THE SECOND INITIAL-BOUNDARY-VALUE PROBLEM FOR A LINEAR PARABOLIC EQUATION WITH A SMALL PARAMETER

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1. INTRODUCTION

Let ε denote a small positive parameter, and let a, b, c, and f be smooth functions with $a > 0$, $c \geq 0$. We shall study the dependence on ε of the solution $u(x, t; \varepsilon)$ of the second initial-boundary-value problem for the equation

(1) $$L_\varepsilon[u] \equiv \varepsilon a(x, t) u_{xx} - b(x, t) u_x - u_t - c(x, t) u = f(x, t)$$

in the domain

$$D = \{(x, t): s_0(t) < x < s_1(t),\ 0 < t \leq 1\}.$$

Here s_0 and s_1 are prescribed functions of class C^4 on [0, 1], with $s_0(t) < s_1(t)$ for $0 \leq t \leq 1$ and $s_0(0) = 0$, $s_1(0) = 1$. Since the change of variables $(x, t) \to (z, t)$ with

$$z = \frac{x - s_0(t)}{s_1(t) - s_0(t)}$$

leaves the form of equation (1) unchanged, we shall assume that D is the unit square Ω (closed on top). We may thus specify the initial and boundary conditions as

(2) $$u(x, 0) = \phi(x) \qquad (0 \leq x \leq 1),$$

(3) $$u_x(0, t) - \beta_0(t) u(0, t) = \psi_0(t) \qquad (0 < t \leq 1),$$

(4) $$-u_x(1, t) - \beta_1(t) u(1, t) = \psi_1(t) \qquad (0 < t \leq 1),$$

where β_0 and β_1 are smooth nonnegative functions of t on [0, 1]. That a smooth solution of problem (1)-(4) exists is well known [4].

The problem (1)-(4) discussed here was considered by Oleĭnik [6] in the case where the two curves $x = s_0(t)$, $x = s_1(t)$ are characteristics of the first-order *degenerate* equation $L_0[u] = f$. We shall be interested in the case where neither of the lateral boundary curves is tangent to such a characteristic.

The first initial-boundary-value problem for a linear second-order parabolic operator with a small parameter multiplying the highest-order derivative has been considered in [1], [2], and [5]. The present study is an extension of the methods of [2] to the second initial-boundary-value problem.

As in [2], several cases arise according to whether the characteristics (of the degenerate equation) through the corners (0, 0) and (0, 1) enter Ω. The situation is

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simplest if neither of these corner characteristics enters Ω; in this case a full asymptotic expansion of $u(x, t; \varepsilon)$ can be obtained by a simplification and extension of the procedure described below in Section 4. This is a straightforward adaption of the method of [3] by means of the majorant-function theorem of Section 3 instead of Lemma 1 of [3], and we shall not further describe it here.

A more interesting case arises if one or both of the corner characteristics enters Ω. The additional difficulties in this case stem from the fact that the solution of the appropriate degenerate problem is not smooth enough near these characteristics to admit the full operator L_ε, and hence the usual methods for establishing an asymptotic expansion for the solution of a singular perturbation problem can not be applied. The problem with both corner characteristics entering Ω can be treated by obvious modifications of the method that we use for the problem with only one corner characteristic entering Ω; hence we restrict ourselves to this latter situation. We shall assume, then, that the characteristic through $(0, 0)$ enters Ω, whereas that through $(0, 1)$ does not. This will be the case if *the coefficient* b *of equation* (1) *satisfies* $b(0, t) > 0$, $b(1, t) > 0$ *for* $0 \leq t \leq 1$, and this we shall henceforth assume.

