A CONVENIENT CATEGORY OF TOPOLOGICAL SPACES

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Dedicated to R. L. Wilder, who taught my first course in analysis situs, suggested my first research problem, and nursed my initial efforts to fruition.

1. INTRODUCTION

For many years, algebraic topologists have been laboring under the handicap of not knowing in which category of spaces they should work. Our need is to be able to make a variety of constructions and to know that the results have good properties without the tedious spelling out at each step of lengthy hypotheses such as countably paracompact, normal, completely regular, first axiom of countability, metrizable, and so forth. It may be good research technique and an enjoyable exercise to analyse the precise circumstances for which an argument works; but if a developing theory is to be handy for research workers and attractive to students, then simplicity of the fundamentals must be the goal.

The demands which a *convenient* category should satisfy are first that it be large enough to contain all of the particular spaces arising in practice. Second, it must be closed under standard operations; these are the formation of subspaces, product spaces $X \times Y$, function spaces Y^X, decomposition spaces, unions of expanding sequences of spaces, and compositions of these operations. Third, the category should be small enough so that certain reasonable propositions about the standard operations are true. These state that the order of performing two operations can be reversed. We adopt the following as test propositions.

(1) $$(Y \times Z)^X = Y^X \times Z^X .$$

(2) $$Z^{Y\times X} = (Z^Y)^X .$$

(3) A product of decomposition spaces is a decomposition space of the product.

(4) A product of unions is a union of products.

(5) A decomposition space of a union is a union of decomposition spaces.

It is well known that (1), (2), and (3) are valid for compact metric spaces, but the category of these is not closed, under several standard operations. It is also known that these propositions do not hold in the category of all Hausdorff spaces. In fact, arguments have been given which imply that there is no convenient category in our sense [13, Appendix]. The arguments are based on a blind adherence to the customary definitions of the standard operations. These definitions are suitable for the category of Hausdorff spaces, but they need not be for a subcategory. The categorical viewpoint enables us to defrost these definitions and bend them a bit.

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In this paper, we propose as convenient the category of spaces we shall call *compactly generated.* Such a space is a Hausdorff space with the property that each subset that intersects every compact set in a closed set is itself a closed set. These spaces are called k-spaces, in the book of Kelley [7]. Subsequent development of the concept by Spanier [12], Weingram [14], and R. Brown [1], [2] has demonstrated its utility. In Sections 2 through 5, we shall present propositions which taken together assert that the category is convenient. A principal reason for this success is that we use a modification of the cartesian product; the cartesian product is categorical for all topological spaces but not for the subcategory of compactly generated spaces; however the modified product is categorical in the subcategory. Similar modifications are made in the definitions of subspace and function space. Sections 2 through 5 are mainly expository; most of the results can be found in the above references.

Next we take up the question of a convenient category of pairs of spaces (X, A). We require of course that X be compactly generated and that A be closed in X. In addition, we shall require that A be a neighborhood deformation retract in X (abbreviated NDR). Sections 6, 7, and 8 present the evidence supporting our claim that this category is convenient.

From pairs one passes to triples, quadruples, and eventually to filtered spaces. This is not an automatic generalization. There are a variety of circumstances in algebraic topology in which one wants to form the union of an expanding sequence of spaces; this passage to a limit is highly nontrivial. It is customary to assign the weak topology to the limit, but this leads to various difficulties involving the Hausdorff property, normality, and behavior under products. In Sections 9 and 10, we show that if our filtered spaces are built as unions of NDR pairs, all compactly generated, then these difficulties disappear.

In Section 11 we compare briefly our category with three others: quasi-topological spaces (Spanier [13]), $\aleph_0$-spaces (Michael [8]), and spaces having the homotopy types of CW-complexes (Milnor [9]).

Most of the material in Sections 6 to 10 was developed during joint work with M. Rothenberg on classifying spaces of topological groups (see [11]).

2. COMPACTLY GENERATED SPACES

We denote by CG the category of compactly generated Hausdorff spaces and their continuous maps. The Hausdorff property is imposed to insure that compact subsets are closed. A useful test for a space to be in CG is the following.

2.1. *If* X *is a Hausdorff space, and if for each subset* M *and each limit point* x *of* M *there exists a compact set* C *in* X *such that* x *is a limit point of* $M \cap C$, *then* $X \in CG$.

Briefly, if each limit relation in X takes place in some compact subset of X, then $X \in CG$. For the proof, assume M meets each compact set in a closed set, and let x be a limit point of M. By the assumption, there exists a compact C such that x is a limit point of $M \cap C$. Since $M \cap C$ is closed, we have the relation $x \in M \cap C$, hence $x \in M$. So M is closed and $X \in CG$.

2.2. *The category* CG *includes all locally compact spaces and all spaces satisfying the first axiom of countability (for example, metrizable spaces).*

These facts are stated in Kelley's book. They are corollaries of 2.1: if X is locally compact, we take C to be the compact closure of a neighborhood of x, and if X is first countable, C is taken to consist of x and a sequence in M converging to x.

These results show that CG is large enough to contain most of the standard spaces. Perhaps the simplest example of a Hausdorff space not in CG is the following.

2.3. *Example.* Let Y denote the ordinal numbers preceding and including the first noncountable ordinal Ω. Give to Y the topology defined by its natural order. Let X be the subspace obtained by deleting all limit ordinals except Ω. The only compact subsets of X are the finite sets, because each infinite set must contain a sequence converging to a limit ordinal of the second kind. Therefore the set $X - \Omega$ meets each compact set in a closed set, but is not closed in X, because it has Ω as a limit point.

Remark. The condition in the hypothesis of 2.1 is not equivalent to $X \in CG$; there is an example of a space in CG for which the condition does not hold.

The example 2.3 shows that a subspace (X) of a compactly generated space (Y) need not be compactly generated. However, the following results show that certain subspaces are in CG.

2.4. *If* X *is in* CG, *then every closed subset of* X *is also in* CG. *An open set* U *of* X *is in* CG *if it is a "regular" open set, that is, if each point* $x \in U$ *has a neighborhood in* X *whose closure lies in* U.

Suppose A is closed in X and $B \subset A$ meets each compact subset of A in a closed set. Let C be a compact set in X. Then $A \cap C$ is a compact set of A; hence $B \cap (A \cap C) = B \cap C$ is closed in A. Since A is closed, $B \cap C$ is closed in X. Because $X \in CG$, it follows that B is closed in X, hence also in A. So $A \in CG$.

Let U be a regular open set in X, suppose $B \subset U$ meets each compact set of U in a closed set, and let $x \in U$ be a limit point of B. By assumption, there is a neighborhood V of x in X with closure $\overline{V} \subset U$. If C is compact in X, then $\overline{V} \cap C$ is a compact set of X in U. Since it is also compact in the relative topology of U, it follows that $B \cap \overline{V} \cap C$ is closed first in U, then in $\overline{V} \cap C$, and finally in X. Because C is any compact set in X and $X \in CG$, it follows that $B \cap \overline{V}$ is closed in X. Since x is a limit point of $B \cap \overline{V}$, we see that $x \in B \cap \overline{V}$, hence $x \in B$, so B is closed in U.

