LOCAL BOUNDEDNESS OF NONLINEAR, MONOTONE OPERATORS

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1. INTRODUCTION

Let X denote a locally convex Hausdorff (topological vector) space over the reals R. Let X^* denote the dual of X, and write $\langle x, x^* \rangle$ in place of $x^*(x)$ for $x \in X$ and $x^* \in X^*$.

A multivalued mapping $T: X \to X^*$ is called a *monotone operator* if

$$(1.1) \qquad \langle x - y, x^* - y^* \rangle \geq 0$$

whenever $x^* \in T(x)$ and $y^* \in T(y)$. It is called a *maximal* monotone operator if, in addition, the graph of T, in other words, the set

$$(1.2) \qquad \{(x, x^*) \mid x^* \in T(x)\} \subset X \times X^*,$$

is not properly contained in the graph of any other monotone operator $T': X \to X^*$. It is said to be *locally bounded* at x if there exists a neighborhood U of x such that the set

$$(1.3) \qquad T(U) = \bigcup \{T(y) \mid y \in U\}$$

is an equicontinuous subset of X^*. (Of course, if X is a Banach space, then the equicontinuous subsets of X^* coincide with the bounded subsets.)

In the case where X is a Banach space, it follows from a result of T. Kato [7] that a monotone operator $T: X \to X^*$ is locally bounded at a point x if x is an interior point of the set

$$(1.4) \qquad D(T) = \{x \in X \mid T(x) \neq \emptyset\}$$

and T is locally hemibounded at x (in other words, for each $u \in X$ there exists an $\varepsilon > 0$ such that the set

$$\bigcup \{T(x + \lambda u) \mid 0 \leq \lambda \leq \varepsilon\}$$

is equicontinuous in X^*). Moreover, Kato showed in [6] that the assumption of local hemiboundedness is redundant when X is finite-dimensional.

In this note, we establish the following more general result, which implies, among other things, that the assumption of local hemiboundedness is redundant even when X is an infinite-dimensional Banach space. (The abbreviations conv, int, and cl denote convex hull, interior, and (strong) closure, respectively.)

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THEOREM 1. *Let* X *be a Banach space, and let* $T: X \to X^*$ *be a maximal monotone operator. Suppose either that*

$$\text{int}(\text{conv } D(T)) \neq \emptyset \tag{1.5}$$

or that X *is reflexive and there exists a point of* D(T) *at which* T *is locally bounded. Then* int D(T) *is a nonempty convex set whose closure is* cl D(T). *Furthermore,* T *is locally bounded at each point of* int D(T), *whereas* T *is not locally bounded at any boundary point of* D(T).

We shall prove Theorem 1 and its corollaries in Section 3.

The following corollary corresponds to the result of Kato [7] that a single-valued, monotone operator T on an open subset of a Banach space X is demicontinuous if it is hemicontinuous.

COROLLARY 1.1. *Suppose the hypothesis of Theorem* 1 *is satisfied, and let* D_0 *denote the subset of* D(T) *where* T *is single-valued. Then* $D_0 \subset \text{int } D(T)$, *and* T *is demicontinuous on* D_0, *in other words, continuous as a single-valued mapping from* D_0 *in the strong topology to* X^* *in the weak** *topology.*

The convexity assertion of Theorem 1 is also worth noting. We have shown elsewhere [11] that if X is a *reflexive* Banach space and $T: X \to X^*$ is a maximal monotone operator, then cl D(T) is a convex set. In fact, if X is also separable, then D(T) itself is a *virtually convex* set, in the sense that for each relatively (strongly) compact subset K of conv D(T) and each $\varepsilon > 0$ there exists a strongly continuous mapping p of K into D(T) such that $\|p(x) - x\| \leq \varepsilon$ for every $x \in K$. In this context, Theorem 1 contributes a condition under which D(T) is virtually convex even though X may not be reflexive.