The following smoothness assumptions will be made throughout:

$$\text{(S)} \qquad a, b, c, f \in C^2(\overline{\Omega}), \qquad \phi \in C^4([0, 1]), \qquad \psi_0, \psi_1 \in C^3([0, 1]).$$

In addition, we shall assume that

$$\text{(T)} \qquad \begin{aligned} \phi_x(0) - \beta_0(0)\phi(0) &= \psi_0(0), \\ -\phi_x(1) - \beta_1(0)\phi(1) &= \psi_1(0); \end{aligned}$$

these conditions, which are necessary to our treatment, mean that the prescription of the linear combination of u and u_x on the lateral boundaries $x = 0$ and $x = 1$ is compatible with the value of u assigned at $t = 0$.

In the next section we discuss a particular problem (related to the full problem given by equations (1)-(4)) for the degenerate equation. In Section 3 we establish a majorant-function theorem (Theorem 3), which will be the primary tool in proving that the asymptotic representation obtained in Section 4 is valid.

The main result is Theorem 4, stated at the beginning of Section 4; it asserts that the expansion $u(x, t; \varepsilon) = U(x, t) + v(x, t; \varepsilon)$ of the solution of the problem (1)-(4) is valid in $\overline{\Omega}$, where U is the solution to the degenerate problem of Section 2, and where $v(x, t; \varepsilon) = O(\varepsilon^{1/2})$ uniformly in $\overline{\Omega}$.

2. THE DEGENERATE PROBLEM

We obtain the degenerate equation from equation (1) by setting $\varepsilon = 0$:

$$\text{(5)} \qquad b(x, t)U_x + U_t + c(x, t)U = -f(x, t);$$

recall that $b(0, t) > 0$. For this equation in the square Ω we wish to consider the nonstandard problem composed of equation (5) and the boundary conditions

$$\text{(6)} \qquad U_x - \beta_0(t)U = \psi_0(t) \quad \text{along } x = 0,$$

$$\text{(7)} \qquad U = \phi(x) \quad \text{along } t = 0.$$

We introduce the characteristic $x = X(t; \xi, \tau)$ through $(\xi, \tau) \in \overline{\Omega}$ as the solution of

$$\frac{dX}{dt} = b(X, t), \qquad X(\tau; \xi, \tau) = \xi \tag{8}$$

in $\overline{\Omega}$; *we assume that every point in* $\overline{\Omega}$ *can be joined to a point of* $t = 0$ *or* $x = 0$ *by a characteristic lying wholly in* $\overline{\Omega}$. Let $\gamma(\xi, \tau)$ be the solution of $X(\gamma(\xi, \tau); \xi, \tau) = 0$ for $(\xi, \tau) \in D_1$, where D_1 is the set of points in $\overline{\Omega}$ lying on or above the curve $x = X(t; 0, 0)$. Set $D_2 = \overline{\Omega - D_1}$.

In D_2, the problem (5), (7) has the unique solution

$$\begin{aligned} U(\xi, \tau) &= \phi(X(0; \xi, \tau)) \exp\left\{-\int_0^\tau c(X(t; \xi, \tau), t)\, dt\right\} \\ &\quad - \int_0^\tau f(X(\sigma; \xi, \tau), \sigma) \exp\left\{\int_\tau^\sigma c(X(t; \xi, \tau), t)\, dt\right\} d\sigma . \end{aligned} \tag{9}$$

In D_1 we seek a solution of (5), (6). Eliminating U_x between these equations and writing $\theta(t)$ for $U(0, t)$, we see that θ satisfies the equation

$$\theta'(t) + [c(0, t) + \beta_0(t) b(0, t)]\, \theta(t) = -f(0, t) - \psi_0(t) b(0, t), \tag{10}$$

which, along with the obvious initial condition $\theta(0) = \phi(0)$, completely determines θ. The solution of (5) with the boundary condition $U(0, t) = \theta(t)$ determines U in D_1:

$$\begin{aligned} U(\xi, \tau) &= \theta(\gamma(\xi, \tau)) \exp\left\{\int_{\gamma(\xi,\tau)}^\tau c(X(t; \xi, \tau), t)\, dt\right\} \\ &\quad - \int_{\gamma(\xi,\tau)}^\tau f(X(\sigma; \xi, \tau), \sigma) \exp\left\{\int_\tau^\sigma c(X(t; \xi, \tau), t)\, dt\right\} d\sigma . \end{aligned} \tag{11}$$