2.5. *Definition.* If X and Y are topological spaces, a mapping $f: X \to Y$ is called *proclusive* (or a *proclusion*) if $fX = Y$ and a set U is open in Y whenever $f^{-1}U$ is open in X.

The name *proclusion* is justified by comparing with an inclusion $X \to Y$; in the latter case, the topology of Y determines that of X, and in the former the topology of X determines that of Y.

When f is proclusive, Y is equivalent to the decomposition space of X by the family of inverse images of points of Y. Also, if a space X is decomposed into a family Y of disjoint closed sets, the topology of the decomposition space Y is defined so that the natural mapping $X \to Y$ is continuous and proclusive.

It is well known that if $f: X \to Y$ is continuous, X and Y are Hausdorff spaces, X is compact, and $fX = Y$, then f is proclusive.

2.6. *If* $f: X \to Y$ *is proclusive,* $X \in CG$ *and* Y *is a Hausdorff space, then* $Y \in CG$.

Suppose $B \subset Y$ meets each compact set of Y in a closed set. Let C be a compact set in X. Then fC is compact, hence $B \cap fC$ is closed, so $f^{-1}(B \cap fC)$ is closed, and therefore $f^{-1}(B \cap fC) \cap C$ is closed. Since this last set coincides with

$f^{-1}B \cap C$, it follows that $f^{-1}B$ meets each compact set of X in a closed set. Because $X \in CG$, this means that $f^{-1}B$ is closed. Since f is proclusive, B must be closed in Y. This shows that $Y \in CG$.

The preceding results show that CG is large in the sense that it contains many spaces. By definition, it contains all continuous maps between any two of its spaces. The following proposition sometimes facilitates the recognition of the continuity of a function.

2.7. *If* $X \in CG$, Y *is a Hausdorff space, and a function* $f: X \to Y$ *is continuous on each compact subset of* X, *then* f *is continuous.*

To prove this, let A be closed in Y, and let C be compact in X. Since Y is a Hausdorff space and $f \mid C$ is continuous, fC is compact, hence closed in Y. This implies that $A \cap fC$ is closed, hence also

$$(f \mid C)^{-1}(A \cap fC) = (f^{-1}A) \cap C.$$

Because $X \in CG$, it follows that $f^{-1}A$ is closed in X, and this shows that f is continuous.

3. THE RETRACTION FUNCTOR

3.1. *Definition.* If X is a Hausdorff space, *the associated compactly generated space* k(X) is the set X with the topology defined as follows: a closed set of k(X) is a set that meets each compact set of X in a closed set. If $f: X \to Y$ is a mapping of Hausdorff spaces, k(f) denotes the same function $k(X) \to k(Y)$.

3.2. THEOREM.

(i) *The identity function* $k(X) \to X$ *is continuous.*

(ii) k(X) *is a Hausdorff space.*

(iii) k(X) *and* X *have the same compact sets.*

(iv) k(X) *is compactly generated.*

(v) *If* $X \in CG$, *then* $k(X) = X$.

(vi) *If* $f: X \to Y$ *is continuous on compact sets, then* k(f) *is continuous.*

(vii) *The identity mapping* $k(X) \to X$ *induces isomorphisms of homotopy groups and singular homology and cohomology groups.*

The theorem can be paraphrased by saying that k is a retraction of the category H of Hausdorff spaces into CG. It is an adjoint of the inclusion of CG in H.

Proof. If A is closed in X, and C is compact in X, then C is closed in X, hence also $A \cap C$. This means that A is also closed in k(X), and this proves (i). Since X is a Hausdorff space, (i) implies (ii). If a set A is compact in k(X), then (i) implies that A is compact in X. Suppose now that C is compact in X, and C' denotes the set C with its relative topology from k(X). By (i), the identity map $C' \to C$ is continuous; we must prove the continuity of its inverse. Let B denote a closed set of C'. By definition, B meets every compact set of X in a closed set; therefore $B \cap C = B$ is closed in C. Thus $C \to C'$ is continuous; this shows that C' is compact, and (iii) is proved. If a set A meets each compact set of k(X) in a closed set, then, by (iii), it meets each compact set of X in a compact (hence, closed) set; therefore A is closed in k(X), and (iv) is proved. (v) follows directly from (iv).

To prove (vi), it suffices by 2.7 to prove that $k(f)$ is continuous on each compact set of $k(X)$. Let C' be compact in $k(X)$; by (iii), the same set with its topology in X (call it C) is compact and the identity map $C' \to C$ is a homeomorphism. Since $f \mid C$ is continuous, fC is compact, and by (iii), so is the same set fC' with its topology in $k(Y)$. Thus the function $k(f) \mid C'$: $C' \to fC'$ factors into the composition of $f \mid C$ and two identity maps: $C' \to C \to fC \to fC'$. Hence $k(f) \mid C'$ is continuous, and (vi) is proved. By (vi), the maps of closed cells into X coincide with those into $k(X)$; this implies (vii), because the groups in question are derived from such mappings. In particular, X and $k(X)$ have the same singular complex.

4. PRODUCT SPACES

Our second criterion for a convenient category of spaces is that it be closed under standard operations—a construction applied to one or more spaces in the category should give a space in the category. The category CG is nearly ideal in this respect: in case a construction applied to spaces in CG gives a Hausdorff space outside CG, we need only combine the construction with the retraction functor k to obtain again a space in CG.

Consider the operation of forming the cartesian product $X \times_c Y$ of two spaces; we find that CG is not closed under this operation. An example of this is due to Dowker [3], who constructed CW-complexes X and Y such that the complex $X \times_c Y$ does not have the weak topology, and hence (as was shown by Spanier [12, 2.6]) cannot be compactly generated.

4.1. *Definition.* If X and Y are in CG, their product $X \times Y$ (in CG) is $k(X \times_c Y)$, where $\times_c$ denotes the product with the usual cartesian topology.

4.2. THEOREM. *The product defined in* 4.1 *satisfies the axioms for a product in the category* CG.

Proof. Since by 3.2 the identity function $X \times Y \to X \times_c Y$ is continuous, and since the projections $X \times_c Y$ into X and Y are continuous, their compositions projecting $X \times Y$ into X and Y are continuous and, hence, belong to CG. Let $W \in$ CG, and let f and g be maps $W \to X$ and $W \to Y$ in CG. As usual, f and g are the components of a unique mapping (f, g): $W \to X \times_c Y$. Applying k and using the facts $k(W) = W$ and $k(X \times_c Y) = X \times Y$, we obtain a unique mapping $k(f, g)$: $W \to X \times Y$ which, when composed with the projections, gives f and g.

The above proof can be omitted by appealing to the general proposition that any adjoint functor (such as k) must preserve products.

It follows from 4.2 that the product $X \times Y$ in CG satisfies the usual commutative and associative laws. We can extend the construction to products having any number of factors, by applying k to the usual product.

