RIEMANN MATRICES FOR HYPERELLIPTIC SURFACES WITH INVOLUTIONS OTHER THAN THE INTERCHANGE OF SHEETS

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Let $\omega_1, \cdots, \omega_g$ form a basis for the holomorphic differentials on a compact Riemann surface S of genus g, and let (a_i, b_i) $(i = 1, \cdots, g)$ form a set of retrosections (one-cycle representatives of a homology basis for S, where $\delta(a_i, b_j) = \delta_{ij}$, $\delta(a_i, a_j) = 0 = \delta(b_i, b_j)$, δ being the bilinear, skew-symmetric intersection number); then the $g \times 2g$ matrix

$$(A\ B) \equiv \left(\left(\int_{a_j} \omega_i\right) \left(\int_{b_j} \omega_i\right)\right)$$

is called a *period matrix* for S. By a change of basis for the holomorphic differentials, the matrix A can be reduced to the multiplicative identity (the new basis is said to be *normalized* with respect to (a_i, b_i)), and then B becomes $A^{-1}B$, which is symmetric with positive-definite imaginary part and is called the *Riemann matrix for* S *with respect to* (a_i, b_i). Torelli's theorem says that if a surface S has the same Riemann matrix with respect to (a_i, b_i) as a surface S' has with respect to (a_i', b_i'), then there exists a conformal homeomorphism from S onto S' taking either a_i to a_i' and b_i to b_i' or a_i to $-a_i'$ and b_i to $-b_i'$ (see [5, pp. 27-28] and the references cited there). If S' (and therefore S) is hyperelliptic, then conformality of one map implies conformality of the other, since the two maps then differ by the "interchange of sheets" on S', which is conformal. Every conformal equivalence class of hyperelliptic surfaces of genus g that have involutions (conformal self-homeomorphisms of order 2) other than the interchange of sheets contains a surface whose equation is $w^2 = f(z^2)$, where $f(x)$ is a complex polynomial of degree $g + 1$. This is a particular case of a result due to Hurwitz [2, p. 257]. The purpose of this note is to show that such surfaces can be classified according to their Riemann matrices.

If a surface S has the equation $w^2 = f(z^2)$, then in addition to the interchange of sheets $\iota: (z, w) \to (z, -w)$, the surface has at least two involutions, namely

$$\tilde{\sigma}: (z, w) \to (-z, w) \quad \text{and} \quad \hat{\sigma} \equiv \iota\tilde{\sigma}: (z, w) \to (-z, -w).$$

The natural projection $\tilde{\pi}$ from S to the quotient surface $\tilde{S} \equiv S/(1, \tilde{\sigma})$ is given concretely by $(z, w) \to (z^2, 2w) \equiv (\tilde{z}, \tilde{w})$, from which we see that $\tilde{S}$ has the equation $\tilde{w}^2 = 4f(\tilde{z})$ and that the differentials $(\tilde{z}^i/\tilde{w})d\tilde{z}$ $(i = 0, 1, \cdots)$ on $\tilde{S}$ lift to the "odd" differentials $(z^{2i+1}/w)dz$ on S. Similarly, the projection $\hat{\pi}$ from S to the quotient surface $\hat{S} \equiv S/(1, \hat{\sigma})$ is given by $(z, w) \to (z^2, 2zw) \equiv (\hat{z}, \hat{w})$, $\hat{S}$ has equation $\hat{w}^2 = 4\hat{z}f(\hat{z})$, and the differentials $(\hat{z}^i/\hat{w})d\hat{z}$ on $\hat{S}$ lift to the "even" differentials $(z^{2i}/w)dz$ on S. Note that if g is even, then both $\tilde{S}$ and $\hat{S}$ are of genus $g/2$, whereas if g is odd, then $\tilde{S}$ is of genus $(g-1)/2$ and $\hat{S}$ is of genus $(g+1)/2$. In either case, we can construct a model for S by pasting together two slit copies of $\tilde{S}$ or $\hat{S}$, and then ι, $\tilde{\sigma}$, $\hat{\sigma}$ appear as rotations through 180°. (See Figure 1 for $g = 4$, which is typical for even genus, and Figure 2 for $g = 5$, which is typical for odd genus.) It