Equations (9) and (11) thus provide a formal solution of the problem posed by (5)-(7). In general, this solution is purely formal, since the constructed function U, although continuous in $\overline{\Omega}$, is not continuously differentiable across the corner characteristic $x = X(t; 0, 0)$, and thus the differential equation (5) cannot be satisfied on this curve. However, we observe that U is a smooth solution of (5)-(7) in the special case where

$$\phi(x) \equiv 0 \qquad \text{near } x = 0,$$

$$\psi_0(t) \equiv -\frac{f(0, t)}{b(0, t)} \qquad \text{near } t = 0,$$

and, in particular, if ϕ, ψ_0, and f vanish in some neighborhood of the origin. In view of the smoothness assumptions (S), these latter conditions guarantee, in fact, that $U \in C^2(\overline{\Omega})$. The importance for our work of this simple observation will presently appear.

3. MAJORANT FUNCTIONS

Throughout this section, we write L for L_1 and $u(x, t)$ for $u(x, t; 1)$. We denote by $C^{2,1}(\Omega)$ the set of all functions that are continuous in $\overline{\Omega}$ and continuously differentiable twice in x and once in t throughout Ω.

THEOREM 1. *If* $u \in C^{2,1}(\Omega)$ *has a positive maximum at* $(x_0, t_0) \in \Omega$, *then* $L[u] \leq 0$ *at* (x_0, t_0).

Proof. If u is a (positive) constant in $[0, 1] \times [0, t_0]$, then $L[u]|_{(x_0,t_0)} \leq 0$, since $c \geq 0$. If u is not a constant, assume that $L[u]|_{(x_0,t_0)} > 0$. Then there exists a neighborhood V of (x_0, t_0) such that $L[u] > 0$ in $V \cap \Omega$; let ω be a square

$$\omega \equiv \{(x, t): |x - x_0| < \delta,\ t_0 - 2\delta < t \leq t_0\}$$

contained in $V \cap \Omega$. Then $L[u] > 0$ in ω, and u assumes a positive maximum in $\overline{\omega}$ at $(x_0, t_0) \in \omega$. But the maximum principle for L in ω implies that $u \equiv$ constant in $\overline{\omega}$; since this constant must be positive, we conclude that $L[u]|_{(x_0,t_0)} \leq 0$, a contradiction.

The following result is proved in [4].

THEOREM 2. *Let* $u \in C^{2,1}(\Omega)$, *and suppose* $L[u] > 0$ *in* Ω. *Assume that* u *has a positive maximum* M *in* $\overline{\Omega}$ *at a point* P_0: (x_0, t_0) *with* $0 < t_0 \leq 1$, $x_0 = 0$ *or* $x_0 = 1$. *Assume further that* $u < M$ *throughout the intersection of* Ω *and some neighborhood* V *of* P_0. *Then for each nontangential inward direction* ν,

$$\frac{\partial u}{\partial \nu} < 0 \quad at\ P_0 .$$

THEOREM 3. *Let* $\Phi, \Psi \in C^{2,1}(\Omega)$, *and assume that*

$$|L[\Phi]| < -L[\Psi] \qquad in\ \Omega, \tag{12}$$

$$|\Phi| \leq \Psi \qquad on\ t = 0,\ 0 \leq x \leq 1, \tag{13}$$

$$|\Phi_x - \beta_0 \Phi| \leq -\Psi_x + \beta_0 \Psi \qquad on\ x = 0,\ 0 < t \leq 1, \tag{14}$$

$$|\Phi_x + \beta_1 \Phi| \leq \Psi_x + \beta_1 \Psi \qquad on\ x = 1,\ 0 < t \leq 1, \tag{15}$$

where $\beta_0(t) \geq 0$ *and* $\beta_1(t) \geq 0$ $(0 \leq t \leq 1)$. *Then* $|\Phi| \leq \Psi$ *throughout* $\overline{\Omega}$.

