq (1, 1)$ commutes with (a, 1) and (1, b); hence $I_{(a,b)} = A \times B$.

1.4. LEMMA. *For each* $x \in G$, *the group* I_x *is the unique largest subgroup of* G *in which* x *is a total element.*

Proof. (a) *The element* x *is total in* I_x. In fact, if J is an isolated subgroup of I_x that contains x, then by transitivity J is isolated in G. Hence $J \supset I_x$, so that $J = I_x$.

(b) *If* x *is total in a subgroup* H *of* G, *then* $H \subset I_x$. Since $I_x \cap H$ is isolated in H and contains x, we see that $I_x \cap H = H$.

Assertion (b) has a very useful consequence.

1.5. LEMMA. *Let* H *be a subgroup of* G *that contains a nontrivial normal subgroup of* G. *If* x *is total in* H, *then* x *is total in* G.

Proof. Since x is total in H it follows that $I_x \supset H$. The result now follows by Lemma 1.1 (5).

Clearly, the preceding lemma generalizes to the case where H is *subnormal* in G, that is, where there exists a sequence of subgroups $H = H_1 \lhd H_2 \lhd \cdots \lhd G$. (In fact, the lemma extends to the case where H is *transfinitely subnormal in* G [6].)

1.6. COROLLARY. *Solvable groups are total.*

Proof. A solvable group has a nontrivial subnormal abelian subgroup, and abelian groups are total.

1.7. LEMMA. *Every finite group is total.*

Proof. Suppose not. Let G be a finite nontotal group of smallest order. In particular, G contains an isolated proper subgroup I. By a consequence of Frobenius's theorem (see Theorem 6.2), G contains a normal subgroup M such that $I \cap M = \{1\}$ and $G = IM$. Hence $1 < |M| < |G|$. By hypothesis, M is total, and thus G .is also total, contrary to our supposition.

A finite (solvable) group need not be I-simple. For example, if n is odd and $n \geq 3$, the dihedral group D_n contains proper isolated subgroups of order 2. On the other hand, if $n \neq 3$, then A_n and S_n are I-simple, since they are not Frobenius groups (see Definition 6.1).

1.8. LEMMA. *If* G *contains a total subgroup* H *of finite index, then* G *is total.*

Proof. Since $[G: H] < \infty$, there exists a normal subgroup N of G such that $N \subset H$ and $[G: N] < \infty$. Thus, if $N = \{1\}$, then G is finite, hence total. If $N \neq \{1\}$, then G is total, by Lemma 1.5.

Lemmas 1.3 (1) and 1.7 can be extended as follows.

1.9. PROPOSITION. *If* G *contains an element* $x \neq 1$ *with only finitely many conjugates, then* G *is total.*

Proof. The hypothesis on x implies that the centralizer C(x) has finite index in G. Since $x \neq 1$ is in the center of C(x), this subgroup is I-simple. Thus, by the preceding lemma, G is total.

2. MONIC GROUPS

A partition $P = \{P_i\}$ of a group G by isolated subgroups is an *isolation* of G. Explicitly, $G = \bigcup \{P_i: i \in I\}$, where each P_i is isolated and $P_i \cap P_j \neq \{1\}$ implies $P_i = P_j$. (For technical reasons, we also suppose that if $G \neq \{1\}$, then no P_i is $\{1\}$.) Thus G is monic provided the only isolation it possesses is the trivial one: $P_i = G$ for all i.

Remarks. (1) If P is an isolation of G, and if for each $P_i \in P$ we have an isolation P_i^* of P_i, then $\bigcup_i P_i^*$ is an isolation of G.

(2) If $\{P_i\}$ is an isolation of G and H is a subgroup of G, then $\{P_i \cap H\}$ is an isolation of H.

The latter fact has a consequence we shall use very often.

2.1. LEMMA. *If* H *is a monic subgroup of* G *and* $\{P_i\}$ *is an isolation of* G, *then* H *is contained in a single* P_i.

For example, we use this lemma to obtain analogues of earlier results on total groups:

2.2. LEMMA. *Let* H *be a monic subgroup of* G. *If* H *has finite index or contains a nontrivial normal subgroup of* G, *then* G *is monic.*

Proof. Let P be an isolation of G. By the preceding lemma, H is contained in P_i, say. If H contains a nontrivial subgroup normal in G, then, since P_i is isolated in G, we see that $G = P_i$. If H has finite index, we argue as in the proof of Lemma 1.8.

For completeness, we give the proof of the following known result.

2.3. LEMMA. *If* G *is minimally covered by subgroups* $H_1, \cdots, H_n$ (*that is, if no* $n - 1$ *of these subgroups cover* G), *then* $H = \bigcap H_i$ *has finite index in* G.