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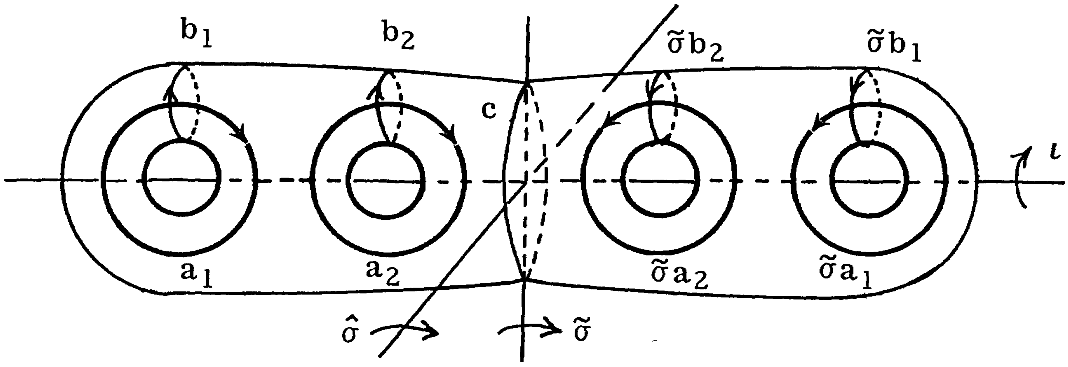

Figure 1. Two slit copies of $\tilde{S}$ pasted together along C.

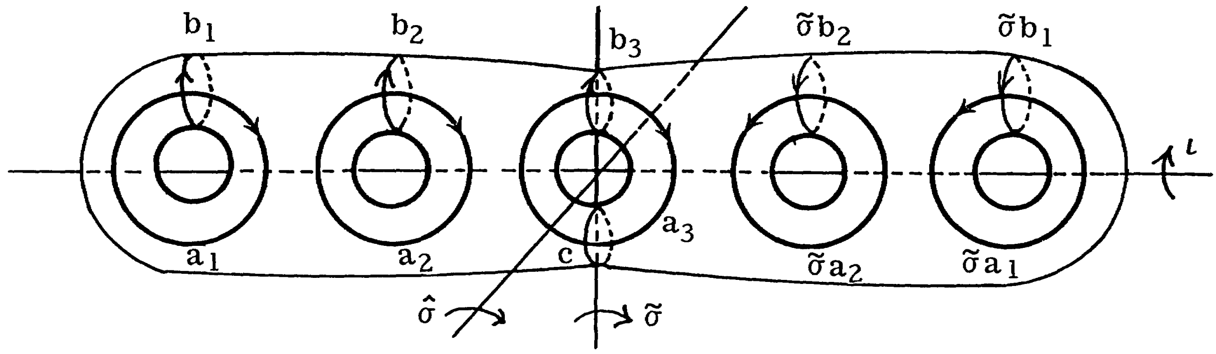

Figure 2. Two slit copies of $\tilde{S}$ pasted together along b_3 and C.

follows from the Riemann-Hurwitz relation (or from the figures) that if g is even, then S is a two-sheeted branched covering of $\tilde{S}$ and $\hat{S}$ with two branch points in each case, while if g is odd, then S is a two-sheeted branched covering of $\tilde{S}$ with four branch points and a two-sheeted smooth covering of $\hat{S}$.

Taking first the case where g is even, we can select retrosections

$$(a_i, \tilde{\sigma}a_i, b_i, \tilde{\sigma}b_i) = (a_i, -\hat{\sigma}a_i, b_i, -\hat{\sigma}b_i) \qquad (i = 1, \cdots, g/2)$$

for S, where the $(\tilde{a}_i, \tilde{b}_i) \equiv (\tilde{\pi}a_i, \tilde{\pi}b_i)$ are retrosections for $\tilde{S}$ and the $(\hat{a}_i, \hat{b}_i) \equiv (\hat{\pi}a_i, \hat{\pi}b_i)$ are retrosections for $\hat{S}$ (see Figure 1). Now, for any hyperelliptic surface $W^2 = f(Z)$, the differentials $(Z^i/W)dZ$ $(i = 0, \cdots, g-1)$ form a basis for the holomorphic differentials. Hence, if a period matrix (A B) for S is constructed with respect to the retrosections $(a_i, \tilde{\sigma}a_i, b_i, \tilde{\sigma}b_i)$, with the integrals of the "even" differentials in the first g/2 rows and the integrals of the "odd" differentials in the remaining g/2 rows, then