COROLLARY 1.2. *Under the hypothesis of Theorem* 1, D(T) *is virtually convex, and in particular* cl D(T) *is convex. If in addition* D(T) *is dense in* X, *then* D(T) *must be all of* X.

When X is reflexive, we can apply Theorem 1 to the maximal monotone operator T^{-1}, where

$$T^{-1}(x^*) = \{x \mid x^* \in T(x)\}. \tag{1.6}$$

Since $D(T^{-1})$ is the same as the *range* of T, that is, the set

$$R(T) = \bigcup \{T(x) \mid x \in X\}, \tag{1.7}$$

one can obtain various corollaries concerning the range of T.

COROLLARY 1.3. *Let* X *be a reflexive Banach space, and let* $T: X \to X^*$ *be a maximal monotone operator. Then* $0 \in \text{int } R(T)$ *if and only if* $0 \in \text{cl } R(T)$ *and there exist positive numbers* α *and* ε *such that*

$$\|x\| \geq \alpha \Rightarrow \|x^*\| \geq \varepsilon \qquad (\forall x^* \in T(x)). \tag{1.8}$$

COROLLARY 1.4. *Let* X *be a reflexive Banach space, and let* $T: X \to X^*$ *be a maximal monotone operator. Suppose there exists a subset* $B \subset X$ *such that*

$$0 \in \text{int}(\text{conv } T(B)).$$

Then there exists an $x \in X$ *such that* $0 \in T(x)$.

COROLLARY 1.5. *Let* X *be a reflexive Banach space, and let* $T: X \to X^*$ *be a monotone operator (not necessarily maximal). Suppose there exist* $x_i^* \in T(x_i)$ $(i = 1, 2, \cdots)$ *such that*

$$\lim_{i\to\infty} \|x_i\| = \infty \quad \textit{and} \quad \lim_{i\to\infty} \|x_i^* - x^*\| = 0.$$

Then x^* *is a boundary point of* R(T).

COROLLARY 1.6. *Let* X *be a reflexive Banach space, and let* $T: X \to X^*$ *be a maximal monotone operator. In order that* R(T) *be all of* X^*, *it is necessary and sufficient that the sequence* $x_1^*, x_2^*, \cdots$ *have no strongly convergent subsequence whenever* $x_i^* \in T(x_i)$ $(i = 1, 2, \cdots)$ *and* $\lim_{i\to\infty} \|x_i\| = \infty$.

We remark that according to [11, Corollary 1 to Theorem 2], the condition $0 \in \mathrm{cl}\, R(T)$ in Corollary 1.3 is equivalent to the nonexistence of a $u \in X$ and a $\delta > 0$ such that $\langle u, x^* \rangle \leq -\delta$ for every $x^* \in R(T)$.

Corollary 1.4 is a generalization of the main existence theorem of G. J. Minty [8], which requires that the unit ball of X is smooth and that, in effect,

$$0 \in \mathrm{int}(\mathrm{conv}\, T_0(B)),$$

where T_0 is some mapping with the properties that $T_0(x) \subset T(x)$ for every x and

$$\sup_{x\in B} \sup_{x^*\in T_0(x)} \langle x, x^* \rangle < \infty.$$

The necessary and sufficient condition in Corollary 1.6 is satisfied, in particular, if the following condition is satisfied:

$$\text{if } x_i^* \in T(x_i)\ (i = 1, 2, \cdots) \text{ and } \lim_{i\to\infty} \|x_i\| = \infty, \text{ then } \lim_{i\to\infty} \|x_i^*\| = \infty.$$

(The two conditions are equivalent, of course, when X is finite-dimensional.) Under the additional assumption that X is uniformly convex and X^* is strictly convex, F. Browder [4, Theorem 4] established that the latter condition is sufficient for R(T) to be all of X^*.