Proof. It suffices to show that each $[H_i: H]$ is finite. Assume not; then $[H_1: H] = \infty$, say. Thus $[H_1 \cap \cdots \cap H_k: H] = \infty$ for $k = 1$. Suppose the assertion is true for $1 \leq k < n$, so that there exist infinitely many distinct cosets $Hg_1, Hg_2, \cdots$, where $g_i \in H_1 \cap \cdots \cap H_k$. By the hypothesis of minimality, there is an element $x \in G \setminus (H_1 \cup \cdots \cup H_k)$. Thus there are infinitely many distinct cosets of the form $Hg_i x$ in some H_i with $i > k$, say H_{k+1}. For these we have the relations

$$g_i x(g_j x)^{-1} = g_i g_j^{-1} \in (H_1 \cap \cdots \cap H_{k+1}) \setminus H$$

if $i \neq j$. Thus $[H_1 \cap \cdots \cap H_{k+1}: H] = \infty$, which leads, by induction, to the contradiction $[H: H] = \infty$.

2.4. COROLLARY. *Let* G *be minimally covered by subgroups* $H_1, \cdots, H_n$. *If one of these subgroups is monic* [*total*], *then* G *is monic* [*total*].

Proof. This follows from the preceding lemma and Lemma 2.2 [Lemma 1.8].

The following result was first proved for us by M. Schacher in a different way.

2.5. COROLLARY. *A nontrivial isolation of* G *is infinite.*

Proof. In fact we can show somewhat more. By Lemma 1.7, G is infinite; but, *if an infinite group* G *is minimally covered by subgroups* $H_1, \cdots, H_n$ $(n \geq 2)$, *then no* H_i *is isolated.* For, by Lemma 2.3, each H_i contains a nontrivial normal subgroup; thus the result follows by Lemma 1.1(5).

2.6. PROPOSITION. *Every group* G *has a unique monic isolation* M, *and* M *consists of all the maximal monic subgroups of* G.

Proof. Let $\mathscr{P}$ be the set of all isolations of G. Fix $x \in G \setminus \{1\}$. For each $P \in \mathscr{P}$, let $P(x)$ be the element of P containing x, and let $M(x) = \bigcap \{P(x): P \in \mathscr{P}\}$. As an intersection of isolated groups, $M(x)$ is isolated. To show that $M = \{M(x): 1 \neq x \in G\}$ is an isolation of G, it suffices to verify that $M(x) \cap M(y) \neq \{1\}$ implies $M(x) = M(y)$. But if $1 \neq z \in M(x) \cap M(y)$, then $P(x) = P(z) = P(y)$ for all $P \in \mathscr{P}$, hence $M(x) = M(y)$.

Clearly, M is the unique finest isolation of G. Hence, by Remark 1 above, M is monic. By Lemma 2.1, every monic isolation of G is finer than M, hence equals M.

It remains to verify that the elements of M are the maximal monic subgroups of G. Suppose H is a maximal monic subgroup; then, by Lemma 2.1, H is contained in some $M(x)$, so that $H = M(x)$. On the other hand, if $M(x)$ is contained in some monic subgroup H, another application of Lemma 2.1 shows that $H \subset M(x)$, hence $H = M(x)$.

We say that two elements x, y of G are *monically equivalent* (notation: $x \sim y$) if both are contained in the same monic subgroup of G. It follows from Proposition 2.6 that $\sim$ is an equivalence relation on $G \setminus \{1\}$. Thus if S is a set of monically equivalent elements of G, the subgroup $\langle S \rangle$ is contained in a monic subgroup of G.

2.7. PROPOSITION. *If a group* G *contains an element of order* 2, *then* G *is monic.*

Proof. Let N be the (nontrivial, normal) subgroup of G generated by all elements of order 2. Now suppose x and y have order 2. Then

$$x(xy)x^{-1} = (xy)^{-1} = y(xy)y^{-1};$$

hence $x, y \in I_{xy}$. Since the latter is monic, we see that $x \sim y$. Therefore, as we noted above, N is contained in a monic subgroup of G. By Lemma 2.2, G is monic.

Thus in a multic group the order of every element is odd or infinite.

2.8. *Example. A monic nontotal group.* Let G be the free product of infinitely many groups $\{1, a_i\}$ of order 2. Then G is monic. To see that it is not total we write an element x of $G \setminus \{1\}$ as a word $a_{i_1} a_{i_2} \cdot \cdots \cdot a_{i_k}$. If $H = \langle a_{i_1}, \cdots, a_{i_k} \rangle$, then $x \in H \neq G$. But H is isolated; for if $g \in G \setminus H$, then g as a word in the elements a_i contains some $a_n \notin H$. Thus $ghg^{-1} \notin H$ for all $h \in H \setminus \{1\}$.

2.9. *Remark.* In Example 2.8, the product cannot be finite, since a group G generated by a total subgroup H and a finite number of elements of order 2 is total.

Proof. Let t be a total element of H. By induction, we can assume that G is generated by H and a single element a of order 2. We assert that the commutator $c = tat^{-1}a$ is total in G. Let I be an isolated subgroup of G that contains c. Since both tat^{-1} and a normalize $\langle c \rangle$, these elements are in I, hence $t \in I$. But then t is in $H \cap I$, which is isolated in H, so that $H \subset I$. Thus $I = G$.

3. $\mathscr{S}$-MULTIC GROUPS

We can generalize previous work by considering only isolated subgroups drawn from a restricted class of groups. This leads to more refined notions of multicity. In the case where the class is that of infinite cyclic groups, this notion of multicity has geometric significance (see Section 5), and it is useful in the construction of a broad class of multic groups (Theorem 4.2).