Having modified the concept of product space, we should note what effect this has on other concepts that are based on products such as topological group G ($G \times G \to G$), transformation group G of X ($G \times X \to X$), and homotopy ($I \times X \to X$). If we restrict ourselves to G and X in CG, any multiplications $G \times_c G \to G$ or actions $G \times_c X \to X$ that are continuous in the old sense remain continuous when we apply k. Thus the effect of the new definition is to allow an increase in the number of groups and actions. The following theorem asserts that in many cases there is no change; in particular, the concept of homotopy is unaltered.

4.3. THEOREM. *If* X *is locally compact and* $Y \in CG$, *then* $X \times_c Y$ *is in* CG; *that is* $X \times Y = X \times_c Y$.

Proof. Let A be a subset of $X \times_c Y$ that meets each compact set in a closed set, and let (x_0, y_0) be a point of its complement. By local compactness, x_0 has a neighborhood whose closure N is compact. Since $N \times_c y_0$ is compact, $A \cap (N \times_c y_0)$ must be closed. It follows that x_0 has a smaller neighborhood U such that $\overline{U} \times_c y_0$ does not meet A. Let B denote the projection in Y of $A \cap (\overline{U} \times_c Y)$. If C is a compact set in Y, then $A \cap (\overline{U} \times_c C)$ is compact, and therefore $B \cap C$ is closed. Since $Y \in CG$, B must be closed in Y. Since y_0 is not in B, it follows that $U \times_c (Y - B)$ is a neighborhood of (x_0, y_0) not meeting A. This proves that A is closed; hence $X \times_c Y$ is in CG.

In the category of compact spaces, it is well known that a product of decomposition spaces has the topology of the decomposition space of the product. It is not difficult to find counterexamples involving noncompact spaces. However, the following theorem asserts that each such uses either spaces not in CG or the wrong product.

4.4. THEOREM. *If* $f: X \to X'$ *and* $g: Y \to Y'$ *are proclusive as mappings in* CG (*see* 2.5), *then* $f \times g: X \times Y \to X' \times Y'$ *is also proclusive in* CG.

Proof. Since $f \times g$ factors into the composition $(f \times 1)(1 \times g)$, and since a composition of proclusions is a proclusion, it suffices to prove the special case where $Y = Y'$ and g is the identity. Suppose then that $A \subset X' \times Y$ and that $(f \times 1)^{-1}A$ is closed in $X \times Y$. Let C be a compact set in $X' \times Y$, and let D and E denote its projections in X' and Y, respectively. Then $D \times E$ is compact. If we can show that $A \cap (D \times E)$ is closed, it will follow that $A \cap C$ is closed, and since $X' \times Y$ is in CG, this will show that A is closed, and the proposition will be proved. Since $(f \times 1)^{-1}(D \times E) = f^{-1}D \times E$ is closed in $X \times Y$, it follows that $(f \times 1)^{-1}(A \cap (D \times E))$ is closed in $f^{-1}D \times E$. Substituting X, X', Y for $f^{-1}D, D, E$, respectively, we have reduced the proof to the case where X' and Y are compact. Then, by 4.3, $X' \times Y = X' \times_c Y$ and $X \times Y = X \times_c Y$.

Suppose then that $W \subset X' \times Y$, $(f \times 1)^{-1}W$ is open in $X \times Y$, and $(x_0', y_0) \in W$. Choose $x_0 \in X$ so that $fx_0 = x_0'$. Since (x_0, y_0) is in the open set $(f \times 1)^{-1}W$ and Y is compact, there exists a neighborhood V of y_0 such that $x_0 \times \overline{V}$ lies in $(f \times 1)^{-1}W$. Let U denote the set of those $x \in X$ such that $(fx) \times \overline{V} \subset W$. To see that U is open in X, let $x_1 \in U$. We can cover $x_1 \times \overline{V}$ by products of open sets contained in $(f \times 1)^{-1}W$, and we can select a finite subcovering; then the intersection of the X-factors of these products is a neighborhood N of x_1 such that $N \times \overline{V}$ lies in $(f \times 1)^{-1}W$. Therefore U is open. By its definition, $U = f^{-1}fU$; hence fU is open in X', because f is proclusive. Since (x_0', y_0) is in $fU \times V$, and since $fU \times V$ is open and contained in W, it follows that W is open.

4.5. LEMMA. *If* X *and* Y *are Hausdorff spaces, then the two topologies* $(kX) \times (kY)$ *and* $k(X \times_c Y)$ *on the product space coincide.*

Proof. Since the identity maps $kX \to X$ and $kY \to Y$ are continuous, so also is the identity map $g: (kX) \times_c (kY) \to X \times_c Y$; hence each compact set of $(kX) \times_c (kY)$ is compact in $X \times_c Y$. Let A be a compact set of $X \times_c Y$. Since its projections B and C in X and Y, respectively, are compact, they are also compact in kX and kY, respectively. Therefore $B \times_c C$ is a compact set of $(kX) \times_c (kY)$; hence $g \mid B \times_c C$ is bicontinuous. Since $A \subset B \times_c C$, it follows that A is compact in $(kX) \times_c (kY)$. Because $(kX) \times_c (kY)$ and $X \times_c Y$ have the same compact sets, it follows from Definition 3.1 of k that their associated topologies in CG coincide.

4.6. *Definition.* If X is a subset of a space $Y \in CG$, it may happen that X with its usual relative topology X_r is not in CG (see Example 2.3); in any case, we define the *subspace* X of Y to be the set X with the topology $k(X_r)$. A mapping $f: X \to Y$ in CG is called *inclusive* (or an inclusion) if f is a homeomorphism of X with the subspace fX of Y.

This concept of an inclusion $f: X \to Y$ has the following property, which characterizes inclusions in a category of sets: if $g: Z \to Y$ is a map in CG such that $gZ \subset fX$, then g factors into the composition fg', where $g': Z \to X$ is in CG (that is, $g' = f^{-1}g$ is continuous). In analogy with 4.4, we have the following result (the proof is routine, and we omit it).

4.7. THEOREM. *If* $f: X \to X'$ *and* $g: Y \to Y'$ *are inclusive in* CG, *then* $f \times g: X \times Y \to X' \times Y'$ *is also inclusive in* CG.

5. FUNCTION SPACES

Most of the results of this section are contained in essence in the paper of R. Brown [2]. Since his formulations do not accord with our viewpoint, we review the material in detail.

For Hausdorff spaces X, Y, let C(X, Y) denote the space of continuous mappings $X \to Y$ with the compact-open topology. We recall the definition: if A is a compact set of X and U is an open set of Y, let W(A, U) denote the set of $f \in C(X, Y)$ such that $fA \subset U$; then the family of W(A, U) for all such pairs (A, U) forms a subbasis for the open sets of C(X, Y). Although X and Y are in CG, it may happen that C(X, Y) is not in CG; for example, take X to be two points; then $C(X, Y) = Y \times_c Y$, and Dowker's example shows that this need not be in CG. So again we must apply the retraction k.

5.1. *Definition.* For Hausdorff spaces X, Y, define $Y^X = kC(X, Y)$.