In the case where T is the subdifferential of a lower-semicontinuous, proper convex function f on X (see [10], [12]), Theorem 1 reduces to known results (see [1], [9]), provided int(conv D(T)) is nonempty; but the fact that the local boundedness of T at some point of D(T) implies the nonemptiness of int(conv D(T)) has not been pointed out previously. (Theorem 1 gives this implication only for reflexive X, but reflexivity is used in the proof only to ensure that cl D(T) is convex, and the latter is true for subdifferential mappings even if X is a nonreflexive Banach space [3].)

2. GENERAL BOUNDEDNESS THEOREM

Theorem 1 will be deduced from a broader result, which is applicable even when X is not a Banach space.

THEOREM 2. *Let* X *be a locally convex (real) Hausdorff space, and let* $T: X \to X^*$ *be a maximal monotone operator. Suppose there exist a subset* $S \subset D(T)$ *and an equicontinuous subset* $A \subset X^*$ *such that* $T(x)$ *meets* A *for every* $x \in S$ *and one of the following two conditions holds:*

(a) $\operatorname{int}(\operatorname{cl} S) \neq \emptyset$,

(b) $\operatorname{int}(\operatorname{cl}(\operatorname{conv} S)) \neq \emptyset$ *and* $\sup_{x \in S} \sup_{x^* \in A} |\langle x, x^* \rangle| < \infty$.

Then $\operatorname{int} D(T)$ *is a nonempty, open, convex set whose closure is* $\operatorname{cl} D(T)$. *Furthermore,* T *is locally bounded at each point of* $\operatorname{int} D(T)$, *whereas* T *is not locally bounded at any boundary point of* D(T).

In proving Theorem 2, we shall use three lemmas.

LEMMA 1. *Let* X *be a locally convex Hausdorff space, and let* $T: X \to X^*$ *be a monotone operator. Let* B *be an equicontinuous subset of* X^*. *Then, for every* $x \in X$, *there exists an* $x^* \in X^*$ *such that*

$$\langle u - x, u^* - x^* \rangle \geq 0 \qquad (\forall u \in X, \ \forall u^* \in T(u) \cap B).$$

Proof. Give X^* the weak* topology. Then B is a relatively compact subset of X^*, and the dual of X^* can be identified with X. Let S be the restriction of T^{-1} to B; thus

$$S(u^*) = \{u \in X \mid u^* \in T(u)\}$$

if $u^* \in B$, while $S(u^*) = \emptyset$ if $u^* \notin B$. Then S is a monotone operator from X^* to X with $D(S) \subset B$. According to the theorem of H. Debrunner and P. Flor [5], there exists for every $x \in X$ an $x^* \in X^*$ such that

$$\langle u - x, u^* - x^* \rangle \geq 0 \qquad (\forall u^* \in B, \ \forall u \in S(u^*)).$$

The latter relation is identical to the one in the lemma, in view of the definition of S.

COROLLARY. *Let* X *be a locally convex Hausdorff space, and let* $T: X \to X^*$ *be a maximal monotone operator. If* T *is globally bounded, in other words, if* R(T) *is an equicontinuous subset of* X^*, *then* D(T) *is all of* X.

LEMMA 2. *Let* X *be a locally convex Hausdorff space, and let* $T: X \to X^*$ *be a maximal monotone operator. Then, for each weak*-closed, equicontinuous subset* B *of* X^*, *the set*

$$T^{-1}(B) = \{x \mid B \cap T(x) \neq \emptyset\}$$

is closed in X.

Proof. Let $y \in \operatorname{cl} T^{-1}(B)$. For each neighborhood U of y, the intersection $T(U) \cap B$ is nonempty; denote the weak* closure of this intersection by B_U. Since B is weak*-closed and equicontinuous, each B_U is weak*-compact. The collection of sets B_U, as U ranges over all neighborhoods U of y, has the property that every finite subcollection has a nonempty intersection, and hence this collection as a whole has a nonempty intersection. Thus there exists some $y^* \in B$ such that y^* belongs to the weak* closure of T(U) for every neighborhood U of y. We shall show that the latter implies that $y^* \in T(y)$, and this will prove the lemma.