Henceforth, $\mathscr{S}$ will denote a nonempty, hereditary class of groups (that is, $\mathscr{S}$ contains every subgroup of each of its members). We can adapt our previous terminology to the case at hand. For example, a subgroup H of G is *$\mathscr{S}$-isolated* if H is isolated and $H \in \mathscr{S}$; an *$\mathscr{S}$-isolation* is a partition of G by $\mathscr{S}$-isolated subgroups; a group G is *$\mathscr{S}$-multic* if G has a nontrivial $\mathscr{S}$-isolation (otherwise, G is *$\mathscr{S}$-monic*). If $\mathscr{S} \subset \mathscr{T}$, then $\mathscr{S}$-multicity implies $\mathscr{T}$-multicity; in particular, $\mathscr{S}$-multicity implies multicity (and monicity implies $\mathscr{S}$-monicity), for each $\mathscr{S}$. The results of the previous section remain valid, *mutatis mutandis*, except for Proposition 2.4, which holds when the group in question has an $\mathscr{S}$-isolation. But this may not obtain; in fact one can easily verify the following assertion.

3.1. LEMMA. *For a group* G *and a class* $\mathscr{S}$, *the following are equivalent:*

(1) G *is* $\mathscr{S}$-*multic or* $G \in \mathscr{S}$,

(2) G *has an* $\mathscr{S}$-*isolation,*

(3) *every monic subgroup of* G *is in* $\mathscr{S}$.

Many natural choices of $\mathscr{S}$ consist solely of monic groups; for example, all abelian groups, all cyclic groups, all finite groups—and also the infinite cyclic class $\{\{1\}, \mathbb{Z}\}$, which we denote simply by $\mathbb{Z}$. For such classes, an $\mathscr{S}$-multic group cannot be in $\mathscr{S}$; thus Lemma 3.1 has this consequence.

3.2. COROLLARY. *Let* $\mathscr{S}$ *consist of monic groups. Then a group* G *is* $\mathscr{S}$-*multic if and only if* $G \notin \mathscr{S}$ *and every monic subgroup of* G *is in* $\mathscr{S}$.

In general, a group may have many $\mathscr{S}$-isolations; but in the monic case there is at most one:

3.3. LEMMA. *If every* $\mathscr{S}$-*subgroup of* G *is monic, then each* $\mathscr{S}$-*isolation of* G *consists of the set of all maximal* $\mathscr{S}$-*subgroups of* G, *and is the monic isolation of* G.

Proof. If P is an $\mathscr{S}$-isolation of G, then obviously P is monic—and by Lemma 3.1, every monic subgroup of G is an $\mathscr{S}$-subgroup. We have hypothesized the reverse. Thus the set of all [maximal] $\mathscr{S}$-subgroups of G is the same as the set of all [maximal] monic subgroups of G. Hence, by Proposition 2.6, the monic isolation of G consists of all the maximal $\mathcal{S}$-subgroups of G.

This leads to a basic criterion for $\mathcal{S}$-multicity:

3.4. PROPOSITION. *If* $G \notin \mathcal{S}$ *and*

(a) *each* $x \in G \setminus \{1\}$ *is contained in a maximal* $\mathcal{S}$*-subgroup,*

(b) *distinct maximal* $\mathcal{S}$*-subgroups are disjoint,*

(c) *maximal* $\mathcal{S}$*-subgroups of* G *are self-normalizing,*

then G *is* $\mathcal{S}$*-multic. The converse holds if* $\mathcal{S}$ *consists of monic groups.*

Proof. (a) and (b) assert that G is partitioned by the maximal $\mathcal{S}$-subgroups of G, and the partition is nontrivial, since $G \in \mathcal{S}$. We need only verify that each maximal $\mathcal{S}$-subgroup H is isolated. But if $xHx^{-1} \cap H \neq \{1\}$, then, since xHx^{-1} is also a maximal $\mathcal{S}$-subgroup, (b) implies that $xHx^{-1} = H$. Thus (c) implies that H is isolated. When $\mathcal{S}$ consists of monic groups, the converse follows from the preceding lemma and Corollary 3.2.

We apply this to some monic classes. For example, for the class of abelian groups we have the following result.

3.5. COROLLARY. *A group* G *is abelian-multic if and only if* G *is nonabelian and for each* $x \in G \setminus \{1\}$ *the centralizer* C(x) *is isolated (or abelian and self-normalizing).*

Proof. The parenthesized formulation follows from Proposition 3.4 in view of the equivalence of the following:

(1) C(x) is abelian for all $x \neq 1$;

(2) C(x) is the largest abelian subgroup containing x;

(3) if $C(x) \cap C(y) \neq \{1\}$ and $x \neq 1$, $y \neq 1$, then $C(x) = C(y)$.

But if C(x) is isolated, then $y \in C(x) \setminus \{1\}$ implies $C(y) \subset C(x)$. Hence, if C(y) is isolated for all $y \neq 1$, then C(x) is abelian.

