# ON FIXED POINTS OF A COMPACT AUTOMORPHISM GROUP

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In this note, we investigate the existence of nontrivial fixed points under a compact, connected group of automorphisms of a Lie group. Although we cannot always hope for the existence of such fixed points in connected groups, the situation for nonconnected groups seems to be more favorable. The following theorem, the proof of which constitutes the main portion of this note, bears this out.

THEOREM 1. *Let* G *be a Lie group such that the identity component* $G_0$ *of* G *contains no nontrivial compact subgroup. Then, for every compact, connected group* C *of automorphisms of* G, *we have the relation* $G = G_0 \cdot F(C)$, *where* F(C) *denotes the collection of all fixed points of* C *in* G.

As an easy consequence of this theorem, we prove that a compact, connected group of automorphisms of G has a nontrivial fixed point, if G is a nonconnected Lie group that contains no compact semisimple subgroup. We also present other applications of the theorem, together with an example to supplement our result.

We note that the topology in the group of automorphisms of a Lie group is understood to be the so-called generalized compact-open topology, under which the group is a topological group. In the rest of this note, we use $G_0$ to denote the identity component of a Lie group G.

## 1. PROOF OF THEOREM 1

LEMMA 1. *Let* H *be a connected semisimple Lie group that contains no compact, semisimple subgroup. Then every compact, connected group of automorphisms of* H *is a torus.*

*Proof.* Let C be a compact, connected group of automorphisms of H, and let S be the commutator subgroup of C. We claim that S is trivial. Since H is semisimple, every automorphism in C is an inner automorphism. Since the adjoint group of H is isomorphic with H/Z, where Z denotes the center of H, we may identify S with a compact, connected subgroup of H/Z. With this identification, let L be the complete inverse image of S under the projection map $H \to H/Z$. Since Z is a discrete subgroup of H, $L_0$ is easily seen to be a covering group of S. Hence $L_0$ is a compact, connected semisimple subgroup of H. Because H contains no such subgroup by our hypothesis, it follows that $L_0$ is trivial. Hence S is trivial, and C is a torus.

LEMMA 2. *If* $G_0$ *is a semisimple group that contains no nontrivial compact subgroup, then the conclusion of Theorem* 1 *holds.*

*Proof.* Let $\pi$: Aut(G) $\to$ Aut($G_0$) be the restricting homomorphism, where, for a Lie group L, Aut(L) denotes the group of automorphisms of L, and let $C_1 = \pi(C)$. Since $\pi$ is continuous, $C_1$ is a compact subgroup of Aut(G ). Choose a maximal

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compact subgroup P of $\mathrm{Aut}(G_0)$ containing $C_1$. Since $G_0$ is semisimple, the identity component of $\mathrm{Aut}(G_0)$ is the adjoint group $\mathrm{Int}(G_0)$ of $G_0$, which is of finite index in $\mathrm{Aut}(G_0)$. By the well-known decomposition theorem (see [1, Theorem 3.1, p. 180]), we have that $\mathrm{Aut}(G_0) = \mathrm{Int}(G_0) \cdot P$. If M denotes the subgroup of G consisting of elements x whose induced inner automorphisms $\mu(x)$ (when restricted to $G_0$) are in P, then we have that $G = G_0 \cdot M$.

Now consider the homomorphism $\mu\colon M \to P$. It is easy to see that the kernel of $\mu$ is discrete and that $\mu(M_0) = P_0$. Thus $M_0$ is a covering group of $P_0$. Since $P_0$ is a torus by Lemma 1, it follows that its covering group $M_0$ is abelian. Note that C leaves M invariant; hence C induces a connected group of automorphisms of the discrete group $M/M_0$. But $\mathrm{Aut}(M/M_0)$ is totally disconnected, and thus, for each $\rho \in C$, we have that

$$\rho(m)m^{-1} \in M_0, \qquad \text{for all } m \in M.$$

Let $\lambda(m) = \rho(m)m^{-1}$, for $m \in M$. Recalling that $M_0$ is abelian and that $C_1$ is a subgroup of the torus $P_0$, we see that C leaves $M_0$ pointwise fixed. It follows that

$$(\lambda(m))^k = \rho^k(m)m^{-1} \qquad (k = 1, 2, \cdots).$$

The compactness of C implies that $\lambda(m)$ generates a cyclic group whose closure is compact. However, $G_0$ has no nontrivial compact subgroup by the assumption of our lemma. Thus $\lambda(m)$ is trivial, and we see that C leaves M pointwise fixed. Therefore we have that $G = G_0 \cdot F(C)$.