Consider any $u \in X$ and $u^* \in T(u)$. For each $\varepsilon > 0$, we can find a neighborhood U of y and a weak* neighborhood U^* of y^* such that

$$|\langle x - y, u^* \rangle| \leq \varepsilon \qquad (\forall\, x \in U), \tag{2.1}$$

$$|\langle u - y, x^* - y^* \rangle| \leq \varepsilon, \qquad (\forall\, x^* \in U^*), \quad \text{and} \tag{2.2}$$

$$|\langle x - y, x^* \rangle| \leq \varepsilon \qquad (\forall\, x \in U,\ \forall\, x^* \in B). \tag{2.3}$$

(Condition (2.3) can be met, because B is equicontinuous.) Let x^* be an element of the set $T(U) \cap U^* \cap B$, which is nonempty by the choice of y^*. Let x be an element of U such that $x^* \in T(x)$. The monotonicity of T gives the relation

$$\langle u - x, u^* - x^* \rangle \geq 0,$$

and hence

$$\begin{aligned} \langle u - y, u^* - y^* \rangle &= \langle u - x, u^* - x^* \rangle + \langle x - y, u^* \rangle \\ &\quad + \langle u - y, x^* - y^* \rangle - \langle x - y, x^* \rangle \geq 0 - \varepsilon - \varepsilon - \varepsilon = -3\varepsilon . \end{aligned} \tag{2.4}$$

Since (2.4) holds for arbitrary $\varepsilon > 0$, we must have that

$$\langle u - y, u^* - y^* \rangle \geq 0 . \tag{2.5}$$

Furthermore, inequality (2.5) holds for each $u \in X$ and $u^* \in T(u)$. Therefore the maximality of T implies that $y^* \in T(y)$.

LEMMA 3. *Let* X *be a locally convex Hausdorff space, and let* $T: X \to X^*$ *be a maximal monotone operator. Suppose that* cl (conv D(T)) *has a nonempty interior and that* x *is a point of* D(T) *not belonging to this interior. Then the set* T(x) *contains at least one half-line (and consequently* T(x) *is not equicontinuous).*

Proof. Since x is a boundary point of cl (conv D(T)), which is a closed, convex set with a nonempty interior, there exists a supporting hyperplane to cl (conv D(T)) at x. Thus there exists a $y^* \in X^*$ $(y^* \neq 0)$ such that

$$\langle x, y^* \rangle \geq \langle u, y^* \rangle \qquad (\forall\, u \in D(T)). \tag{2.6}$$

Let x^* be an element of T(x). By (2.6) and the monotonicity of T, each vector $x^* + \lambda y^*$ $(\lambda \geq 0)$ satisfies the condition

$$\begin{aligned} \langle u - x, u^* - (x^* + \lambda y^*) \rangle &= \langle u - x, u^* - x^* \rangle + \lambda \langle x - u, y^* \rangle \geq 0 \\ &\qquad (\forall\, u \in D(T),\ \forall\, u^* \in T(u)). \end{aligned} \tag{2.7}$$

Since T is maximal, (2.7) implies that $x^* + \lambda y^* \in T(x)$. Thus T(x) contains the half-line

$$\{x^* + \lambda y^* \mid \lambda \geq 0\} .$$

Proof of Theorem 2. If condition (a) holds for S, then condition (b) holds for $S_\alpha = S \cap \alpha A_1^0$, where α is a sufficiently large, positive number and A_1^0 is the polar of $A_1 = A \cup (-A)$. (Since A_1 is equicontinuous, A_1^0 is a neighborhood of the origin in X.) Thus it suffices to prove the theorem with condition (b).

Condition (b) implies in particular that cl (conv D(T)) has a nonempty interior. Let $\bar{x}$ be a point in this interior. We shall prove that T is locally bounded at $\bar{x}$ and that $\bar{x} \in D(T)$. This will establish Theorem 2 except for the assertion that T is not locally bounded at boundary points of D(T).