Proof. Since (12) implies that $L[\Phi - \Psi] > 0$ in Ω, it follows from Theorem 1 that $\Phi - \Psi$ does not have a positive maximum in Ω. Thus, if $\Phi - \Psi$ has a strictly positive maximum at a point P_0: (x_0, t_0) in $\overline{\Omega}$, then P_0 must be on $\Gamma \equiv \overline{\Omega} - \Omega$; from (13) we see that $\Phi - \Psi \leq 0$ on $t = 0$, whence $t_0 > 0$. Denoting the hypothetical positive maximum by M, we have the inequalities $(\Phi - \Psi)(P_0) = M > 0$ and $(\Phi - \Psi) < M$ in $V \cap \Omega$, where V is some neighborhood of P_0. Let us assume first that $x_0 = 0$. Then, applying Theorem 2 with the direction ν along the x-axis, we see that

$$\frac{\partial(\Phi - \Psi)}{\partial x}(P_0) < 0 .$$

But (14) implies that

$$\frac{\partial(\Phi - \Psi)}{\partial x}(P_0) \geq \beta_0(t_0)(\Phi - \Psi)(P_0) \geq 0,$$

and this is a contradiction. Suppose then that $x_0 = 1$. By Theorem 2,

$$-\frac{\partial(\Phi - \Psi)}{\partial x}(P_0) < 0,$$

whereas (15) yields

$$-\frac{\partial(\Phi - \Psi)}{\partial x}(P_0) \geq \beta_1(t_0)(\Phi - \Psi)(P_0) \geq 0,$$

again a contradiction. Hence $\Phi - \Psi \leq 0$ in $\overline{\Omega}$.

The other inequality in the conclusion of the theorem is proved similarly.

The function Ψ of Theorem 3 has been called a *barrier function* for Φ [3], but since this term has another meaning in potential theory, we shall use the term *majorant function.*

4. ASYMPTOTIC REPRESENTATION

THEOREM 4. *Assume that the conditions* (S) *and* (T) *hold, that* $\beta_0, \beta_1 \geq 0$, *and that the characteristics of the degenerate equation* (5) *cover* $\overline{\Omega}$. *Then the expansion*

$$u(x, t; \varepsilon) = U(x, t) + v(x, t; \varepsilon) \tag{16}$$

of the solution of (1)-(4) *is valid in* $\overline{\Omega}$, *where* U *is the function given by* (9) *and* (11), *and where* $v(x, t; \varepsilon) = O(\varepsilon^{1/2})$ *uniformly in* $\overline{\Omega}$.

We may assume that the initial datum ϕ of equation (2) is in fact identically zero. For, introducing $v = u - \phi$, we see that v satisfies

$$\varepsilon v_{xx} - bv_x - v_t - cv = f' - \varepsilon\phi_{xx}, \tag{17}$$

where $f' = f + b\phi_x + c\phi$, and

$$v_x(0, t) - \beta_0(t)v(0, t) = \psi_0'(t) \equiv \psi_0(t) - \phi_x(0) + \beta_0(t)\phi(0), \tag{18}$$

$$-v_x(1, t) - \beta_1(t)v(1, t) = \psi_1'(t) \equiv \psi_1(t) + \phi_x(1) + \beta_1(t)\phi(1), \tag{19}$$

$$v(x, 0) = 0. \tag{20}$$

This problem is again of the form (1)-(4) except for the ε-term on the right of equation (17). To treat this, we write $v = v^0 + \varepsilon v^1$, where v^0 satisfies (18)-(20) and

$$\varepsilon v^0_{xx} - bv^0_x - v^0_t - cv^0 = f',$$

and where v^1 satisfies the equation

$$\varepsilon v^1_{xx} - bv^1_x - v^1_t - cv^1 = -\phi_{xx}$$

and homogeneous boundary conditions of the form (18)-(20). If we set

$$\mu = 1 + \sup_{[0,1]} |\phi_{xx}|,$$

then Theorem 3 implies that $|v^1| \leq \mu t \leq \mu$. It is now clear that the term v^1 can be absorbed by the function v of Theorem 4; thus we may assume that our problem is of the form (1)-(4) with $\phi \equiv 0$.

