ZEROS OF PARTIAL SUMS OF POWER SERIES

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1. INTRODUCTION

Let $\mathscr{F}$ denote the family of functions that are analytic in the unit disk $|z| < 1$ but not in any disk $|z| < 1 + \varepsilon$ $(\varepsilon > 0)$. If $f(z) = \sum a_k z^k$ belongs to $\mathscr{F}$, we write

$$S_n(z) = S_n(z; f) = \sum_{k=0}^{n} a_k z^k,$$

and we denote by $\rho_n(f)$ the largest of the moduli of the zeros of the polynomial S_n. We write

$$\rho(f) = \liminf_{n\to\infty} \rho_n(f) \qquad \text{and} \qquad P = \sup_{f\in\mathscr{F}} \rho(f).$$

In 1906, M. B. Porter [3] proved that $1 \leq \rho(f) \leq 2$ for all $f \in \mathscr{F}$. Porter showed that his lower bound for $\rho(f)$ is best possible, but he made no similar claim for his upper bound. Quite recently, it has been shown that the constant 2 is *not* best possible. J. Clunie and P. Erdös [1] proved that $P < 2$. In the other direction, they constructed an example to show that $P > \sqrt{2}$. Determination of the exact value of P remains an open problem [2, Problem 7.7].

In the present paper, I prove that $1.7 < P \leq 12^{1/4} = 1.861 \cdots$. The method used to obtain the upper bound is essentially a refinement of the method used by Clunie and Erdös. The lower bound is derived from the remarkably simple example

$$g(z) = \frac{1 + iz - iz^2 - z^3}{1 + z^4}.$$

In Section 3, I prove that $\rho(g) > 1.7$ and indicate why the choice of g is not entirely fortuitous.

2. THE UPPER BOUND

LEMMA. *If* $0 \leq x < 12^{-1/4}$, *then*

$$\sum_{k=1}^{\infty} x^{k+1} |e^{ik\theta} - 1| < 1$$

for all real numbers θ.

Proof. From the Cauchy-Schwarz inequality we get the estimate

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$$\left\{ \sum_{k=1}^{\infty} x^{k+1} |e^{ik\theta} - 1| \right\}^2 \leq \left\{ \sum_{k=1}^{\infty} 2^{-k} \right\} \left\{ x^2 \sum_{k=1}^{\infty} (2x^2)^k |e^{ik\theta} - 1|^2 \right\}$$

$$= x^2 \sum_{k=1}^{\infty} (2x^2)^k \{2 - 2\cos k\theta\}$$

$$= \frac{4x^4}{1 - 2x^2} - 2x^2 \Re \left\{ \frac{2x^2 e^{i\theta}}{1 - 2x^2 e^{i\theta}} \right\}.$$

This expression is largest at $\theta = \pi$; its value there is

$$\frac{4x^4}{1 - 2x^2} + \frac{4x^4}{1 + 2x^2} = \frac{8x^4}{1 - 4x^4} < 1,$$

since $x < 12^{-1/4}$.

THEOREM 1. *If* $\{A_k\}_{k=1}^{\infty}$ *is a sequence of complex numbers such that* $|A_k| \leq 1$ *for* $k \geq 2$, *then either*

$$1 + A_1 z + A_2 z^2 + \cdots \quad \text{or} \quad A_1 + A_2 z + A_3 z^2 + \cdots$$

does not vanish in the disk $|z| < 12^{-1/4}$.

Proof. Suppose $\max\{|z_0|, |z_1|\} = x < 1$ and

$$1 + \sum_{k=1}^{\infty} A_k z_0^k = 0, \qquad \sum_{k=1}^{\infty} A_k z_1^{k-1} = 0.$$

If we multiply the second equation by z_0 and subtract it from the first, we obtain the equation

$$1 + \sum_{k=2}^{\infty} A_k (z_0^k - z_0 z_1^{k-1}) = 0.$$

Therefore

$$1 = \left| \sum_{k=1}^{\infty} A_{k+1} (z_0^{k+1} - z_0 z_1^k) \right| \leq \max_{|z| \leq x, |w| \leq x} \left| \sum_{k=1}^{\infty} A_{k+1} (z^{k+1} - zw^k) \right|$$

$$= \max_{|z| = |w| = x} \left| \sum_{k=1}^{\infty} A_{k+1} (z^{k+1} - zw^k) \right|,$$

by a double application of the maximum modulus theorem. If we write $z/w = e^{i\theta}$ and make use of the triangle inequality and the inequalities $|A_{k+1}| \leq 1$, we obtain the inequality

$$1 \leq \max_{\theta} \sum_{k=1}^{\infty} x^{k+1} |e^{ik\theta} - 1|.$$

In view of the preceding lemma, this implies that $x \geq 12^{-1/4}$, which completes the proof.