Next we consider the cyclic-multic case. By using Zorn's lemma, we can replace (a) in Proposition 3.4 with various equivalent conditions, for example, with the condition that every ascending chain of cyclic groups is finite. In context with (2) of the following Corollary, (1) can replace (a).

3.6. COROLLARY. *A group* G *is cyclic-multic if and only if* G *is not cyclic and*

(1) *every* $x \in G \setminus \{1\}$ *has only a finite number of roots,*

(2) *if* $x^i = y^j \neq 1$ *for some* i *and* j, *then* $\langle x, y \rangle$ *is cyclic,*

(3) *if* $xyx^{-1} \in \langle y \rangle$, *then* $\langle x, y \rangle$ *is cyclic.*

The following corollary gives another criterion.

3.7. COROLLARY. *A group* G *is cyclic-multic if and only if* G *is not cyclic and the group* I_x *is cyclic for each* $x \in G$.

Proof. The conditions are necessary, by Lemma 3.2, because I_x is total, hence monic, hence cyclic. For the sufficiency, it suffices to note that (in the case at hand) distinct subgroups I_x are disjoint.

We now consider the $\mathbb{Z}$-multic case, showing in particular that every free noncyclic group is $\mathbb{Z}$-multic. Clearly, the two preceding corollaries give necessary and sufficient conditions for G to be $\mathbb{Z}$-multic provided we add the hypothesis that G is torsion-free. In a torsion-free group, the *height* of an element $x \in G$ is

$$\sup \{n: \text{there is a } y \in G \text{ such that } y^n = x\}.$$

It is easy to see that condition (1) in Corollary 3.6 can now be replaced by the hypothesis of finite height. Condition (3) of the corollary can be simplified by the following observation. (Recall that a subgroup $S \subset G$ is *pure* provided $x^n \in S \setminus \{1\}$ implies $x \in S$.)

Remark. Suppose that G has no elements of order 2, and that A is a pure, infinite cyclic subgroup of G. If A is self-centralizing, then A is isolated. In fact, if $xAx^{-1} \cap A \neq \{1\}$ for $x \neq 1$, then $xAx^{-1} = A$, since both A and xAx^{-1} are pure. Since $A = \langle a \rangle$ is infinite cyclic, xax^{-1} is either a or a^{-1}. In either case, x^2 and a commute. But $x^2 \neq 1$, hence $x \in A$.

Thus we obtain the following result.

3.8. COROLLARY. *A torsion-free group* G *is* $\mathbb{Z}$-*multic if and only if* G *is noncyclic and*

(1) *each element of* G *has finite height,*

(2) *if* $x^i = y^j \neq 1$ *for some* i *and* j, *then* $\langle x, y \rangle$ *is cyclic,*

(3) *if* $xy = yx$, *then* $\langle x, y \rangle$ *is cyclic.*

3.9. COROLLARY. *A group* G *is* $\mathbb{Z}$-*multic if and only if it is noncyclic and the centralizer of each element is infinite cyclic.*

Proof. Suppose G is $\mathbb{Z}$-multic with $\mathbb{Z}$-isolation $\{P_i\}$. It suffices to prove that the centralizer $C(x)$ of $x \in G \setminus \{1\}$ is cyclic. If $x \in P_i$, then $P_i \subset C(x)$. But since P_i is isolated, it follows that $y \in P_i$ for each y in $C(x)$. Thus $P_i = C(x)$.

Conversely, since each $C(x)$ is abelian, it follows as in the proof of Corollary 3.5 that $\{C(x): x \in G \setminus \{1\}\}$ is a partition of G—and thus, in the case at hand, a nontrivial $\mathbb{Z}$-partition. But each $C(x)$ is then a pure, infinite cyclic group and $CC(x) = ZC(x) = C(x)$. Hence, by the Remark before Corollary 3.8, each $C(x)$ is isolated.

By either of the preceding corollaries, *every free noncyclic group is* $\mathbb{Z}$-*multic.* This provides the last of the four examples promised after Definition 1.2. Many further examples of multic groups appear in the next two sections.

CONJECTURE (M. Schacher). *All torsion groups are monic.*

We give some positive evidence for this conjecture. First, note that monicity is a "two-element" condition, that is, if every pair of elements of G lies in some monic subgroup, then (by remarks preceding Proposition 2.5) G is monic. In particular, a 2-finite group—that is, a group such that each pair of elements generates a finite subgroup—is monic.

3.10. COROLLARY. *There exists no nontrivial isolation by cyclic groups* $\mathbb{Z}_n$ *for any* n *for which the Burnside conjecture holds.*

In particular, if G is a group in which every nontrivial element has order 3, then G is I-simple, for it is known that each element of G commutes with all its conjugates.

4. MULTIC GROUPS

We describe some methods for constructing or recognizing multic groups. A *partial isolation* A of G is a collection of mutually disjoint, isolated subgroups of G. Partial isolations A and B are *complementary* if $\bigcup A \cup \bigcup B = G$ and $\bigcup A \cap \bigcup B = \{1\}$. If A and B are complementary, then $A \cup B$ is an isolation of G.