We deal first with the case where

$$\bar{x} \in \operatorname{int}(\operatorname{cl}(\operatorname{conv} S)) . \tag{2.8}$$

For each equicontinuous subset B of X^*, we let $T_B(x)$ denote the set of all $x^* \in X^*$ such that

$$\langle u - x, u^* - x^* \rangle \geq 0 \qquad (\forall\, u \in X,\ \forall\, u^* \in T(u) \cap B). \tag{2.9}$$

By Lemma 1 and the monotonicity of T, we have that

$$T(x) \subset T_B(x) \neq \emptyset \qquad (\forall\, x \in X). \tag{2.10}$$

Note that $T_B(x)$ is always a weak*-closed set, since by definition it is the intersection of a certain collection of weak*-closed half-spaces in X^*.

To prove that T is locally bounded at $\bar{x}$, we consider the mapping $T_B: x \to T_B(x)$ in the case where $B = A$. Choose a convex neighborhood V of the origin in X such that

$$\bar{x} + 2V \subset \operatorname{cl}(\operatorname{conv} S), \tag{2.11}$$

as is possible by (2.8). Let

$$\mu = \sup_{x \in S} \sup_{u^* \in A} |\langle x, u^* \rangle| . \tag{2.12}$$

(Note that μ is finite by hypothesis.) For each $u^* \in A$, the closed, convex set

$$\{x \mid |\langle x, u^* \rangle| \leq \mu\}$$

contains S, and hence it contains cl (conv S). Thus (2.12) actually implies the inequality

$$|\langle x, u^* \rangle| \leq \mu \qquad (\forall\, x \in \operatorname{cl}(\operatorname{conv} S),\ \forall\, u^* \in A). \tag{2.13}$$

Select an $x \in (\bar{x} + V)$ and an $x^* \in T_A(x)$. Relations (2.11) and (2.13) imply that

$$\langle u - x, x^* \rangle \leq \langle u - x, u^* \rangle \leq |\langle u, u^* \rangle| + |\langle x, u^* \rangle| \leq 2\mu ,$$

for every $u \in S$ and $u^* \in T(u) \cap A$. Thus $S \subset \{u \mid \langle u - x, x^* \rangle \leq 2\mu\}$, and it follows that

$$x + V \subset \bar{x} + 2V \subset \operatorname{cl}(\operatorname{conv} S) \subset \{u \mid \langle u - x, x^* \rangle \leq 2\mu\} .$$

Therefore $\langle v, x^* \rangle \leq 2\mu$ for every $v \in V$; hence

$$x^* \in (2\mu + 1)V^0 ,$$

where V^0, the polar of a neighborhood of the origin in X, is an equicontinuous subset of X^*. Since x was any element of $\bar{x} + V$ and x^* was any element of $T_A(x)$, we may conclude from (2.10) that

$$T(\bar{x} + V) = \bigcup \{T(x) \mid x \in (\bar{x} + V)\}$$
$$\subset \bigcup \{T_A(x) \mid x \in (\bar{x} + V)\} \subset (2\mu + 1)V^0 .$$

Thus $T(\bar{x} + V)$ is equicontinuous, and by definition T is locally bounded at $\bar{x}$.

To show that in fact $\bar{x} \in D(T)$, we consider the collection of all the (nonempty, weak*-closed) sets $T_B(\bar{x})$, where B is an equicontinuous subset of X^* containing A. This collection has the property that every finite subcollection has a nonempty intersection. Moreover, every $T_B(\bar{x})$ in the collection is contained in $T_A(\bar{x})$, which is equicontinuous (and hence weak*-compact) according to the preceding paragraph. The collection therefore has a nonempty intersection. Let $\bar{x}^*$ be an element in the intersection. By the definition of the sets $T_B(\bar{x})$, we must have that

$$\langle u - \bar{x}, u^* - \bar{x}^* \rangle \geq 0 \quad (\forall\, u \in X, \ \forall\, u^* \in T(u)).$$

But T is a maximal monotone operator, so this implies $\bar{x}^* \in T(\bar{x})$. Thus $T(\bar{x}) \neq \emptyset$ and $\bar{x} \in D(T)$.