$$(A\ B) = \begin{pmatrix} \hat{A} & -\hat{A} & \hat{B} & -\hat{B} \\ \tilde{A} & \tilde{A} & \tilde{B} & \tilde{B} \end{pmatrix},$$

where $(\hat{A}\ \hat{B})$ is a period matrix for $\hat{S}$ with respect to $(\hat{a}_i, \hat{b}_i)$, and where $(\tilde{A}\ \tilde{B})$ is a period matrix for $\tilde{S}$ with respect to $(\tilde{a}_i, \tilde{b}_i)$. The corresponding Riemann matrix is

$$A^{-1}B = \frac{1}{2}\begin{pmatrix} \tilde{M} + \hat{M} & \tilde{M} - \hat{M} \\ \tilde{M} - \hat{M} & \tilde{M} + \hat{M} \end{pmatrix},$$

where $\tilde{M} \equiv \tilde{A}^{-1}\tilde{B}$ and $\hat{M} \equiv \hat{A}^{-1}\hat{B}$ are the corresponding Riemann matrices for $\tilde{S}$ and $\hat{S}$, respectively. Matrices of this form will be denoted by $\langle \tilde{M}, \hat{M} \rangle$. If new retrosections (a_i', b_i') $(i = 1, \cdots, g)$ are defined by

$$a_i' = a_i - \tilde{\sigma} a_i, \quad a_{i+g/2}' = b_i + \tilde{\sigma} b_i, \quad b_i' = b_i, \quad b_{i+g/2}' = -\tilde{\sigma} a_i \quad (i = 1, \cdots, g/2),$$

then the corresponding Riemann matrix is

$$\frac{1}{2}\begin{pmatrix} \hat{M} & I \\ I & -\tilde{M}^{-1} \end{pmatrix},$$

where $-\tilde{M}^{-1}$ is the Riemann matrix for $\tilde{S}$ with respect to the retrosections $(-\tilde{b}_i, \tilde{a}_i)$ $(i = 1, \cdots, g/2)$ and I is the $(g/2) \times (g/2)$ multiplicative identity matrix. Matrices of this form will be denoted by $(\tilde{M}, \hat{M})$. A surface has a Riemann matrix of this form if and only if it has one of the form $\langle \tilde{M}, \hat{M} \rangle$. That there exists a Riemann matrix of the form $(\tilde{M}, \hat{M})$ for hyperelliptic surfaces of even genus having involutions other than the interchange of sheets generalizes a previous result for $g = 2$ due to Oskar Bolza [1], [3, pp. 12-22].

On the other hand, if a surface S has a Riemann matrix of the form $\langle \tilde{M}, \hat{M} \rangle$ with respect to some retrosections (a_i, b_i), then S has the same Riemann matrix with respect to the retrosections (a_i', b_i'), where

$$a_i' = a_{i+g/2}, \quad a_{i+g/2}' = a_i, \quad b_i' = b_{i+g/2}, \quad b_{i+g/2}' = b_i \quad (i = 1, \cdots, g/2).$$

If S is hyperelliptic, it follows from Torelli's theorem (where $S = S'$) that the retrosections (a_i, b_i) are of the form $(a_i, \tilde{\sigma} a_i, b_i, \tilde{\sigma} b_i)$ $(i = 1, \cdots, g/2)$, where $\tilde{\sigma}$ is a conformal self-homeomorphism of S. Furthermore, $\tilde{\sigma}^2$ induces the identity automorphism on the first homology group of S, and therefore, being conformal, it is the identity mapping [4, p. 737]. Finally, $\tilde{\sigma}$ is not the interchange ι of sheets, since $\iota: a_i \to -a_i$, $b_i \to -b_i$, and $(a_i, -a_i, b_i, -b_i)$ are not homologously independent. Hence, S has an involution other than the interchange of sheets. If $\omega_1, \cdots, \omega_g$ are the normalized holomorphic differentials on S with respect to (a_i, b_i) giving rise to $\langle \tilde{M}, \hat{M} \rangle$, then