Let z(y) be a C^∞-function, vanishing for $y \geq 2$, identically 1 for $y \leq 1$, and with $0 \leq z(y) \leq 1$. For any positive number δ we set $U(x, t) = U^1(x, t; \delta) + U^2(x, t; \delta)$, where U^1 and U^2 satisfy the conditions

$$\text{(21)} \qquad \begin{cases} L_0[U^1] = (1 - z(t/\delta))f(x, t), & \\ U^1_x(0, t) - \beta_0(t)U^1(0, t) = (1 - z(t/\delta))\psi_0(t) & (0 < t \leq 1), \\ U^1(x, 0) = 0 & (0 \leq x \leq 1), \end{cases}$$

and

$$\begin{cases} L_0[U^2] = z(t/\delta)f(x, t), & \\ U^2_x(0, t) - \beta_0(t)U^2(0, t) = z(t/\delta)\psi_0(t) & (0 < t \leq 1), \\ U^2(x, 0) = 0 & (0 \leq x \leq 1), \end{cases}$$

respectively. Then, as we noted at the end of Section 2, $U^1(x, t; \delta)$ is twice differentiable in $\overline{\Omega}$.

Similarly, we set $u(x, t; \varepsilon) = u^1(x, t; \varepsilon, \delta) + u^2(x, t; \varepsilon, \delta)$, where u^1 and u^2 are solutions of the problems

$$\text{(22)} \qquad L_\varepsilon[u^1] = (1 - z(t/\delta))f(x, t),$$

$$\text{(23)} \qquad u^1_x(0, t) - \beta_0(t)u^1(0, t) = (1 - z(t/\delta))\psi_0(t) \qquad (0 < t \leq 1),$$

$$\text{(24)} \qquad -u^1_x(1, t) - \beta_1(t)u^1(1, t) = \psi_1(t) \qquad (0 < t \leq 1),$$

$$u^1(x, 0) = 0 \qquad (0 \leq x \leq 1);$$

$$L_\varepsilon[u^2] = z(t/\delta)f(x, t),$$

$$u^2_x(0, t) - \beta_0(t)u^2(0, t) = z(t/\delta)\psi_0(t) \qquad (0 < t \leq 1),$$

$$-u^2_x(1, t) - \beta_1(t)u^2(1, t) = 0 \qquad (0 < t \leq 1),$$

$$u^2(x, 0) = 0 \qquad (0 \leq x \leq 1).$$

We show that both u^2 and U^2 are $O(\delta)$ uniformly with regard to ε and with regard to $(x, t) \in \overline{\Omega}$. For U^2, this is an elementary consequence of the explicit representation (9), (11); for u^2 we proceed as follows. We define $\Psi(x, t)$ by

$$\Psi(x, t) = Pte^{t/\delta} + Q\left(x - \frac{1}{2}\right)^2 te^{t/\delta},$$

where P and Q are positive constants to be determined. Provided (say) $0 \leq \varepsilon \leq 1$ and δ is sufficiently small (independent of ε), we have the inequalities

$$L[-\Psi] \geq -2\varepsilon a Q\delta\cdot\frac{t}{\delta}\,e^{t/\delta} + 2bQ\left(x-\frac{1}{2}\right)\delta\cdot\frac{t}{\delta}\,e^{t/\delta} + Pe^{t/\delta} \geq \frac{P}{2}\,e^{t/\delta} \geq \frac{P}{2} \qquad (0 \leq t \leq 2\delta),$$

since xe^x is bounded for $0 \leq x \leq 2$. If we choose

$$P = 2\sup_{\overline{\Omega}} |f| + 1,$$

then inequality (12) of Theorem 3 is satisfied for $0 \leq t \leq 2\delta$. On $x = 0$, $0 \leq t \leq 2\delta$, we have the relations

$$-\Psi_x + \beta_0\Psi = -2Q\left(x-\frac{1}{2}\right)te^{t/\delta} + \beta_0\Psi \geq Qt\,e^{t/\delta} \geq Qt.$$