We deal now with the general case where $\bar{x}$ is an interior point of cl (conv D(T)), not necessarily satisfying (2.8). We shall reduce this case to the previous case by demonstrating the existence of a subset $S' \subset D(T)$ and an equicontinuous subset $A' \subset X^*$ such that T(x) meets A' for every $x \in S'$, condition (b) holds, and

(2.14) $$\bar{x} \in \operatorname{int}(\operatorname{cl}(\operatorname{conv} S')) .$$

According to the argument already given, D(T) contains a nonempty, open set on which T is locally bounded, namely the interior of cl (conv S). Thus there exists a nonempty, open, convex set $W \subset D(T)$ such that T(W) is equicontinuous. Let

$$E = \bigcup_F \operatorname{int}(\operatorname{conv}(W \cup F)),$$

where the union is taken over all finite subsets F of D(T). Then E is a nonempty, open, convex set whose closure contains D(T). It follows that

$$\operatorname{int}(\operatorname{cl}(\operatorname{conv} D(T))) \subset \operatorname{int}(\operatorname{cl} E) = E ,$$

and hence that $\bar{x} \in E$. Thus there exist elements $x_1, \cdots, x_n$ of D(T) such that

(2.15) $$\bar{x} \in \operatorname{int}(\operatorname{conv}(W \cup \{x_1, \cdots, x_n\})).$$

Choose an arbitrary $x_i^* \in T(x_i)$ $(i = 1, \cdots, n)$, and let

$$A' = T(W) \cup \{x_1^*, \cdots, x_n^*\} .$$

Then A' is equicontinuous. Since W is convex, relation (2.15) implies the existence of an $x_0 \in W$ such that

$$\bar{x} \in \operatorname{int}(\operatorname{conv}(U \cup \{x_1, \cdots, x_n\})),$$

for every neighborhood U of x_0. Take U to be a neighborhood of x_0, contained in W, on which the linear functionals in A' are uniformly bounded, and let

$$S' = U \cup \{x_1, \cdots, x_n\} \subset D(T).$$

Then (2.14) is satisfied, T(x) meets A' for every $x \in S'$, and condition (b) holds for S' and A' as desired.

The proof of Theorem 2 will be complete as soon as we demonstrate that T is not locally bounded at any boundary point of D(T). Let y be a boundary point, and suppose that U is a neighborhood of y such that T(U) is equicontinuous. We shall derive a contradiction. Let B be the weak* closure of T(U). According to Lemma 2, $T^{-1}(B)$ is closed. Since

$$D(T) \cap U \subset T^{-1}(B) \subset D(T)$$

and $y \in \operatorname{cl} D(T)$, it follows that actually $y \in D(T)$. Now $y \notin \operatorname{int} D(T)$, and we have shown above that

$$\operatorname{int} D(T) = \operatorname{int}(\operatorname{conv} D(T)) \neq \emptyset.$$

Lemma 3 then implies that T(y) is not an equicontinuous set, contrary to the assumption that T(U) is equicontinuous. This proves Theorem 2.

COROLLARY 2.1. *Let* X *be a locally convex Hausdorff space, and let* $T: X \to X^*$ *be a maximal monotone operator. Suppose there exists an* $x \in \operatorname{int}(\operatorname{cl} D(T))$ *such that* T *is locally bounded at* x. *Then the conclusions of Theorem* 2 *hold.*

Proof. Let U be a neighborhood of x, contained in cl D(T), such that T(U) is an equicontinuous subset of X^*. Then the sets $S = U \cap D(T)$ and $A = T(U)$ satisfy the hypothesis of Theorem 2.

COROLLARY 2.2. *Let* X *be a locally convex Hausdorff space, and let* $T: X \to X^*$ *be a monotone operator (not necessarily maximal). Let* C *be an open subset of* cl D(T). *If* T *is locally bounded at some point of* C, *then* T *is locally bounded at every point of* C.