$$\int_{a_i, \tilde{\sigma} a_i, b_i, \tilde{\sigma} b_i} \tilde{\sigma} \omega_i = \int_{\tilde{\sigma} a_i, a_i, \tilde{\sigma} b_i, b_i} \omega_i = \int_{a_i, \tilde{\sigma} a_i, b_i, \tilde{\sigma} b_i} \omega_{i+g/2} \quad (i = 1, \cdots, g/2),$$

so that $\omega_{i+g/2} = \tilde{\sigma}\omega_i$ $(i = 1, \cdots, g/2)$. The $\omega_i + \tilde{\sigma}\omega_i$ are invariant with respect to $\tilde{\sigma}$; therefore they are defined on the quotient surface $\tilde{S} \equiv S/(1, \tilde{\sigma})$, and in fact they form a basis for the holomorphic differentials on this surface, normalized with respect to the projection of (a_i, b_i) $(i = 1, \cdots, g/2)$. Hence, $\tilde{M}$ is a Riemann matrix for S. Similarly, by considering $\omega_i - \tilde{\sigma}\omega_i$, we see that $\hat{M}$ is a Riemann matrix for the quotient $\hat{S} \equiv S/(1, \iota\tilde{\sigma})$. S is a two-sheeted branched covering of each quotient space, and since each quotient is of genus $g/2$, the Riemann-Hurwitz relation implies that there are two branch points in each case. We summarize:

THEOREM 1. *Let* S *be a hyperelliptic Riemann surface of even genus* g. *Then* S *has an involution other than the interchange of sheets if and only if* S *has a Riemann matrix of the form*

$$\frac{1}{2}\begin{pmatrix} \hat{M} & I \\ I & \tilde{M} \end{pmatrix},$$

and then $\hat{M}$ *and* $\tilde{M}$ *are Riemann matrices for surfaces* $\hat{S}$ *and* $\tilde{S}$, *respectively, each of genus* $g/2$, *and* S *is a two-sheeted branched covering of* $\hat{S}$ *and of* $\tilde{S}$, *with two branch points in each case.*

If S has equation $w^2 = f(z^2)$ and is of odd genus, one selects retrosections

$$(a_i, \tilde{\sigma} a_i, b_i, \tilde{\sigma} b_i) \quad \text{and} \quad (a_{(g+1)/2}, b_{(g+1)/2}) \quad (i = 1, \cdots, (g-1)/2)$$

for S, where the $(\tilde{a}_i, \tilde{b}_i) \equiv (\tilde{\pi} a_i, \tilde{\pi} b_i)$ are retrosections for $\tilde{S}$ and

$$\tilde{\pi} a_{(g+1)/2} = 0 = \tilde{\pi} b_{(g+1)/2};$$

$(\hat{a}_i, \hat{b}_i)$ $(i = 1, \cdots, (g+1)/2)$ are retrosections for $\hat{S}$, where

$$(\hat{a}_i, \hat{b}_i) \equiv (\hat{\pi} a_i, \hat{\pi} b_i) \quad (i = 1, \cdots, (g-1)/2), \quad \hat{b}_{(g+1)/2} \equiv \hat{\pi} b_{(g+1)/2}, \quad 2\hat{a}_{(g+1)/2} = \hat{\pi} a_{(g+1)/2}$$

(see Figure 2). Proceeding as in the case of even genus, we find that the Riemann matrix $\langle \tilde{M}, \hat{M} \rangle$ for odd genus is

$$\frac{1}{2}\left(\begin{array}{c|c|c} \hat{M}^* + \tilde{M} & R^t & \tilde{M} - \hat{M}^* \\ \hline R & m & -R \\ \hline \tilde{M} - \hat{M}^* & -R^t & \hat{M}^* + \tilde{M} \end{array}\right),$$