Since the match conditions (T) imply that $\psi_0(0) = 0$, and since ψ_0 is continuous at 0, we can choose Q so large that $Qt \geq |\psi_0(t)|$ $(0 \leq t \leq 1)$; then (14) of Theorem 3 is satisfied for $0 \leq t \leq 2\delta$. Inequalities (13) and (15) are trivially satisfied by Ψ; consequently, Ψ is a majorant function for u^2 on $0 \leq t \leq 2\delta$. (Theorem 3 obviously holds for t in an arbitrary interval.) Thus

$$|u^2| \leq \delta\left\{P + Q\left(x-\frac{1}{2}\right)^2\right\}\frac{t}{\delta}\,e^{t/\delta} \leq M\delta$$

for some M, for $0 \leq t \leq 2$. That $M\delta$ is a majorant function for u^2 on $2\delta \leq t \leq 1$ is now an easy consequence of Theorem 3. Thus $|u^2| = O(\delta)$ uniformly in $\overline{\Omega}$ and ε. Since

$$u(x, t; \varepsilon) = u^1(x, t; \varepsilon, \delta) + O(\delta) \quad \text{and} \quad U(x, t) = U^1(x, t; \delta) + O(\delta)$$

uniformly in $\overline{\Omega}$ and ε, it remains only to consider the degeneration of u^1 to U^1 as ε, $\delta \to 0$.

Using the explicit representation of Section 2, we easily derive the following estimates for the δ-dependence (as $\delta \to 0$) of U^1 and its x-derivatives:

$$U^1(x, t; \delta) = O(1), \quad U^1_x(x, t; \delta) = O(1), \quad U^1_{xx}(x, t; \delta) = O(\delta^{-1}), \tag{25}$$

uniformly in $\overline{\Omega}$. Since the full operator L_ε can be applied to the smooth function U^1, we immediately obtain the result

$$L_\varepsilon[U^1] = \varepsilon U^1_{xx} + L_0[U^1] = \varepsilon O(\delta^{-1}) + (1 - z(t/\delta))f(x, t), \tag{26}$$

uniformly in $\overline{\Omega}$.

Since $U^1(x, t; \delta)$ and $u^1(x, t; \varepsilon, \delta)$ have in general different values along the portion of the boundary $x = 1$, $0 < t \leq 1$, we shall construct a compensating boundary-layer term. We shall see that this boundary-layer term has the form εg, where g is a bounded function; consequently, the boundary-layer term can ultimately be absorbed by the error term v of the theorem. Insignificant boundary-layer terms of this type appear to be unavoidable in the boundary-layer method of Višik and Lyusternik [7] (see also [3]).

We shall seek this boundary layer in the form $w = w(x, t; \varepsilon, \delta)$; since δ is presently to be related to ε, we shall ultimately write $w(x, t; \varepsilon)$ where convenient. We introduce the local coordinate ρ by $\varepsilon\rho = x - 1$; in terms of ρ, the homogeneous differential equation for $\tilde{w}(\rho, t; \varepsilon, \delta) = w(\varepsilon\rho + 1, t; \varepsilon, \delta)$ corresponding to equation (1) is

$$\frac{1}{\varepsilon} a\tilde{w}_{\rho\rho} - \tilde{w}_t - \frac{1}{\varepsilon} b\tilde{w}_\rho - c\tilde{w} = 0.$$