5.2. LEMMA. *The evaluation mapping* $e: C(X, Y) \times_c X \to Y$, *defined by* $e(f, x) = fx$, *is continuous on compact sets* (see [2, Lemma 1.3]). *If* X *and* Y *are in* CG, *then* e *is continuous as a mapping* $Y^X \times X \to Y$.

Proof. Since any compact set of the product is contained in the product of its projections, it suffices to show that e is continuous on any set of the form $F \times A$, where F is compact in C(X, Y), and A is compact in X. Let $(f_0, x_0) \in F \times A$, and let U be an open set of Y containing f_0x_0. Since f_0 is continuous, there exists a neighborhood N of x_0 in A whose closure satisfies $f_0\bar{N} \subset U$. Therefore $(W(\bar{N}, U) \cap F) \times N$ is open in $F \times A$, it contains (f_0, x_0), and it is mapped by e into U. This shows that e is continuous on compact sets.

By 3.2vi, if we apply k to $e: C(X, Y) \times_c X \to Y$, we obtain a continuous mapping. When $X \in CG$, the left side gives, by 4.5,

$$k(C(X, Y) \times_c X) = kC(X, Y) \times kX = Y^X \times X;$$

and when $Y \in CG$, the right side becomes $kY = Y$. This proves the lemma.

5.3. LEMMA. *If* X *is in* CG, *and* Y *is a Hausdorff space, then* C(X, kY) *and* C(X, Y) *are equal as sets, and the two topologies have the same compact sets, hence* $kC(X, kY) = kC(X, Y)$ *as spaces in* CG (see [2, Proposition 3.4]).

Proof. If $f: X \to kY$ is continuous, so is its composition with $kY \to Y$, and therefore $f \in C(X, Y)$. Conversely, if $f: X \to Y$ is continuous, then $kf: kX \to kY$ is continuous from X to kY. Thus C(X, kY) and C(X, Y) coincide as sets of functions. Since $kY \to Y$ is continuous, it follows that the identity map $C(X, kY) \to C(X, Y)$ is continuous. This implies that each compact set in $C(X, kY)$ is also compact in $C(X, Y)$.

Now let $F \subset C(X, Y)$ be a compact set in its relative topology in $C(X, Y)$. Let F' denote the same set with its relative topology in $C(X, kY)$. We wish to prove that F' is compact. It suffices to show that each open set W of $C(X, kY)$ meets F' in an open set of F, because this implies that the inverse correspondence $F \to F'$ is continuous, whence F is compact. It obviously suffices to prove this when W is a subbasic open set $W(C, U)$, where C is compact in X, and U is open in kY. Suppose then that $f_0 \in W(C, U) \cap F$. Since $F \times C$ is compact, and since by 5.2, the evaluation mapping $e: F \times C \to Y$ is continuous, it follows from 3.2vi that it is also continuous as a mapping $F \times C \to kY$. Hence $e^{-1} U$ is an open set of $F \times C$. Since C is compact and $f_0 \times C \subset e^{-1} U$, there exists an open set V of F containing f_0 such that $V \times C \subset e^{-1} U$. It follows that $f_0 \in V \subset W(C, U)$. Hence $W(C, U) \cap F$ is open in F. This shows that F' is compact, and it completes the proof that the two topologies have the same compact sets. It follows now from Definition 3.1 of k that their associated topologies in CG are equal.

5.4. THEOREM. *If* X, Y, *and* Z *are in* CG, *then*

$$(Y \times Z)^X = Y^X \times Z^X \qquad \text{(see [2, Proposition 3.6])}.$$

Proof. The correspondence associates with $f: X \to Y \times Z$ the pair (pf, qf), where p and q are the projections of $Y \times Z$ into Y and Z, respectively. Clearly, the continuity of f implies that of pf and qf. Conversely, if pf and qf are continuous, it follows that $f: X \to Y \times_c Z$ is continuous; then, by 3.2, $kf = f$ is continuous from $kX = X$ to $k(Y \times_c Z) = Y \times Z$. Thus the correspondence is one-to-one.

We prove first the equality of the CO-topologies:

$$C(X, Y \times_c Z) = C(X, Y) \times_c C(X, Z). \tag{5.5}$$

Consider first a subbasic open set on the right, of the form $W(C, U) \times_c W(D, V)$, where C, D are compact in X, and U, V are open in Y, Z, respectively. This corresponds exactly to the open set $W(C, U \times_c Z) \cap W(D, Y \times_c V)$ on the right. Conversely, a subbasic open set on the left of the form $W(C, U \times V)$ corresponds exactly to the open set $W(C, U) \times_c W(C, V)$ on the right. In the case of a general subbasic open set $W(C, S)$ on the left, we choose a point $f_0 \in W(C, S)$, and proceed as follows. Since $f_0 C$ is compact and is contained in the open set $S \subset Y \times_c Z$, there are compact subsets C_i of C and open sets $U_i \times V_i$ of $Y \times_c Z$ $(i = 1, \cdots, r)$ such that $C = \bigcup_{i=1}^r C_i$, and $f_0 C_i \subset U_i \times V_i \subset S$ for $i = 1, \cdots, r$. Then

$$f_0 \in \bigcap_{i=1}^{r} W(C_i, U_i \times V_i) \subset W(C, S).$$

Since each $W(C_i, U_i \times V_i)$ is open in the topology on the right of 5.5, so also is their intersection. Hence $W(C, S)$ is open in the topology on the right. This proves 5.5.

We now apply k to both sides of 5.5. By 5.3, the left side gives

$$kC(X, Y \times_c Z) = kC(X, k(Y \times_c Z)) = (Y \times Z)^X,$$

and the right side, by 4.5, becomes

$$k(C(X, Y) \times_c C(X, Z)) = kC(X, Y) \times kC(X, Z) = Y^X \times Z^X.$$

This completes the proof.

5.6. THEOREM. *If* X, Y, *and* Z *are in* CG, *then* $Z^{Y \times X} = (Z^Y)^X$.

Proof. We shall prove first the natural equivalence

(5.7) $\mu\colon C(Y \times X, Z) = C(X, C(Y, Z))$ (see [2, Theorem 1.6]).

Corresponding to an $f \in C(Y \times X, Z)$, define $\mu f\colon X \to C(Y, Z)$ by $((\mu f)x)y = f(y, x)$. To see that for each x, $(\mu f)x$ is continuous from Y to Z, suppose it carries y_0 into the open set U of Z. Then $f(y_0, x) \in U$, and the continuity of f gives an open set V of Y containing y_0 such that $f(V \times x) \subset U$; therefore $(\mu f)x$ maps V into U. We must now prove

(5.8) if $f \in C(Y \times X, Z)$, then $\mu f\colon X \to C(Y, Z)$ is continuous.

Let $W(B, U)$ be a subbasic open set of $C(Y, Z)$, and suppose that $(\mu f)x_0 \in W(B, U)$. Then $f(B \times x_0) \subset U$. Since U is open and B is compact, there is a neighborhood N of x_0 such that $f(B \times N) \subset U$. This implies $(\mu f)N \subset W(B, U)$, and it proves 5.8.