Proof. By Zorn's Lemma, there exists a maximal monotone operator $T': X \to X^*$ such that $T'(x) \supset T(x)$ for every x. Let U be a nonempty, open subset of C such that T(U) is an equicontinuous subset of X^*. Then $S = U \cap D(T)$ and $A = T(U)$ satisfy the hypothesis of Theorem 2, with T' in place of T. It follows that T' is locally bounded on the interior of cl D(T'). In particular, T is locally bounded throughout C.

Remark. Corollary 2.2 is the analogue for monotone operators of a familiar result about convex functions: if f is a real-valued, convex function on an open, convex subset C of X, and if f is continuous at some point of C, then f is continuous at every point of C. For the connection between these results in the case of subdifferential mappings, see a result of J. J. Moreau [9, p. 79] and its formulation as Theorem 2 of [1].

COROLLARY 2.3. *Let* X *be a locally convex, reflexive Hausdorff space, and let* $T: X \to X^*$ *be a maximal monotone operator. Suppose there exists a bounded subset* A *of* X *such that one of the following two conditions holds:*

(a) $0 \in \operatorname{int}(\operatorname{cl} T(A))$,

(b) *for some* $S \subset T(A)$, *one has* $0 \in \text{int}(\text{cl}(\text{conv } S))$ *and*

$$\sup_{x \in A} \sup_{x^* \in S} |\langle x, x^* \rangle| < \infty .$$

Then there exists an $x \in X$ *such that* $0 \in T(x)$.

Proof. Apply Theorem 2 to T^{-1}. Since X is reflexive, the bounded subsets of X are equicontinuous subsets of X^{**}.

3. PROOFS OF THEOREM 1 AND ITS COROLLARIES

Proof of Theorem 1. We consider first the case where the set

$$C = \text{int}(\text{conv } D(T))$$

is nonempty. For each positive integer n, let S_n denote the set of all $x \in D(T)$ such that $\|x\| \leq n$ and $T(x)$ contains an x^* with $\|x^*\| \leq n$. Since $D(T) = \bigcup_{n=1}^{\infty} S_n$, we have the inclusion

$$C \subset \bigcup_{n=1}^{\infty} \text{conv } S_n . \tag{3.1}$$

Of course C, being a nonempty, open, convex subset of a Banach space, is of the second Baire category. The sets $C \cap \text{conv } S_n$ therefore cannot all be nowhere-dense. Thus

$$\text{int}(\text{cl}(\text{conv } S_n)) \neq \emptyset$$

for some n. The sets $S = S_n$ and

$$A = \{x^* \in X^* \,|\, \|x^*\| \leq n\}$$

then satisfy the hypothesis of Theorem 2, and the conclusion of Theorem 1 follows.

Next we consider the case where X is reflexive and T is locally bounded at some point of D(T). In this case, there exists an open, convex set U, meeting D(T), such that T(U) is norm-bounded in X^*. If $U \subset D(T)$, the hypothesis of Theorem 2 is satisfied and the conclusions of Theorem 1 again follow. Suppose therefore that U is not included in D(T). Then U contains a boundary point of D(T). We shall show that this is impossible.

Let B be the weak* closure of T(U). By Lemma 2, $T^{-1}(B)$ is closed. We have that

$$U \cap D(T) \subset T^{-1}(B) \subset D(T),$$

and this implies the inclusion

$$\text{cl}[U \cap D(T)] \subset D(T).$$

Therefore

$$U \cap D(T) = U \cap \text{cl } D(T), \tag{3.2}$$

in other words, every boundary point of D(T) in U belongs to D(T). Since X is a reflexive Banach space and T is a maximal monotone operator, cl D(T) is a convex set (Rockafellar [11, Theorem 2]); hence the set of points where cl D(T) has a supporting hyperplane is dense in the boundary of cl D(T) (E. Bishop and R. R. Phelps [2, Theorem 1]). Since U contains a boundary point of D(T), it follows from (3.2) that U actually contains a point $x \in D(T)$ such that cl D(T) has a supporting hyperplane at x, in other words, such that (2.6) holds for some nonzero $y^* \in X^*$. The argument given to prove Lemma 3 now implies that T(x) is unbounded, contrary to the assumption that T(U) is bounded. This completes the proof of Theorem 1.