We shall not require that $\tilde{w}$ satisfy this equation, but only that it satisfy

$$\frac{1}{\varepsilon} a\tilde{w}_{\rho\rho} - \tilde{w}_t - \frac{1}{\varepsilon} b\tilde{w}_\rho - c\tilde{w} = \varepsilon O(\varepsilon) \tag{27}$$

for negative values of ρ. Moreover, we require that $\tilde{w}$ tend to zero with ε for $\rho < 0$ and satisfy the boundary condition

$$\begin{aligned} -\frac{1}{\varepsilon} \tilde{w}_\rho(0, t; \varepsilon, \delta) - \beta_1(t)\tilde{w}(0, t; \varepsilon, \delta) \\ = \psi_1(t) + U_x^1(1, t; \delta) + \beta_1(t) U^1(1, t; \delta) + \varepsilon O(\varepsilon). \end{aligned} \tag{28}$$

Thus $\tilde{w} + U^1$ satisfies the boundary condition (24) imposed on u^1, except for a term $\varepsilon O(\varepsilon)$. Taylor's theorem with remainder implies that

$$a(x, t) = a(1, t) + \varepsilon\rho a_1(t) + \varepsilon^2 \rho^2 a_2(\varepsilon\rho + 1, t),$$

$$b(x, t) = b(1, t) + \varepsilon\rho b_1(t) + \varepsilon^2 \rho^2 b_2(\varepsilon\rho + 1, t),$$

$$c(x, t) = c(1, t) + \varepsilon\rho c_1(\varepsilon\rho + 1, t).$$

Substituting these expressions and the expansion $\tilde{w} = \tilde{w}^0 + \varepsilon\tilde{w}^1$ into equation (27), and equating to zero the coefficients of ε^{-1} and ε^0, we get for $\tilde{w}^0$ and $\tilde{w}^1$ the differential equations

$$a(1, t)\tilde{w}^0_{\rho\rho} - b(1, t)\tilde{w}^0_\rho = 0,$$

$$a(1, t)\tilde{w}^1_{\rho\rho} - b(1, t)\tilde{w}^1_\rho = -\rho a_1(t)\tilde{w}^0_{\rho\rho} + \tilde{w}^0_t + \rho b_1(t)\tilde{w}^0_\rho + c(1, t)\tilde{w}^0.$$

Similarly, we have the boundary conditions

$$\tilde{w}^0_\rho = 0 \quad (\rho = 0),$$

$$\tilde{w}^1_\rho + \beta_1(t)\tilde{w}^0 = -\psi_1(t) - U_x^1(1, t; \delta) - \beta_1(t) U^1(1, t; \delta) \equiv Q(t; \delta) \qquad (\rho = 0). \tag{29}$$

Clearly, we can meet all the conditions on $\tilde{w}^0$ by taking $\tilde{w}^0 \equiv 0$ (moreover, zero is the unique solution); for $\tilde{w}^1$ we now have the problem

$$\begin{aligned} a(1, t)\tilde{w}^1_{\rho\rho} - b(1, t)\tilde{w}^1_\rho = 0, \\ \tilde{w}^1_\rho = Q(t; \delta) \qquad (\rho = 0,\ 0 \le t \le 1). \end{aligned} \tag{30}$$

Requiring that $\tilde{w}^1$ tend to zero with ε for $\rho < 0$ determines $\tilde{w}^1$ uniquely as

$$\tilde{w}^1(\rho, t; \varepsilon, \delta) = \frac{1}{\lambda} Q(t; \delta) e^{\lambda\rho},$$

where

$$\lambda \equiv \frac{b(1, t)}{a(1, t)} > 0.$$

Since we need the boundary-layer term w only near $x = 1$, we shall take for our boundary layer

$$\hat{w}(x, t; \varepsilon, \delta) = z(n(1-x))w(x, t; \varepsilon, \delta) = \varepsilon z(n(1-x))\frac{1}{\lambda(t)}Q(t; \delta)\exp\{\lambda(t)(x-1)/\varepsilon\},$$

where n is some (fixed) large integer. L_ε can obviously be applied to $\hat{w}$, and we see that