To prove the continuity of μ, we start with the continuity of the evaluation mapping rearranged as

$$e\colon Y \times X \times C(Y \times X, Z) \to Z \qquad \text{(see 5.2)}.$$

If we apply 5.8 with X replaced by $X \times C(Y \times X, Z)$, we find that

$$\mu e\colon X \times C(Y \times X, Z) \to C(Y, Z)$$

is continuous. Apply 5.8 again, with X replaced by $C(Y \times X, Z)$, Y by X, and Z by $C(Y, Z)$; then

$$\mu(\mu e)\colon C(Y \times X, Z) \to C(X, C(Y, Z))$$

is continuous. It is readily verified that $\mu(\mu e)$ coincides with μ of 5.7.

To show that μ has a continuous inverse, let

$$e\colon X \times C(X, C(Y, Z)) \to C(Y, Z), \qquad e'\colon Y \times C(Y, Z) \to Z$$

be evaluation mappings. By 5.2, the composition

$$e'(1 \times e)\colon Y \times X \times C(X, C(Y, Z)) \to Z$$

is continuous. Applying 5.8, with X replaced by $C(X, C(Y, Z))$, Y by $Y \times X$, and Z by Z, we see that

$$\mu(e'(1 \times e))\colon C(X, C(Y, Z)) \to C(Y \times X, Z)$$

is defined and continuous. It is readily verified that $\mu(e'(1 \times e))$ is the inverse of μ in 5.7.

We now apply the functor k to both sides of 5.7. On the right side we use 5.3 to obtain

$$kC(X, C(Y, Z)) = kC(X, kC(Y, Z)) = (Z^Y)^X .$$

On the left side we obtain $kC(Y \times X, Z) = Z^{Y \times X}$. This completes the proof of 5.6.

5.9. THEOREM. *For* X, Y, *and* Z *in* CG, *the composition of mappings* $X \to Y \to Z$ *is a continuous function* $Z^Y \times Y^X \to Z^X$.

Proof. By 5.2, the mappings

$$Z^Y \times Y^X \times X \xrightarrow{1 \times e} Z^Y \times Y \xrightarrow{e'} Z$$

are continuous, hence $e'(1 \times e)$ is also continuous. Applying 5.8 with X replaced by $Z^Y \times Y^X$, Y by X, Z by Z, and f by $e'(1 \times e)$, we see that

$$\mu(e'(1 \times e)): Z^Y \times Y^X \to C(X, Z)$$

is continuous. Then $k\mu(e'(1 \times e)): Z^Y \times Y^X \to Z^X$ is also continuous.

5.10. *Definition.* We denote by $C((X, A), (Y, B))$ the space of continuous mappings of pairs $(X, A) \to (Y, B)$. It is the subspace of $C(X, Y)$ of maps f such that $fA \subset B$. We abbreviate $kC((X, A), (Y, B))$ by $(Y, B)^{(X,A)}$. A *pointed space* is a pair (X, x_0), where x_0 is a point of X called the base point. We abbreviate (X, x_0) by X_0. A *mapping* $f: X_0 \to Y_0$ in the category of pointed spaces is a mapping of pairs $f: (X, x_0) \to (Y, y_0)$. The *smash* product $X_0 \wedge Y_0$ is obtained from $X \times Y$ by collapsing the wedge $(X \times y_0) \cup (x_0 \times Y)$ to a point that is the base point of $X_0 \wedge Y_0$. Define the function space of mappings of pointed spaces by

$$Y_0^{X_0} = kC((X, x_0), (Y, y_0)),$$

where its base point is the constant map $f_0 X = y_0$.

Our objective is to prove the analog of the exponential rule 5.6 in the category of pointed spaces; but we need a preliminary result. Let $X \in CG$, and let A be a closed subspace of X such that the space X/A obtained by collapsing A to a point a_0 (the base point) is a Hausdorff space. Let $h: (X, A) \to (X/A)_0$ denote the collapsing map. Let Y_0 be a pointed space in CG. By composing a map $f: (X/A)_0 \to Y_0$ with h, we obtain $fh \in C((X, A), (Y, y_0))$, and this defines a mapping of function spaces

$$h^*: C((X/A)_0, Y_0) \to C((X, A), (Y, y_0)).$$

5.11. LEMMA. *The above mapping* h^* *is continuous and one-to-one (bijective), and it sets up a one-to-one correspondence between compact subsets. Hence, applying the functor* k, *we obtain an induced natural equivalence*

$$Y_0^{(X/A)_0} = (Y, y_0)^{(X,A)}.$$

Proof. The continuity and bijective properties are readily proved. The crucial point is to show that if F is a compact subset of $C((X, A), (Y, y_0))$, then $h^{*-1}(F)$ is compact. It suffices to show that h^{*-1} is continuous on F. Suppose $g_0 \in F$ and $W(C, U)$ is a subbasic open set of $C((X/A)_0, Y_0)$ containing $h^{*-1}g_0$. This means that $g_0 h$ maps C into U. In case C does not contain the base point a_0, then $h^{-1}C$ is compact in X, and $W(h^{-1}C, U)$ is an open set that contains g_0, and is mapped into $W(C, U)$ by h^{*-1}.

Suppose therefore that C contains a_0. Since F is compact, the evaluation mapping $e: F \times X \to Y$ is continuous, by 5.2. Since $e(F \times A) = y_0$ and $F \times (X/A)$ is the decomposition space of $F \times X$ obtained by collapsing $F \times A$ to $F \times a_0$ (see 4.4), it follows that e induces a continuous mapping $e': F \times (X/A) \to Y$. Since $e'(g_0, a_0) \in U$, there exist a neighborhood V of g_0 in F and a neighborhood N of a_0 in X/A such that e' maps $V \times N$ into U. Set $C' = C - C \cap N$; then C' is compact and does not contain a_0. It follows that $V \cap W(h^{-1}C', U)$ is a neighborhood of g_0 in F, and any g in this neighborhood will map $h^{-1}N$ into U because $g \in V$, and it will map $h^{-1}C'$ into U because $g \in W(h^{-1}C', U)$. Since $C \subset C' \cup N$, it follows that $h^{*-1}g \in W(C, U)$. This completes the proof of the lemma.

5.12. THEOREM. *If* X_0, Y_0, Z_0 *are pointed spaces in* CG (see 5.10), *then* $Z_0^{Y_0 \wedge X_0} = (Z_0^{Y_0})^{X_0}$.

Proof. Abbreviate the wedge $(Y \times x_0) \cup (y_0 \times X)$ by W. If in 5.11 we replace Y by Z and (X, A) by $(Y \times X, W)$, we obtain the natural equivalence

$$Z_0^{Y_0 \wedge X_0} = (Z, z_0)^{(Y \times X, W)}. \tag{5.13}$$

The space on the right of 5.13 is a subspace of $Z^{Y \times X}$ which, by 5.6, is equivalent to $(Z^Y)^X$. It is readily verified that, under the latter equivalence, $(Z, z_0)^{(Y \times X, W)}$ corresponds exactly to $(Z_0^{Y_0})^{X_0}$. This completes the proof.

