# MANIFOLDS ADMITTING A ONE-PARAMETER GROUP OF CONFORMAL TRANSFORMATIONS

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## 1. INTRODUCTION

Let M be an n-dimensional, compact, connected Riemannian manifold with metric tensor g, and let C(M) be its group of conformal transformations, and I(M) its group of isometries. We denote by D the operation of covariant differentiation with respect to g, and by R the Ricci tensor. The manifold is said to be an *Einstein space* if it carries an *Einstein metric*, that is, if

$$R = \lambda g$$

for some scalar field $\lambda$. Let r denote the scalar curvature of (M, g); that is, let $r = \text{trace } Q$, where Q is the Ricci operator. Then, for $n > 2$, $\lambda$ is a constant equal to $r/n$. If r is a nonpositive constant, it is easily shown that C(M) coincides with I(M). It is an outstanding conjecture that if M is a compact Riemannian manifold of dimension $n > 2$ with $r = \text{const.}$, then either $C(M) = I(M)$ or M is globally isometric with a sphere. In the latter case, if X is a conformal vector field on M and $\xi$ is its covariant form, then by the duality defined by the metric, the Hessian of the function divergence X, namely Hess $\delta\xi$, is $-\frac{r}{n(n-1)}\,\delta\xi \cdot g$. We shall prove the following result.

THEOREM 1. *Let* M *be a compact manifold, of dimension* $n > 2$ *and admitting an infinitesimal, nonisometric conformal transformation field* X. *Then, if* r *is a (positive) constant,*

$$2n(n-1)^2 \frac{\|\text{Hess } \delta\xi\|^2}{\|\delta\xi\|^2} \geq r^2 \qquad (\xi = g(X, \cdot\,)),$$

*equality holding if and only if* M *is globally isometric with a sphere.*

Thus, if r is a constant equal to some appropriate value (depending on n and X), the conjecture is valid. In attempting to generalize by replacing the Einstein condition (Corollary 1.1) with constant scalar curvature, we apparently need to prescribe the constant, for otherwise we may not obtain the constant curvature sphere (see for example [2, Theorem 1]). It would be nice, however, if we could rid ourselves of the dependence of r on X.

## 2. MISCELLANEOUS RESULTS

Let $C_0(M)$ (respectively, $I_0(M)$) denote the component of the identity of C(M) (of I(M)), and let $C_0^*(M)$ be the dual space of $C_0(M)$, that is, the set of all $\xi = g(X, \cdot\,)$ with $X \in C_0(M)$. Denote by d the exterior differential operator, and by $\delta$ its adjoint with respect to g. The following additional statements support the above conjecture.

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PROPOSITION 1. *Let* M *be a compact manifold, of dimension* $n > 2$, *and admitting a Riemannian metric of constant scalar curvature for which* $C_0(M) \neq I_0(M)$. *If*

$$\int_M \langle T\, d\delta\xi, d\delta\xi \rangle \, dV \geq 0,$$

*that is, if* T *defines a positive semidefinite quadratic form on* $d\delta C_0^*(M)$, *with respect to the global scalar product, where* $T = R - \frac{r}{n} g$, *then* M *is globally isometric with a sphere.*

T is viewed here as a tensor field of type (1, 1), that is, as a linear transformation field.

The restriction on r is not essential, since any Riemannian metric may be conformally deformed to a metric having constant scalar curvature, and this is the metric given to M [6].

COROLLARY 1.1. *A compact Einstein space of dimension* $n > 2$ *admitting an infinitesimal, nonisometric conformal transformation is globally isometric with a sphere* [7].

COROLLARY 1.2. *Let* M *be a compact Riemannian manifold, of dimension* $n > 2$, *and admitting a metric of constant scalar curvature such that* trace $Q^2$ = const. *If* $C_0(M) \neq I_0(M)$, *then* M *is globally isometric with a sphere* [5].

COROLLARY 1.3. *A compact homogeneous Riemannian manifold* M *of dimension* $n > 3$ *for which* $C_0(M) \neq I_0(M)$ *is globally isometric with a sphere* [3].

## 3. CONFORMAL FIELDS ON MANIFOLDS OF CONSTANT SCALAR CURVATURE

Let $X \in C_0(M)$, and let $\xi \in C_0^*(M)$ be the covariant form of X. Then

$$\Delta\xi + \left(1 - \frac{2}{n}\right) d\delta\,\xi = 2Q\xi, \tag{1}$$

where $\Delta = d\delta + \delta d$ is the Laplace-Beltrami operator. Conversely, since M is compact, a solution of (1) is an element of $C_0^*(M)$.