4.1. LEMMA. *Let* A *be a partial isolation of a group* G *that has no elements of order* 2. *Let* $X = \left(G \setminus \bigcup A\right) \cup \{1\}$. *Suppose*

(1) X *is torsion-free,*

(2) *each* $x \in X \setminus \{1\}$ *has finite height in* X,

(3) *if* $x, y \in X$ *and* $x^i = y^j \neq 1$ *for some* i *and* j, *then* $\langle x, y \rangle$ *is cyclic,*

(4) *if* $xy = yx$ *for* $x, y \in X$, *then* $\langle x, y \rangle$ *is cyclic.*

Then A *has a complementary partial isolation consisting of all the maximal cyclic subgroups contained in* X.

We omit the proof, which is a minor variant of earlier arguments.

4.2. THEOREM. *Let* $\{G_\alpha\}$ *be a collection of at least two groups with no elements of order* 2. *Then the free product* $G = \prod^* G_\alpha$ *is multic, with an isolation consisting of* (a) *all subgroups of* G *conjugate to any* G_α, *and* (b) *all maximal cyclic subgroups of* G *that meet no conjugate of any* G_α.

Proof. By well-known properties of free products, the collection (a) is a partial isolation of G; thus it suffices to verify that conditions 1 to 4 in Lemma 4.1 are satisfied. The first condition is clearly satisfied, since an element of G has finite order only if it is conjugate to an element of some G_α. To show that conditions 2 to 4 are satisfied, we recall some facts about free products.

Each element $x \neq 1$ of G has a unique expression $x = g_1 \cdots g_n$, where each g_i is a nontrivial element of some G_α, and where adjacent letters g_i belong to different groups G_α. Then $n = L(x)$ is the *length* of x. If $\alpha \neq \beta$, let $W_{\alpha\beta}$ consist of all x with the first letter in G_α and the last in G_β. For each α, let W_α consists of those x with $L(x) \geq 3$ and with first and last letter in G_α that are not conjugate to an element of some G_β. Finally, let C consist of all elements of G conjugate to an element of some G_α. Then G has the following two properties.

(i) If $x \notin C$, then x is conjugate to an element of some $W_{\alpha\beta}$, and $L(x^n) \geq 2n$.

(ii) G is the disjoint union of C and all the sets $W_{\alpha\beta}$ and W_α, each of which is closed under powers—hence pure.

We now prove properties 2, 3, 4 of the lemma in the case at hand where $X = (G \setminus C) \cup \{1\}$. Property (2) follows immediately from (i).

(3) If $x^i = y^j \neq 1$ for $x, y \in X$, then $\langle x, y \rangle$ is cyclic.

Note that if this is true for a certain x (and all y satisfying the hypothesis), then it is true for each conjugate of x. Thus by (i) we can assume $x \in W_{\alpha\beta}$. Hence it follows from (ii) that $y \in W_{\alpha\beta}$. But then the word $x^i = y^j$ is reduced as written, hence the conclusion that $\langle x, y \rangle$ is cyclic follows by the same formal argument as in a free group.

(4) If $x, y \in X$ and $xy = yx$, then $\langle x, y \rangle$ is cyclic.

As above, if this holds for x it holds for any conjugate; therefore we can assume that $x \in W_{\alpha\beta}$. Let $y \in W_{\gamma\delta}$ (which we understand to be W_γ if $\delta = \gamma$). *Case* 1. $\gamma \neq \beta$. Here xy is reduced as written; that is, $L(xy) = L(x) + L(y)$. Hence the same is true for yx, and the conclusion follows as in a free group. *Case* 2. $\gamma = \beta$. Here $L(xy) < L(x) + L(y)$. The same must be true for yx, hence $\delta = \alpha$. Thus $y \in W_{\beta\alpha}$. But then $y^{-1} \in W_{\alpha\beta}$, hence we have reduced the problem to Case 1.

The preceding theorem lets us construct a variety of multic groups. In particular, it gives a characterization of the groups that appear in isolations.

4.3. COROLLARY. *A group* H *is isomorphic to an element of a nontrivial isolation of some group if and only if* H *has no elements of order* 2.

Let G be a group acting (by means of permutations) on a set X. We say that G acts *multicly* on X provided that

(1) each $g \in G \setminus \{1\}$ has exactly one fixed point in X, and

(2) not all $g \in G$ have the same fixed point.

4.4. LEMMA. *A group* G *is multic if and only if* G *acts multicly on some set.*

Proof. Suppose G is multic. If D is the monic isolation of G, then G acts by inner automorphisms on D. That (1) above holds follows immediately from the definition of isolated subgroup, and (2) holds since D is nontrivial.

Conversely, suppose G acts multicly on a set X. Let G_x be the stability group $\{g \in G: gx = x\}$ of the element x of X. Then (1) implies that $\{G_x: x \in X\}$ is a partition of G. Furthermore, each G_x is isolated; for if $gG_xg^{-1} \cap G_x \neq \{1\}$, then $G_{gx} \cap G_x \neq \{1\}$ (since $gG_xg^{-1} = G_{gx}$). Hence (1) implies that $gx = x$; that is, $g \in G_x$. Finally, (2) shows that this isolation is nontrivial.