Proof of Corollary 1.1. Since $\operatorname{int} D(T) = \operatorname{int}(\operatorname{conv} D(T)) \neq \emptyset$ by Theorem 1, Lemma 3 implies that T(x) is an unbounded set for each $x \in D(T) \setminus \operatorname{int} D(T)$. Therefore $D_0 \subset \operatorname{int} D(T)$, and, by Theorem 1, T is locally bounded at each point of D_0. Let x_0 be a point of D_0, and let x_0^* be the unique element of $T(x_0)$. Let U be a neighborhood of x_0 such that T(U) is equicontinuous, and let U^* be some weak*-open neighborhood of x_0^*. Let B be the intersection of the weak* closure of T(U) with the complement of U^* in X^*. Thus B is a weak*-closed, equicontinuous set, so that $T^{-1}(B)$ is closed by Lemma 2. If there did not exist a neighborhood W of x_0 such that $T(W) \subset U^*$, then $T^{-1}(B)$ would meet every neighborhood W of x_0. This would imply that $x_0 \in T^{-1}(B)$, contrary to the fact that $T(x_0)$ contains no element of B. Thus $T(W) \subset U^*$ for some neighborhood W of x_0. This shows, in particular, that the restriction of T to D_0 is strong-to-weak continuous at x_0.

Proof of Corollary 1.2. Every set containing the interior of its convex hull is virtually convex—see the proof of the lemma in [11]. According to Theorem 1, we have that

$$\operatorname{cl} D(T) = \operatorname{cl}(\operatorname{int} D(T)),$$

where int D(T) is convex. Thus

$$\operatorname{int}(\operatorname{cl} D(T)) = \operatorname{int} D(T),$$

and if cl D(T) = X, it follows that

$$X = \operatorname{int} D(T) \subset D(T).$$

Proof of Corollary 1.3. Applying Theorem 1 to T^{-1}, we obtain that $0 \in \operatorname{int} D(T^{-1})$ if and only if $0 \in \operatorname{cl} D(T^{-1})$ and T^{-1} is locally bounded at 0. The latter means that there exist positive numbers α and ε such that $\|x\| < \alpha$ whenever $x \in T^{-1}(x^*)$ and $\|x^*\| < \varepsilon$.

Proof of Corollary 1.4. Since $T(B) \subset R(T) = D(T^{-1})$, the corollary follows immediately from applying Theorem 1 to T^{-1}.

Proof of Corollary 1.5. Because $x_i \in T^{-1}(x_i^*)$ for every i, the set $T^{-1}(U)$ is a nonempty, unbounded subset of X for every neighborhood U of x^* in X^*. Thus x^* belongs to the closure of $D(T^{-1})$, but T^{-1} is not locally bounded at x^*. By Zorn's Lemma, there exists a maximal monotone operator $S\colon X^* \to X$ such that $S(y^*) \supset T^{-1}(y^*)$ for every $y^* \in X^*$. This operator S cannot be locally bounded at x^*, and hence Theorem 1 implies that $x^* \notin \operatorname{int} D(S)$ and, in particular, that $x^* \notin \operatorname{int} D(T^{-1})$. Thus x^* is a boundary point of $D(T^{-1}) = R(T)$.

Proof of Corollary 1.6. The stated condition means that T^{-1} is locally bounded at every point of X^*. By Theorem 1 (applied to T^{-1}), this is equivalent to $D(T^{-1})$ being an open, convex subset of X^* with no boundary points, and the only such nonempty subset is X^* itself.

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