COROLLARY. *If* $a_0 + a_1 z + \cdots + a_n z^n$ *is a polynomial of degree* n *and* r *is a positive number such that*

$$|a_n| r^n \geq |a_k| r^k \qquad (k = 0, 1, \cdots, n - 1),$$

then either $a_0 + a_1 z + \cdots + a_n z^n$ *or* $a_0 + a_1 z + \cdots + a_{n-1} z^{n-1}$ *has all of its zeros in the disk* $|z| \leq r \cdot 12^{1/4}$.

Proof. Write

$$\sum_{k=0}^{n} a_k z^k = a_n z^n \left\{ 1 + \sum_{k=1}^{n} \frac{a_{n-k} r^{n-k}}{a_n r^n} \left(\frac{r}{z}\right)^k \right\}$$

and apply Theorem 1 to the function

$$T(z) = 1 + \sum_{k=1}^{n} \frac{a_{n-k} r^{n-k}}{a_n r^n} z^k .$$

THEOREM 2. $P \leq 12^{1/4}$.

Proof. Suppose that $f(z) = \sum a_k z^k$ belongs to $\mathscr{F}$ and that $r > 1$. Then $\{|a_n| r^n\}$ is unbounded, which implies that there exist infinitely many integers n such that

$$\max_{0 \leq k < n} |a_k| r^k \leq |a_n| r^n .$$

The preceding corollary guarantees that

$$\min \{\rho_n(f), \rho_{n-1}(f)\} \leq r \cdot 12^{1/4}$$

for such integers n. Therefore $\rho(f) \leq r \cdot 12^{1/4}$, and consequently $P \leq 12^{1/4}$, since f and r are arbitrary.

3. AN EXAMPLE

For a fixed complex number α ($|\alpha| = 1$), let

$$F_\alpha(z) = \sum_{k=0}^{\infty} \alpha^{k(k+1)/2} z^k ,$$

and let $r_0(\alpha)$ denote the modulus of the zero (or zeros) of F_α nearest to the origin; in case F_α has no zero in the disk $|z| < 1$, take $r_0(\alpha)$ to be 1.

Let S_n denote the nth partial sum of the power series of F_α. It is easily verified that

$$\text{(3.1)} \qquad \frac{\alpha^{n(n+1)/2}}{z^n} S_n\left(\frac{z}{\alpha^{n+1}}\right) = S_n\left(\frac{1}{z}\right) \qquad (n = 1, 2, \cdots).$$

The moduli of the zeros of the left member of (3.1) are the same as the moduli of the zeros of S_n; furthermore, the sequence of functions on the right of (3.1) converges uniformly to $F_\alpha(1/z)$ on closed subsets of $|z| > 1$. The last observation, together with Hurwitz' theorem, determines the behavior of the zeros of (3.1) in $|z| > 1$ and allows us to conclude that

$$\rho(F_\alpha) = \lim_{n \to \infty} \rho_n(F_\alpha) = 1/r_0(\alpha).$$

The problem of determining $\rho(F_\alpha)$ thus reduces to the problem of locating the zeros of F_α. This in turn is facilitated by the observation that if α is a root of unity, then F_α is a rational function. To see this, we first note the identity

$$F_\alpha(z) = S_{k-1}(z) + \alpha^{k(k+1)/2} z^k F_\alpha(\alpha^k z).$$

If $\alpha^k = 1$, then

$$F_\alpha(z) = \frac{S_{k-1}(z)}{1 - \beta z^k}, \qquad \text{where } \beta = \alpha^{k(k+1)/2},$$

and in the special case where $\alpha = i$, we find (with $k = 4$) that

$$F_i(z) = \frac{1 + iz - iz^2 - z^3}{1 + z^4}.$$

Now $1 + iz - iz^2 - z^3 = (1 - z)(1 + (1 + i)z + z^2)$, and an easy computation shows that one zero of the quadratic factor has modulus less than $(1.7)^{-1}$. Therefore $P \geq \rho(F_i) > 1.7$.

Since the above was written, considerably better numerical bounds for P have been obtained. J. L. Frank has shown that $1.7818 < P < 1.82$. The lower bound was obtained by using an IBM 360 to compute values of $r_0(\alpha)$; the upper bound was obtained from a theorem which differs from Theorem 1 in that it involves zeros of $A_2 + A_3 z + A_4 z^2 + \cdots$ as well as those of the two functions in Theorem 1.

REFERENCES

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