Remark. In the work of Fuks [5], [6] on duality in homotopy theory, the natural equivalence of 5.12 plays a central role: the functors $Y_0^{X_0}$ and $Y_0 \wedge X_0$ for a fixed X_0 and variable Y_0 are adjoint.

6. NEIGHBORHOOD DEFORMATION RETRACTS AND THEIR PRODUCTS

Our objective is to define a category of pairs (X, A) (where X is in CG and A is closed in X) having certain useful properties. Most important is that each (X, A) should have the homotopy extension property; this restricts severely the possible subsets A of X one may take. Also, we require that the category be closed under the operations of forming products and adjunction spaces.

It is not enough to require that A be a neighborhood retract in X, because neighborhood retracts do not behave well under products, as we now show.

6.1. *Example.* Let X be the unit interval $[0, 1]$ and A the point 1. Let Y be the transfinite line $[0, \Omega]$, where Ω is the first uncountable ordinal, and let $B = \Omega$. It is clear that A is a retract of X, and B of Y. We claim that in $X \times Y$ the subset $(X \times B) \cup (A \times Y)$ is not even a neighborhood retract. For suppose $f: U \to X \times B \cup A \times Y$ were a neighborhood retraction. Let

$$V = f^{-1}((X - A) \times B) \quad \text{and} \quad W = f^{-1}(A \times (Y - B)).$$

Then, in the space $X \times Y$ with the point $A \times B$ deleted, the sets V and W are disjoint open sets separating the closed subsets $(X - A) \times B$ and $A \times (Y - B)$. But this is one of the standard examples of a nonnormal space and two closed subsets that cannot be separated (see for example the book of D. W. Hall and G. L. Spencer, *Elementary Topology*, p. 291, Ex. 3.5).

6.2. *Definition.* A closed subspace A of a space X in CG is called a *neighborhood deformation retract* in X (briefly, an NDR in X) if there exist a mapping

$u: X \to I$ ($I = [0, 1]$) and a homotopy $h: I \times X \to X$ such that $A = u^{-1}(0)$, and $h(0, x) = x$ for all $x \in X$, $h(t, x) = x$ for $(t, x) \in I \times A$, and $h(1, x) \in A$ for all x such that $ux < 1$. The pair (X, A) is called an NDR pair. If, in addition, h may be chosen so that $h(1 \times X) \subset A$, we say that A is a *deformation retract* (DR) of X and (X, A) is a DR pair.

The existence of u such that $A = u^{-1}(0)$ implies that A is a closed G_δ-set in X; thus, in Example 6.1, (Y, B) is not an NDR. If U is the open set where $ux < 1$, then the mapping $g: U \to A$ defined by $gx = h(1, x)$ retracts U into A, and hence A is an NR in X.

If $\emptyset$ denotes the empty set, the pair $(X, \emptyset)$ is an NDR for any X, because we may set $ux = 1$ and $h(t, x) = x$ for all $x \in X$, $t \in I$. Also, (X, X) is a DR, hence an NDR, for any X, because we may set $ux = 0$ and $h(t, x) = x$ for all x and t.

6.3. THEOREM. *If* (X, A) *and* (Y, B) *are* NDR *pairs, then so is their product*

$$(X, A) \times (Y, B) = (X \times Y, X \times B \cup A \times Y).$$

If one is a DR *and the other an* NDR, *then their product is a* DR.

Proof. Let $u: X \to I$ and $h: I \times X \to X$ represent A as an NDR in X, and let $v: Y \to I$ and $j: I \times Y \to Y$ represent B as an NDR in Y. Define $w: X \times Y \to I$ by $w(x, y) = (ux)(vy)$. Clearly, $w^{-1}(0)$ is $X \times B \cup A \times Y$. Define the homotopy $q: I \times X \times Y \to X \times Y$ by

$$q(t, x, y) = \begin{cases} (x, y) & \text{if } x \in A \text{ and } y \in B, \\ \left(h(t, x), j\left(\frac{ux}{vy} t, y\right)\right) & \text{if } vy \geq ux \text{ and } vy > 0, \\ \left(h\left(\frac{vy}{ux} t, x\right), j(t, y)\right) & \text{if } ux \geq vy \text{ and } ux > 0. \end{cases}$$

The domains of definition of the last two lines intersect in the relatively closed set where $vy = ux > 0$, and both definitions reduce to $(h(t, x), j(t, y))$; thus they define a continuous function on $I \times (X \times Y - A \times B)$. Hence the proof of continuity of q reduces to proving continuity at a point (t, x, y) in $I \times A \times B$. Let U, V be open sets containing x, y in X, Y, respectively. Since $x \in A$, we have the inclusion relation $I \times \{x\} \subset h^{-1} U$. Since I is compact and $h^{-1} U$ is open, there is an open set S of X containing x such that $I \times S \subset h^{-1} U$. Similarly, there is an open set T of Y containing y such that $I \times T \subset j^{-1} V$. It follows that q maps $I \times S \times T$ into $U \times V$. This shows that q is continuous with respect to the standard topology of the products. By 3.2 vi, q remains continuous after the functor k is applied.

When $t = 0$, all three lines defining q reduce to (x, y). When $x \in A$, we have $ux = 0$, so that $q(t, x, y)$ is given by line 1 or 2, and line 2 reduces to $(h(t, x), j(0, y)) = (x, y)$. Hence $q(t, x, y) = (x, y)$ whenever $x \in A$. Similarly, $q(t, x, y) = (x, y)$ whenever $y \in B$.

Now let $t = 1$, and suppose (x, y) is such that $0 < w(x, y) < 1$. There are two similar cases, according as $0 < ux < 1$ or $0 < vy < 1$. Consider the first. In the subcase $ux \leq vy$, $q(1, x, y)$ is given by line 2, and since $h(1, x) \in A$, $q(1, x, y)$ is in $A \times Y$. In the subcase $vy < ux$, we must use line 3, and since $j(1, y) \in B$, it follows that $q(1, x, y) \in X \times B$. This shows that w and q represent the product pair as an NDR.

Suppose now that u, h represent (X, A) as a DR. Replace u by $u' = \frac{1}{2}u$; then u', h also represent (X, A) as a DR. Make the above constructions with u' in place of u. It follows that $w(x, y) < 1$ for all (x, y), hence $q(1, x, y) \in X \times B \cup A \times Y$. Thus the product pair is a DR, and the theorem is proved.

7. THE HOMOTOPY EXTENSION PROPERTY

The following is a modification of a theorem of Dowker [4]; his assumption that X is normal is replaced by our conditions on $u: X \to I$ (see also Young [15] and Puppe [10]).