We can improve the lemma as follows. If $f: X \to X$, let $F(f)$ denote the fixed-point set of f.

4.5. PROPOSITION. *Suppose* G *acts on a set* X *so that*

(1) *each element of* G *has a fixed point, but not all* $g \in G \setminus \{1\}$ *have the same fixed-point set, and*

(2) *if* $h(F(g)) = F(g)$ *for* $g \neq 1$, $h \neq 1$, *then* $F(h) \subset F(g)$.

Then G *is multic.*

Proof. Under our hypotheses, $F(g) \subset F(h)$ implies $F(g) = F(h)$, for then $hF(g) = F(g)$, hence $F(h) \subset F(g)$.

Let $Y = \{F(g): g \in G \setminus \{1\}\}$. The group G acts on Y with $h \in G$ sending $F(g)$ to $hF(g) = F(hgh^{-1})$. We assert that this action is multic. First, $g \in G$ obviously has the fixed point $F(g) \in Y$. If $g \neq 1$, this is the only fixed point; for if $gF(h) = F(h)$ with $h \neq 1$, then by (2) and the observation at the beginning of the proof, $F(g) = F(h)$. Finally, (1) implies that not all $g \in G$ have a common fixed point in Y.

For example, if G is a subgroup of the rotation group SO(3), then by considering the usual action on the sphere S^2 one can show that the necessary conditions

(i) G nonabelian and

(ii) G contains no elements of order 2

are sufficient for G to be multic.

Another application of the proposition: if a torsion-free group G acts on a set X so that every $g \in G \setminus \{1\}$ has exactly n fixed points but not all have the same set of fixed points, then G is multic.

5. GEOMETRIC APPLICATIONS

The geometric properties of a Riemannian manifold, notably its sectional curvature K, are closely related to the monicity of its fundamental group $\pi_1(M)$.

5.1. COROLLARY. *If* M *is a complete Riemannian manifold with sectional curvature* $K \geq 0$, *then* $\pi_1(M)$ *is total.*

Proof. By a theorem of J. Cheeger and D. Gromohl [2], the group $\pi = \pi_1(m)$ contains a finite normal subgroup ϕ such that π/ϕ is a crystallographic group. If ϕ is nontrivial, then π is total, by Lemmas 1.5 and 1.7. If ϕ is trivial, then, being crystallographic, π contains a normal abelian subgroup of finite index. Hence again π is total, by Lemmas 1.3 and 1.8.

5.2. THEOREM. *If* M *is a compact Riemannian manifold with* $K < 0$, *then* $\pi_1(M)$ *is* $\mathbb{Z}$*-multic.*

This result is implicit in Théorème 7, Chapitre 3, of Preismann [5]. He argued as follows: Consider $\pi = \pi_1(M)$ as the *decktransformation* group of the simply connected Riemannian covering $\tilde{M} \to M$ of M, and let X be the set of geodesics of $\tilde{M}$ (considered as 1-dimensional submanifolds of $\tilde{M}$). Preismann showed that (in our language) π acts multicly on X. The stability groups π_x are well known to be infinite cyclic; hence the result follows by Lemma 4.4. Preismann's actual conclusion was that $\pi_1(M)$ *is not cyclic and that every* abelian *subgroup of* M *is infinite cyclic.* In view of Corollary 3.2, the conclusion above is stronger to the extent of replacing *abelian* by *monic.*

These ideas were extended in [1] and [3], where a complete Riemannian manifold M with curvature $K \leq c < 0$ is called *parabolic, axial,* or *fuchsian* depending on whether the number of closed geodesics in M is 0, 1, or ∞ (these being the only possibilities). If M is parabolic, nothing significant is known about its fundamental group; if M is axial, $\pi_1(M)$ is infinite cyclic. However, *if* M *is fuchsian, then* $\pi_1(M)$ *is multic.* This case may almost be considered typical, since M if fuchsian if one of the following holds:

(1) M is compact,

(2) M has at least two closed geodesics,

(3) M has a closed but not simply closed geodesic,

(4) M is not diffeomorphic to a product $L \times R^1$ or to a vector bundle over a circle,

(5) M has more than two ends.

By a *surface* we mean a connected, paracompact, 2-dimensional manifold. The fundamental groups of surfaces go to extremes in their supplies of isolated subgroups:

5.3. COROLLARY. *The fundamental group* $\pi_1 M$ *of a surface* M *is* I-*simple if* M *is one of the following: a plane, a sphere, a projective plane, a torus, a Klein bottle, a cylinder, an open Möbius band. Otherwise,* $\pi_1 M$ *is* $\mathbb{Z}$*-multic.*

Proof. The fundamental groups of the seven listed surfaces are abelian, except for the group of the Klein bottle, which has nontrivial center. Hence all are I-simple, by Lemma 1.3.