$$L_\varepsilon[\hat{w}] = z(n(1-x))L_\varepsilon[w] + O(\varepsilon),$$

since $z''(n(1-x))$ and $z'(n(1-x))$ are nonzero only for $1/2n < 1 - x < 1/n$, and in this interval the exponential factor of w tends to zero more rapidly than any power of ε. In view of the estimates in (25) and the fact that w satisfies $L_\varepsilon[w] = O(\varepsilon)$ for each $\delta > 0$, we have the relation

$$L_\varepsilon[\hat{w}] = O(\varepsilon). \tag{31}$$

From equations (22), (26), and (31) we see that the function $\Phi \equiv u^1 - U^1 - w$ satisfies

$$L_\varepsilon[\Phi] = O(\varepsilon\delta^{-1}) \tag{32}$$

uniformly for $(x, t) \in \Omega$. Since $Q(0; \delta) = 0$ by assumption (T) of Section 1, we see that

$$(U^1 + w)(x, 0; \varepsilon, \delta) = u^1(x, 0; \varepsilon, \delta) = 0,$$

whence $\Phi(x, 0; \varepsilon, \delta) = 0$ $(0 \le x \le 1)$. By equations (21) and (23) and the fact that w vanishes near $x = 0$,

$$\Phi_x(0, t; \varepsilon, \delta) - \beta_0(t)\Phi(0, t; \varepsilon, \delta) = 0 \qquad (0 < t \le 1);$$

equations (24), (28), (29), and (30) imply that

$$-\Phi_x(1, t; \varepsilon, \delta) - \beta_1(t)\Phi(1, t; \varepsilon, \delta) = \varepsilon\beta_1(t)\hat{w}^1(1, t; \varepsilon, \delta) = O(\varepsilon) \qquad (0 < t \le 1) \tag{33}$$

uniformly in δ (since $\hat{w}^1$ depends on δ only through U^1 and U_x^1, which are $O(1)$ as $\delta \to 0$). For convenience, we rewrite (32) as

$$|L_\varepsilon[\Phi]| < M\varepsilon\delta^{-1} \qquad \text{in } \Omega,$$

and (33) as

$$|\Phi_x(1, t; \varepsilon, \delta) + \beta_1(t)\Phi(1, t; \varepsilon, \delta)| < M\varepsilon \qquad (0 < t \le 1), \tag{34}$$

where M is some constant; (34) holds uniformly in δ and t.

For Φ we shall seek a majorant function Ψ of the form

$$\Psi(x, t; \varepsilon, \delta) = P\frac{\varepsilon}{\delta}t + \varepsilon Q\left(x - \frac{1}{2}\right)^2;$$

as in an earlier calculation, it is easy to see that Ψ satisfies (12)-(15) of Theorem 3, provided ε and δ are sufficiently small and $Q \ge M$, $P \ge 2M$. It follows from Theorem 3 that

$$|u^1 - U^1 - \hat{w}| = |\Phi| \leq |\Psi| = O(\varepsilon/\delta)$$

holds uniformly in $\overline{\Omega}$.

Combining this estimate with our earlier estimates for u^2 and U^2, we finally conclude that

$$u(x, t; \varepsilon) = U(x, t) + \hat{w}(x, t; \varepsilon, \delta) + O(\varepsilon/\delta) + O(\delta) = U(x, t) + O(\varepsilon) + O(\varepsilon/\delta) + O(\delta)$$

uniformly in $\overline{\Omega}$. The best possible choice of δ is clearly $\delta = \varepsilon^{1/2}$, whence Theorem 4 follows.

Remarks. 1. The restriction to two independent variables has led to some simplifications; but the methods employed extend readily to several independent variables, as well as to the third initial-boundary-value problem.

2. As in [2], it is possible to treat the case of characteristic lateral boundaries by a nonconstructive boundary-layer technique.

3. Our methods can easily be adapted to treat the second boundary-value problem for elliptic equations in domains with corners.

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