Let $\iota(A)$ and $\theta(A)$ denote the interior product and Lie derivative operators with respect to the vector field A. If $\alpha$ is the 1-form dual to A, we shall occasionally write $\iota(\alpha)$ for $\iota(A)$ and $\theta(\alpha)$ for $\theta(A)$.

Let $\{X_1, \cdots, X_n\}$ be an orthonormal basis of $M_m$, the tangent space to M at m. Then the *co-differential* $\delta S$ at m of a symmetric tensor field S of order p is defined by

$$\delta S(Y_1, \cdots, Y_{p-1}) = -\sum_i (D_{X_i} S)(X_i, Y_1, \cdots, Y_{p-1}).$$

The *Hessian* of a $C^\infty$-function f on M is the symmetric 2-form

$$\text{Hess } f = Ddf.$$

LEMMA 1. *If* r = const., *then*

$$T d\delta\xi = -\frac{n}{n-2}\,\delta\,\theta(\xi)T \qquad (\xi \in C_0^*(M)),$$

*where* $T\alpha = \iota(\alpha)T$.

*Proof.* This is a consequence of the formula

$$\theta(X)T = \frac{n-2}{n}\left(\text{Hess}\ \delta\xi + \frac{1}{n}\Delta\delta\xi\cdot g\right) \tag{2}$$

and the fact that $\Delta\delta\xi = \frac{r}{n-1}\,\delta\xi$ (see [5]), since

$$\frac{n}{n-2}\,\delta\,\theta(\xi)T = \delta D d\delta\xi - \frac{r}{n(n-1)}\,d\delta\xi = -Qd\delta\xi + d\delta\, d\delta\xi - \frac{r}{n(n-1)}\,d\delta\xi$$

$$= \frac{r}{n-1}\,d\delta\xi - \frac{r}{n(n-1)}\,d\delta\xi - Qd\delta\xi = \frac{r}{n}\,d\delta\xi - Qd\delta\xi\,.$$

Let $\langle\ ,\ \rangle$ (respectively, $(\ ,\ )$) denote the local (global) scalar product of symmetric or skew-symmetric tensor fields; that is, if $\alpha$ and $\beta$ are both symmetric or skew-symmetric of order p, let

$$\langle\alpha,\beta\rangle = \frac{1}{p!}\,\alpha^J\beta_J \qquad\text{and}\qquad (\alpha,\beta) = \int_M \langle\alpha,\beta\rangle\, dV,$$

where $\alpha^J$ and $\beta_J$ $(J = j_1\cdots j_p)$ are the contravariant and covariant components of $\alpha$ and $\beta$, respectively, with respect to a given system of local coordinates, and where dV is the volume element. The norm of $\alpha$, denoted by $\|\alpha\|$, is defined by $\|\alpha\| = (\alpha,\alpha)^{1/2}$.

Observe that since

$$\Delta f = -2\langle\text{Hess}\ f, g\rangle\,,$$

$$\theta(X)T = \frac{n-2}{n}\left(\text{Hess}\ \delta\xi - \frac{2}{n}\langle\text{Hess}\ \delta\xi, g\rangle g\right). \tag{2'}$$

LEMMA 2. *If* r = const., *then*

$$(\theta(X)T, \theta(X)T) = -\left(\frac{n-2}{n}\right)^2 (Td\delta\xi, d\delta\xi)\,.$$

*Proof.* By (2), the relation $\langle\theta(X)T, g\rangle = 0$ implies

$$\langle\theta(X)T, \theta(X)T\rangle = \frac{n-2}{n}\langle\theta(X)T, Dd\delta\xi\rangle\,. \tag{3}$$

Integrating, we obtain the equation

$$(\theta(X)T, \theta(X)T) = \frac{n-2}{n}\,(\delta\,\theta(X)T, d\delta\xi)\,.$$

The result now follows from Lemma 1.

Observe that if $\theta(X)T$ lies in the kernel of $\delta^2$, then $\theta(X)T$ vanishes, so that $Qd\delta\xi = \frac{r}{n} d\delta\xi$.

We shall now show that Proposition 1 is a consequence of Lemmas 1 and 2. Since $T$ is positive semidefinite on $d\delta C_0^*(M)$, it follows from Lemma 2 that $\theta(X)T = 0$, and from Lemma 1 we conclude that $Qd\delta\xi = \frac{r}{n} d\delta\xi$. Applying (1), we see that $d\delta\xi \in C_0^*(M)$; since

$$\Delta\delta\xi = \frac{r}{n-1}\delta\xi,$$

we conclude that

$$d\delta\xi = \frac{r}{n-1}\xi + g(Y, \cdot),$$

where $Y$ is a Killing vector field. That this can only hold if $M$ has constant curvature follows from the proof of Theorem 2 of [1]. The result is now a consequence of Proposition 4 of [1].