Now suppose M is a surface not on the list above. If M is not compact, then $\pi_1(M)$ is free, hence $\mathbb{Z}$-multic. If M is compact, then M is a sphere with two or more handles, or is double-covered by such a surface. But all such surfaces admit a Riemannian structure of constant negative curvature; hence $\pi_1 M$ is $\mathbb{Z}$-multic, by Theorem 5.2. (The fundamental groups in this case are well-known [7], and they have presentations with a single relation.)

6. FROBENIUS GROUPS

6.1. *Definition.* A finite group G with a nontrivial isolated subgroup is called a *Frobenius group.* The nontrivial isolated subgroups are known as *Frobenius complements* of G.

Particularly simple examples of Frobenius groups are the following.

(i) The group G of linear mappings $x \to ax + b$ $(a \neq 0)$ of a finite field GF(q) onto itself. The isolated subgroups G_α consist of those mappings that leave an element α fixed. Since G is transitive, all the G_α are conjugates in G, and, since only the identity has more than one fixed point, $G_\alpha \cap G_\beta = \{1\}$ for $\alpha \neq \beta$. The complement of $\bigcup G_\alpha$ together with the identity forms the normal subgroup T of translations $x \to x + b$.

(ii) The dihedral group D_n with odd n. Here the n subgroups of order 2 are isolated, since each is its own normalizer. Again, the complement of these isolated subgroups together with the identity is the cyclic normal subgroup of order n.

The study of Frobenius groups is quite extensive, and we list here a few of the more important results, referring to D. S. Passman's book [4] for proofs and references.

6.2. THEOREMS (Frobenius, Thompson, Higman, Zassenhaus, and others).

1. *All Frobenius complements* H_i *of a Frobenius group* G *are conjugates. The elements of* G *not contained in any* H_i, *together with the identity, form a normal subgroup* M *of* G, *called the* F r o b e n i u s k e r n e l *of* G.

2. *The Frobenius kernel acts transitively and fixed-point-free (under conjugation) on the set of Frobenius complements of* G. *Thus the number of nontrivial isolated subgroups is* $|M|$, *and* $|G| = |H|\ |M|$.

3. *The Frobenius complements* H *act fixed-point-free (under conjugation) on the elements of the Frobenius kernel* M. *Thus the number of elements in any normal class of* M, *other than* $\{1\}$, *is a multiple of* $|H|$, *and* $|M| \equiv 1 \pmod{|H|}$. *In particular* $N < M$, $N \lhd G$ *implies* $|N| \equiv 1 \pmod{|H|}$.

4. *The Frobenius kernel* M *is nilpotent. If* p *is a prime divisor of* $|H|$, *then there exists an upper bound* k(p), *depending on* p *alone, of the nilpotency class of* M.

The Sylow subgroups of M are thus normal Sylow subgroups of the Frobenius group.

5. *If a subgroup* G_1 *of a Frobenius group* G *is not contained in either a Frobenius complement* H *or in the Frobenius kernel* M, *then* G_1 *is itself a*

Frobenius group with Frobenius complements $H_1 = G_1 \cap H$ *and Frobenius kernel* $M_1 = G_1 \cap M$.

6. *Any homomorphic image* G^ϕ *of a Frobenius group* G, *where the kernel* K_ϕ *of* ϕ *does not contain the Frobenius kernel* M, *is itself a Frobenius group. The relation* $K_\phi < M$ *holds, and the Frobenius complements* H^ϕ *of* G^ϕ *are isomorphic to the Frobenius complements* H *of* G, *while the Frobenius kernel is* $M^\phi \cong M/K_\phi$.

7. *The Sylow subgroup* S_p *of a Frobenius complement* H *is cyclic for* $p > 2$ *and cyclic or quaternion for* $p = 2$. *If* p *and* q *are distinct primes, then each subgroup of* H *of order* pq *is cyclic. If* $|H|$ *is even, then* H *contains a unique (central) element of order* 2.

8. *If the Frobenius complement* H *is solvable, then it has a normal subgroup* $H_0 \lhd H$ *such that* H/H_0 *is isomorphic to a subgroup of the symmetric group* S_4, *and* H_0 *has cyclic Sylow subgroups.*

9. *If the Frobenius complement* H *is not solvable, then* H *has a normal subgroup* H_0 *of index* 1 *or* 2 *in* H *with* $H_0 \cong SL(2, 5) \times H_1$, *where* H_1 *has cyclic Sylow subgroups and* $(|H_1|, 30) = 1$.

10. *A group is a Frobenius complement if and only if it can be faithfully represented as a fixed-point-free group of linear transformations on a finite vector space.*

The Frobenius complements of G share an important property with Sylow subgroups.

6.3. THEOREM. *A subgroup* K *of the Frobenius group* G *with* $|K| = |H|$, *where* H *is a Frobenius complement of* G, *is itself a Frobenius complement, and hence a conjugate of* H.

Proof. Since the orders of the elements of K are prime to the order $|M|$ of the Frobenius kernel, each element of K belongs to a Frobenius complement. We may therefore choose a complement H such that $H \cap K \neq \{1\}$. If $H_1 = H \cap K \neq K$, then K is itself a Frobenius group with complement H_1 and kernel M_1, where $(|H_1|, |M_1|) = 1$. According to Theorem 6.2, part 1, we have the relations $|H_1'| = |H' \cap K| = |H_1|$ whenever $H' \cap K \neq \{1\}$, contrary to the fact that each element of M_1 must be contained in one of the Frobenius complements of G. Thus the assumption $1 < |H \cap K| < |K|$ leads to a contradiction, and we conclude that $H = K$ for some Frobenius complement H.