7.1. THEOREM. *If* $X \in CG$ *and* A *is closed in* X, *then the following properties are equivalent:*

(i) (X, A) *is an* NDR,

(ii) $0 \times X \cup I \times A$ *is a* DR *of* $I \times X$,

(iii) $0 \times X \cup I \times A$ *is a retract of* $I \times X$,

(iv) (X, A) *has the homotopy extension property (in other words,* $A \to X$ *is a cofibration).*

Proof. If we assume (i) and note that (I, 0) is a DR, then 6.3 asserts that $(I, 0) \times (X, A)$ is a DR; therefore (i) implies (ii). It is trivial that (ii) implies (iii). The equivalence of (iii) and (iv) is well known. Therefore it remains to prove that (iii) implies (i).

Let r be a retraction of $I \times X$ into $0 \times X \cup I \times A$, and let $p: I \times X \to X$ be the projection into the second coordinate. Let $h: I \times X \to X$ be the composition $h(t, x) = pr(t, x)$. Then

$$h(0, x) = pr(0, x) = p(0, x) = x \qquad \text{for all } x.$$

If $x \in A$, then $h(t, x) = pr(t, x) = p(t, x) = x$. Thus it remains to construct $u: X \to I$ and to verify the last condition on h. Let $w: I \times X \to I$ denote projection into the first factor. For each $x \in X$ and for $m = 0, 1, 2, \cdots$, set

$$v_m x = \mathrm{Min}(1/2^m, wr(1/2^m, x)),$$

so that v_m is a mapping $X \to [0, 1/2^m]$. Define u by

$$ux = 1 - \sum_{m=1}^{\infty} (v_0 x)(v_m x) = \sum_{m=1}^{\infty} \left(\frac{1}{2^m} - (v_0 x)(v_m x)\right) \qquad \text{for all } x \in X.$$

Since the product function $v_0 v_m$ also maps X into $[0, 1/2^m]$, the series converges, and u is a mapping $X \to I$. For an $x \in A$, we have

$$wr(1/2^m, x) = w(1/2^m, x) = 1/2^m,$$

hence $v_m x = 1/2^m$, so that $ux = 0$. If x is not in A, there is a neighborhood V of (0, x) in $I \times X$ such that $rV \subset 0 \times (X - A)$, and there is an m such that $(1/2^m, x)$ is in V, whence $v_m x = 0$. By the second formula for ux above, this implies $ux > 0$. Thus $u^{-1}(0) = A$. If x is such that $ux < 1$, there is some $m \geq 1$ such that

$(v_0 x)(v_m x) > 0$, hence $v_0 x > 0$; this means $wr(1, x) > 0$, implying $r(1, x) \in I \times A$, and finally $h(1, x) \in A$. This completes the proof.

7.2. LEMMA. *If* $A \subset B \subset X$, *and* (B, A) *and* (X, B) *are* NDR's, *then* (X, A) *is an* NDR.

Proof. By 7.1, we have retractions

$$f: I \times X \to 0 \times X \cup I \times B \quad \text{and} \quad g: I \times B \to 0 \times B \cup I \times A.$$

Extend g to $g': 0 \times X \cup I \times B \to 0 \times X \cup I \times A$ by setting $g'(0, x) = (0, x)$ for $x \in X - B$. Then g' is a retraction, and also $g'f: I \times X \to 0 \times X \cup I \times A$; hence, by 7.1, (X, A) is an NDR.

7.3. LEMMA. *If* (X, A) *and* (Y, B) *are* NDR *pairs, then so also are the nine nontrivial pairs formed from the five spaces*

$$X \times Y, \quad X \times B \cup A \times Y, \quad X \times B, \quad A \times Y, \quad A \times B.$$

Proof. Theorem 6.3 applies directly to the five pairs

$$(X, A) \times (Y, B), \quad (X, \emptyset) \times (Y, B), \quad (X, A) \times (Y, \emptyset), \quad (X, A) \times (B, \emptyset), \quad (A, \emptyset) \times (Y, B).$$

The preceding lemma applies now to $X \times Y \supset X \times B \supset A \times B$, and it shows that $(X \times Y, A \times B)$ is an NDR. Choose a representation u, h of $(X \times B, A \times B)$ as an NDR. Extend the representation over $A \times (Y - B)$ by letting u be zero and h constant, there. These extensions represent $(X \times B \cup A \times Y, A \times Y)$ as an NDR. Symmetrically, $(X \times B \cup A \times Y, X \times B)$ is an NDR. Finally apply the preceding lemma to

$$X \times B \cup A \times Y \supset A \times Y \supset A \times B$$

to conclude that $(X \times B \cup A \times Y, A \times B)$ is an NDR.

8. FURTHER PROPERTIES OF NDR's

A space in CG need not be normal; the space in the Example 6.1 is nonnormal and locally compact, hence belongs to CG. We observe now that this deficiency is not important if we confine ourselves to subsets that are NDR's.

8.1. LEMMA. *Let* A, B *be closed disjoint subsets of* X *that are* NDR's *in* X; *then* A, B *form a normal pair of sets in* X; *that is, there exist open sets* U, V *in* X *such that* $A \subset U$, $B \subset V$, *and* $U \cap V = \emptyset$.

Proof. The NDR-hypothesis gives us functions $u, v: X \to I$ such that $u^{-1}(0) = A$ and $v^{-1}(0) = B$. Let U (V) denote the set of $x \in X$ such that $ux < vx$ $(vx < ux)$. It is readily verified that these are the required open sets.

8.2. LEMMA. *Let* Y *be the decomposition space of a space* X *in* CG *by an upper-semicontinuous collection of closed disjoint sets. If the sets of the collection other than points are* NDR's *in* X, *then* Y *is a Hausdorff space.*

Proof. Let $f: X \to Y$ be the natural map, let a, b be distinct points of Y, and let $A = f^{-1}a$, $B = f^{-1}b$. If A, B are points, we use the Hausdorff property of X to obtain open sets $U \supset A$ and $V \supset B$ such that $U \cap V = \emptyset$. Let U' (V') denote the union of all elements of the collection contained in U (V). By upper-semicontinuity, U' and V' are open sets. Since $f^{-1}fU' = U'$ and Y has the decomposition topology, fU'

is open in Y. Similarly fV' is open in Y. These are the required open sets separating a and b.

Suppose now that A is a point and B is not, so that B is an NDR in X. Let $v\colon X \to I$ be such that $v^{-1}(0) = B$. Then $v(A) > 0$. Let

$$U = \{x \mid vx > (vA)/2\} \quad \text{and} \quad V = \{x \mid vx < (vA)/2\}.$$

Construct U', V' as in the preceding paragraph; then fU', fV' are the required open sets.

If neither A nor B is a point, then by 8.1 we have open sets $U \supset A$ and $V \supset B$ such that $U \cap V = \emptyset$. Construct U', V' as above; then fU', fV' are the required open sets.

8.3. *Definition.* A mapping of pairs $f\colon (X, A) \to (Y, B)$ is called a *relative homeomorphism* if, as a map $X \to Y$, f is proclusive (see 2.5), and $f \mid (X - A)$ is a homeomorphism with $Y - B$.