*Proof of Corollary* 1.3. Since trace $Q^2$ is constant, the same is true of trace $T^2$. The evaluation of $\theta(X)\langle T, T\rangle$ gives

$$\langle \theta(X)T, T\rangle = -\frac{2}{n}\delta\xi\langle T, T\rangle .$$

Applying (2), we get the relation

$$\langle \theta(X)T, T\rangle = \frac{n-2}{n}\langle \text{Hess }\delta\xi, T\rangle = \frac{n-2}{n}\langle Dd\delta\xi, T\rangle = -\frac{n-2}{2n}\delta T d\delta\xi .$$

Consequently,

$$(n-2)(\delta T d\delta\xi, \delta\xi) = 4\int_M (\delta\xi)^2\langle T, T\rangle\, dV, \tag{4}$$

which implies that

$$(Td\delta\xi, d\delta\xi) \geq 0 .$$

Corollary 1.2 is now a consequence of Proposition 1.

Note that by Lemma 2, formula (4) may be written in the form

$$\|\theta(X)T\|^2 = -\frac{4(n-2)}{n^2}\int_M (\delta\xi)^2\langle T, T\rangle\, dV .$$

It is interesting to note the similarity in the proofs of Corollary 1.3 and Theorem 1 in [4]. The tensor fields are symmetric in the first case and skew-symmetric in the other case; indeed, $T$ is a symmetric tensor of order 2, whereas in the latter case we treat harmonic tensors $\alpha$. Both $T$ and $\alpha$ depend on the metric, and they are both assumed to have constant length, which is essential to the proof. The fact that trace $Q = r =$ constant does not imply by itself that trace $Q^2 =$ constant.

PROPOSITION 2. *Let* M *be a compact manifold, of dimension* $n > 2$ *and admitting a Riemannian metric* g *of constant scalar curvature. If* $X \in C_0(M)$ *is an infinitesimal, nonisometric conformal transformation field, then*

$$\|\mathrm{Hess}\ \delta\xi\|^2 \geq \frac{2}{n}\int_M \langle \mathrm{Hess}\ \delta\xi, g\rangle^2\, dV \qquad (\xi = g(X, \cdot)),$$

*equality holding if and only if* M *is globally isometric with a sphere.*

*Proof.* From (2') and (3) it follows that

$$\begin{aligned}\|\theta(X)T\|^2 &= \left(\frac{n-2}{n}\right)^2\left(\mathrm{Hess}\ \delta\xi, \mathrm{Hess}\ \delta\xi - \frac{2}{n}\langle \mathrm{Hess}\ \delta\xi, g\rangle g\right)\\ &= \left(\frac{n-2}{n}\right)^2\left[\|\mathrm{Hess}\ \delta\xi\|^2 - \frac{2}{n}\int_M \langle \mathrm{Hess}\ \delta\xi, g\rangle^2\, dV\right].\end{aligned}$$

Theorem 1 is now an immediate consequence of Proposition 2 and the relation $\Delta\delta\xi = -2\langle \mathrm{Hess}\ \delta\xi, g\rangle$. For if

$$r^2 = 2n(n-1)^2\frac{\|\mathrm{Hess}\ \delta\xi\|^2}{\|\delta\xi\|^2},$$

then $\theta(X)T$ vanishes.

The same conclusion prevails if

$$r^2 = 2n(n-1)^2\frac{\langle \mathrm{Hess}\ \delta\xi, \mathrm{Hess}\ \delta\xi\rangle}{(\delta\xi)^2}$$

for some $\xi \in C_0^*(M) - I_0^*(M)$.

*Note.* By applying the operator $\Delta$ to the length function $|T|^2$, or, what is the same (since r is constant), to $|R|^2$, we may ask whether

$$\Phi = g^{rs}R^{ij}D_s D_r R_{ij} \geq 0,$$

or under what conditions this is so. The conjecture is true if $\Phi$ is nonnegative, by virtue of Corollary 1.2, since it turns out that $|R|^2$ is constant. In the special cases where (M, g) is locally symmetric, or, more generally, Ricci symmetric ($DR = 0$), the scalar $\Phi$ vanishes. Even more generally, if (M, g) is a recurrent space, or $DR = a \otimes R$, then $\Phi \geq 0$ provided $|a|^2 \geq \delta a$ for some vector field a.

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