Some of the theory of Frobenius groups can be extended to groups that are not necessarily finite.

6.4. *Definition.* A group G will be called a *Frobenius group* if

(i) G contains nontrivial isolated subgroups all of which are isomorphic under automorphisms of G (these will be called the *Frobenius complements* of G),

(ii) those elements of G that are not contained in a Frobenius complement, together with the identity, form a normal subgroup, the *Frobenius kernel* of G.

Examples of infinite Frobenius groups are quite easy to construct.

6.5. *Examples.* 1. *The infinite dihedral group* D_∞, which is the free product of two groups of order 2, say $\{1, a\} * \{1, b\}$. Here the Frobenius complements are the groups of order 2, that is, $\{1, g^{-1}ag\}$, $\{1, g^{-1}bg\}$. Note that these form *two* conjugacy classes, in contrast to the finite case. The Frobenius kernel is the cyclic group $\langle ab \rangle$.

2. Any multiplicative subgroup H of a division ring R acts fixed-point-free on the additive group M of that ring. Thus the semidirect product

$$\{(h, m) \mid h \in H,\ m \in M\}$$

with

$$(h_1, m_1)(h_2, m_2) = (h_1 h_2, m_1 h_2 + m_2)$$

is a Frobenius group whose complements are

$$(h_1, m_1)(h_2, m_2) = (h_1 h_2, m_1 h_2 + m_2) \qquad (m \in R_+),$$

while the kernel is $\{(1, m) \mid m \in M\} \cong R_+$.

6.6. *Remark.* Each Frobenius group G is total. In fact, each element $m \neq 1$ of the Frobenius kernel M is total.

Note that I_m is isolated and contains m. By hypothesis, no m in M belongs to a nontrivial isolated subgroup of G. Thus $I_m = G$.

We can now strengthen our previous remark that 2-finite groups are monic.

6.7. THEOREM. *A 2-finite group* G *with a nontrivial isolated subgroup is a Frobenius group (which is total).*

Proof. Let H and H' be two nontrivial conjugate isolated subgroups of G. Pick $h \in H \setminus \{1\}$ and $h' \in H' \setminus \{1\}$; then $G_1 = \langle h, h' \rangle$ is a finite Frobenius group with Frobenius complements $H_1 = H \cap G_1$ and $H_1' = H' \cap G_1$. Let M_1 be the Frobenius kernel of G_1. Then $M_1 \cap H'' = \{1\}$ for any conjugate H'' of H; for otherwise M_1 would have nontrivial intersection with the Frobenius complement $H_1'' = H'' \cap G_1$ of G_1.

We have thus proved that the conjugates of H do not cover G. We next prove that the elements not contained in any conjugate of H, together with the identity, form a characteristic subgroup consisting of those elements whose order is prime to the orders of the elements of H. To see this, let m_1 and m_2 be two elements not contained in any conjugate of H, and set $G_1 = \langle m_1, m_2 \rangle$. If $G_1 \cap H' = \{1\}$ for all $H' \sim H$, then either $m_1 m_2 = 1$ or $m_1 m_2$ is not contained in any conjugate of H.

If $G_1 \cap H' \neq \{1\}$, then G_1 is a finite Frobenius group with complement $H_1 = H' \cap G_1$. Since m_1 and m_2 are not contained in any conjugate of H_1, they are contained in the Frobenius kernel, and so is $m_1 m_2$. The elements m thus do form a group M whose intersection with the conjugates of H is trivial.

Let $m \in M \setminus \{1\}$ and $h \in H \setminus \{1\}$; then in the finite Frobenius group $G_1 = \langle m, h \rangle$, the element m is in the Frobenius kernel M_1 of G_1, and h is in the Frobenius complement $H_1 = H \cap G_1$ of G_1. Thus (ord m, ord h) = 1.

It remains to show that every nontrivial isolated subgroup is conjugate to H. Let K be such an isolated subgroup. Choose $k \in K \setminus \{1\}$ with (ord k) | (ord m) for some $m \in M$, and choose $h \in H \setminus \{1\}$. Then $G_1 = \langle k, h \rangle$ is a finite Frobenius group with nonconjugate Frobenius complements $K \cap G_1$ and $H \cap G_1$, a contradiction. Thus every element of K is conjugate to an element of H. Let $k_1 \in K \cap H' \setminus \{1\}$ and $k_2 \in K \cap H'' \setminus \{1\}$, where H' and H'' are distinct conjugates of H. Then $G_1 = \langle k_1, K_2 \rangle$ is a finite Frobenius group with complement $H' \cap G_1$. By what we proved above, the Frobenius kernel M_1 of G_1 is a subgroup of M, and since $G_1 < K$, the intersection $K \cap M \neq \{1\}$, contrary to our remark on the relative primeness of the orders of the elements of K and M.

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