where $\hat{M}^*$ is $\hat{M}$ with the last row and last column deleted, R is the last row of $\hat{M}$ with the last element deleted, m is the last element in R, and t indicates the transpose. If new retrosections (a_i', b_i') $(i = 1, \cdots, g)$ for S are defined by

$$a_i' = a_i - \tilde{\sigma} a_i, \quad a'_{(g+1)/2} = a_{(g+1)/2}, \quad a'_{i+(g+1)/2} = b_i + \tilde{\sigma} b_i, \quad b_i' = b_i,$$

$$b'_{(g+1)/2} = b_{(g+1)/2}, \quad b'_{i+(g+1)/2} = -\tilde{\sigma} a_i \quad (i = 1, \cdots, (g-1)/2),$$

then the Riemann matrix $(\tilde{M}, \hat{M})$ obtained for odd genus is

$$\frac{1}{2}\left(\begin{array}{c|c} \hat{M} & I' \\ \hline I'^t & -\tilde{M}^{-1} \end{array}\right),$$

where I' is the $((g+1)/2) \times ((g+1)/2)$ multiplicative identity matrix with the last column deleted.

If one starts with the assumption that a hyperelliptic surface S of odd genus g has a Riemann matrix of the form $\langle \tilde{M}, \hat{M} \rangle$ (and therefore of the form $(\tilde{M}, \hat{M})$) with respect to some retrosections (a_i, b_i), then a change to retrosections (a_i', b_i'), where

$$a_i' = a_{i+(g+1)/2}, \quad a'_{i+(g+1)/2} = a_i, \quad b_i' = b_{i+(g+1)/2}, \quad b'_{i+(g+1)/2} = b_i$$

for $i = 1, \cdots, (g-1)/2$, and where

$$a'_{(g+1)/2} = -a_{(g+1)/2} \quad \text{and} \quad b'_{(g+1)/2} = -b_{(g+1)/2},$$

leaves the matrix invariant. Proceeding as in the case of even genus, we find that the retrosections giving rise to $\langle \tilde{M}, \hat{M} \rangle$ are of the form $(a_i, \tilde{\sigma} a_i, b_i, \tilde{\sigma} b_i)$ $(i = 1, \cdots, (g-1)/2)$ and $(a_{(g+1)/2}, b_{(g+1)/2})$, with corresponding normalized differentials ω_i, $\tilde{\sigma}\omega_i$, and $\omega_{(g+1)/2}$, where $\tilde{\sigma}$ is an involution other than the interchange ι of sheets, and where

$$\tilde{\sigma} a_{(g+1)/2} = -a_{(g+1)/2}, \quad \tilde{\sigma} b_{(g+1)/2} = -b_{(g+1)/2}.$$

Furthermore, S is a two-sheeted branched covering of the quotient surface $\tilde{S} \equiv S/(1, \tilde{\sigma})$ with four branch points, and $\tilde{M}$ is a Riemann matrix for $\tilde{S}$ with respect to the projections of (a_i, b_i) $(i = 1, \cdots, (g-1)/2)$, the corresponding normalized differentials being the projections of the $\tilde{\sigma}$-invariant $\omega_i + \tilde{\sigma}\omega_i$. However, because $b_{(g+1)/2}$ is invariant under $\iota\tilde{\sigma}$ and therefore need not project to a simple cycle on the quotient surface $\hat{S} \equiv S/(1, \iota\tilde{\sigma})$, the same technique cannot be used to show that $\hat{M}$ is a Riemann matrix for $\hat{S}$. Indeed, it is not difficult to show that the group of matrices of changes in retrosections that preserve the form $\langle \tilde{M}, \hat{M} \rangle$ does not preserve the Siegel modular orbit of $\hat{M}$. We summarize:

THEOREM 2. *Let* S *be a hyperelliptic surface of odd genus* g. *Then* S *has an involution other than the interchange of sheets if and only if* S *has a Riemann matrix of the form*

$$\frac{1}{2}\begin{pmatrix} \hat{M} & I' \\ I'^t & \tilde{M} \end{pmatrix},$$

and then $\tilde{M}$ *is a Riemann matrix for a surface* $\tilde{S}$ *of genus* $(g-1)/2$, *and* S *is a two-sheeted branched covering of* $\tilde{S}$ *with four branch points.*

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