We are now ready to prove Theorem 1 by induction on the dimension of $G_0$. By Lemma 2, we may assume that $G_0$ is not semisimple. Let A be a closed, invariant vector subgroup of $G_0$ that is also invariant under C, and let $C_1$ be the compact, connected group of automorphisms of G/A that is induced by C. Applying the induction assumption on $C_1$, we have that

$$G/A = (G_0/A) \cdot F(C_1).$$

Let H be the complete inverse image of $F(C_1)$ under the projection map $G \to G/A$. Then clearly $G = G_0 \cdot H$. Since $C_1 = 1$ on $F(C_1)$, it follows that $\rho(h)h^{-1} \in A$, for all $\rho \in C$ and all $h \in H$. Hence, for each $h \in H$, we may define a mapping $\mu_h\colon C \to A$ by $\mu_h(\rho) = \rho(h)h^{-1}$ $(\rho \in C)$. Then, for $\rho_i \in C$ $(i = 1, 2)$, we have the equalities

$$\mu_h(\rho_1\rho_2) = \rho_1\rho_2(h)h^{-1} = \rho_1(\rho_2(h))h^{-1} = \rho_1(\mu_h(\rho_2)h)h^{-1} = \mu_h(\rho_1)\rho_1(\mu_h(\rho_2)).$$

Regarding the vector group A as a continuous C-module, we find that $\mu_h$ is a continuous 1-cocycle of C with values in A. Since the cohomology group $H^1(C, A)$ is trivial by the compactness of C (see, for example, [2, Theorem 2.8, p. 18]), there exists an element $a \in A$ such that $\mu_h(\rho) = \rho(a)a^{-1}$, for all $\rho \in C$. This readily implies that $a^{-1}h \in F(C)$. Thus $h \in A \cdot F(C)$, and since $h \in H$ is arbitrary, H is contained in $A \cdot F(C)$. It follows that $G = G_0 \cdot H = G_0 \cdot F(C)$.

## 2. SOME APPLICATIONS OF THEOREM 1

In order to present the announced result on the existence of fixed points, we first prove the following lemma.

LEMMA 3. *Let* C *be a compact group of automorphisms of a connected Lie group* H. *Then there exists a maximal compact subgroup* K *of* H *that is invariant under* C.

*Proof.* Let L be the semidirect product of H and C with respect to the action of C on H, and let M be a maximal compact subgroup of L that contains C. Since $M \cap H$ is an invariant subgroup of M, it is, in particular, invariant under C. Thus it suffices to show that $K = M \cap H$ is a maximal compact subgroup of H. Suppose that $K_1$ is a maximal compact subgroup of H that contains K. By the well-known conjugacy theorem of compact subgroups of L (see, for example, [1, Theorem 3.1, p. 180]), there exists an element $x \in H$ such that $xK_1x^{-1} \subseteq M$. Thus

$$xK_1x^{-1} \subseteq H \cap M = K,$$

and comparison of dimensions of K and $K_1$ gives the result that $K = K_1$, which proves the maximality of K in H.

THEOREM 2. *Let* G *be a nonconnected Lie group such that* $G_0$ *contains no compact semisimple subgroup. Then every connected, compact group of automorphisms of* G *has a nontrivial fixed point.*

*Proof.* By Lemma 3, C leaves a maximal compact subgroup of $G_0$ invariant. Since such a maximal compact subgroup is a torus by our assumption, the connectedness of C implies that C leaves every element of this torus fixed. Hence we may assume that $G_0$ contains no compact subgroups. Since G is not connected, F(C) is not trivial by Theorem 1, and the assertion is established.

Using Theorem 2, we prove the following result.

PROPOSITION. *Let* G *be a connected Lie group, and let* N *be an invariant closed subgroup of* G *such that* G/N *is compact. If the centers of* G *and* N *meet each other trivially, then the center of* N *is connected.*

*Proof.* Let Z denote the center of N, and define $\lambda: G/N \to \mathrm{Aut}(Z)$ by $\lambda(gN)(x) = gxg^{-1}$ ($gN \in G/N$ and $x \in Z$). Clearly, $\lambda(G/N)$ is a compact, connected group of automorphisms of Z. Thus we see that if $\lambda(G/N)$ has no nontrivial fixed points, then Z is connected (Theorem 2). However, the subgroup consisting of all fixed points of $\lambda(G/N)$ in Z is easily seen to coincide with the intersection of Z with the center of G, which is assumed to be trivial. Hence Z is connected.

*Example.* Let H be some compact semisimple connected Lie group such that Aut(H) is not connected (for example, SU(n) is such a group). Let G = Aut(H), and consider the homomorphism $\psi: H \to \mathrm{Aut}(G)$ defined by $\psi(h)(\beta) = I_h \cdot \beta \cdot I_h^{-1}$ ($h \in H$ and $\beta \in G$), where $I_h$ is the inner automorphism of H induced by h. The compact, connected group $\psi(H)$ of automorphisms of G has no nontrivial fixed point in the nonconnected group G. In fact, if $\beta$ is fixed under $\psi(H)$, then $I_h \cdot \beta \cdot I_h^{-1} = \beta$, for all $h \in H$. Hence, for each $x \in H$, we have the relation

$$\beta(x) = I_h \cdot \beta \cdot I_h^{-1}(x) = h\beta(h)^{-1}\beta(x)\beta(h)h^{-1},$$

which implies that $\beta(h)h^{-1}$ is in the center Z of H. But Z is discrete and H is connected. Hence the continuity of the map $h \to \beta(h)h^{-1}$ from H into Z implies that $\beta(h) = h$; hence $\beta = 1$.

This example shows that the restriction imposed in the hypotheses of Theorems 1 and 2 is inevitable.

## REFERENCES

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