8.4. LEMMA. *If* $f\colon (X, A) \to (Y, B)$ *is a relative homeomorphism, and* (X, A) *is an* NDR *pair, then* (Y, B) *is an* NDR *pair, and every representation* u, h *of* (X, A) *as an* NDR *induces a representation* v, j *of* (Y, B) *as an* NDR *so that the diagrams*

$$\begin{tikzcd} X \arrow[dr, "u"] \arrow[dd, "f"'] & \\ & I \\ Y \arrow[ur, "v"'] & \end{tikzcd} \qquad\qquad \begin{tikzcd} I \times X \arrow[r, "h"] \arrow[d, "1\times f"] & X \arrow[d, "f"] \\ I \times Y \arrow[r, "j"] & Y \end{tikzcd}$$

are commutative.

Proof. As required by commutativity, we define $vy = u(f^{-1}y)$ and $j(t, y) = fh(t, f^{-1}y)$. When $y \in Y - B$, then $f^{-1}y$ is single-valued, hence vy and $j(t, y)$ are also single-valued. When $y \in B$, choose an $x \in f^{-1}y \subset A$; then $vy = ux = 0$ and $j(t, y) = fh(t, x) = fx = y$. Thus v and j are uniquely defined, $v^{-1}(0) = B$, and j leaves fixed the points of B. If y is such that $vy < 1$, and $fx = y$, then $ux < 1$, hence $j(1, y) = fh(1, x) \in fA = B$. It remains to verify continuity. Suppose V is open in Y. Since $1 \times f$ is proclusive (see 4.4), $j^{-1}V$ will be open if $(1 \times f)^{-1} j^{-1} V$ is open. But this set coincides with $f^{-1} h^{-1} V$, which is open because f and h are continuous. Similarly, since f and u are continuous, $u = fv$, and f is proclusive, it follows that v is continuous.

8.5. LEMMA. *Let* (X, A) *be an* NDR *pair, and* $h\colon A \to Y$ *a mapping in* CG; *then the space obtained by adjoining* X *to* Y *by the mapping* h *gives an* NDR *pair* $(Y \cup_h X, Y)$.

Proof. Form the disjoint union of X and Y, obtaining the pair $(Y \cup X, Y \cup A)$. Let u, h represent (X, A) as an NDR pair. Extend u over Y by setting $u(Y) = 0$, and extend h over $I \times Y$ by setting $h(t, y) = y$ for all (t, y). It is trivial to verify that $(Y \cup X, Y \cup A)$ is an NDR pair. We obtain the adjunction space from $Y \cup X$ by collapsing to a point each set consisting of a $y \in Y$ and $h^{-1} y \subset A$, and by giving $Y \cup_h X$ the decomposition space topology. Therefore the natural map $f\colon (Y \cup X, Y \cup A) \to (Y \cup_h X, Y)$ is a relative homeomorphism. It is readily seen that $Y \cup_h X$ is a Hausdorff space. The conclusion follows now from 8.4.

9. UNIONS OF EXPANDING SEQUENCES OF SPACES

9.1. *Definition.* An *expanding sequence of spaces* $\{X_n\}$ consists of the indicated sequence of spaces for $n = 0, 1, 2, \cdots$, together with an inclusion map (or imbedding) for each n of X_n in X_{n+1} as a closed subspace. The *union* $X = \bigcup_{n=0}^{\infty} X_n$ is the indicated set with the topology (called the *topology of the union* or *weak topology*) defined as follows: a subset A of X is *closed* if $A \cap X_n$ is closed in X_n for every n.

It is readily verified that each X_n is closed in X, hence also each closed subset of X_n; thus X_n is imbedded in X as a subspace with the relative topology.

9.2. LEMMA. *If each* X_n *is in* CG, *and* X *is a Hausdorff space, then* X *is in* CG.

Proof. Suppose $A \subset X$ meets each compact subset of X in a closed set. For any n, let C be a compact set in X_n. Since X_n is imbedded in X, C is also compact in X. Therefore $A \cap C = (A \cap X_n) \cap C$ is closed in X, hence in X_n. Since X_n is in CG, it follows that $A \cap X_n$ is closed in X_n. Since this holds for each n, A is closed in X. This proves that X is in CG.

9.3. LEMMA. *If* C *is a compact subset of* X, *then there is an integer* n *such that* $C \subset X_n$.

Proof. Let A be a subset of X that is not contained in any X_n. It suffices to prove that A is not compact. By assumption, we may choose a definite point $x_n \in A \cap (X - X_n)$. Let $T_m = \{x_n | \, n \geq m\}$. Then $T_m \supset T_{m+1}$ for each m, and $\bigcap_m T_m = \emptyset$. Now T_m meets every X_n in a finite set, and this is closed because X_n is a Hausdorff space (the T_1-axiom is enough). Since X has the topology of the union, T_m is closed in X; hence $X - T_m$ is open. Thus the $\{X - T_m\}$ form an expanding sequence of open sets whose union is X. They cover A, but no finite number of them can cover A, because any finite collection is contained in the largest, and no single $X - T_m$ covers A. Hence A is not compact, and the lemma is proved.

9.4. THEOREM. *If each* (X_{n+1}, X_n) *is an* NDR *pair, then* X *is a Hausdorff space, and each* (X, X_n) *is an* NDR *pair.*

Proof. Given two distinct points x, y of X, we choose first an integer m such that $x, y \in X_n$ for $n \geq m$. Next we shall construct a sequence $\{U_n, V_n\}$ of sets for $n \geq m$ such that U_n, V_n are open in X_n, $x \in U_n$, $y \in V_n$, $U_n \cap V_n = \emptyset$, and

$$U_{n+1} \cap X_n = U_n, \qquad V_{n+1} \cap X_n = V_n.$$

Using the Hausdorff property of X_m, we select U_m, V_m as required. Suppose the sequence has been constructed for $m \leq n \leq p$. Since (X_{p+1}, X_p) is an NDR, there is a retraction g of a neighborhood W of X_p in X_{p+1} into X_p. Set $U_{p+1} = g^{-1} U_p$ and $V_{p+1} = g^{-1} V_p$. Since W is open in X_{p+1}, these sets are open and separate x and y, as required. Now let $U = \bigcup_{n=m}^{\infty} U_n$ and $V = \bigcup_{n=m}^{\infty} V_n$. These sets are open, because they meet each X_n in open sets of X_n. It is clear that they separate x and y as required, hence X is a Hausdorff space, and, by 9.2, X is in CG.

By 7.1, the NDR hypothesis provides a retraction

$$r_m\colon I \times X_{m+1} \to (0 \times X_{m+1}) \cup (I \times X_m) \qquad \text{for each } m \geq 0.$$

Extend r_m to a retraction

$$s_m\colon (0 \times X) \cup (I \times X_{m+1}) \to (0 \times X) \cup (I \times X_m)$$

by setting $s_m(0, x) = (0, x)$ for $x \in X$. Define

$$s\colon I \times X \to (0 \times X) \cup (I \times X_n)$$

to be the composition $s_n s_{n+1} \cdots s_{m-2} s_{m-1}$ when restricted to $(0 \times X) \cup (I \times X_m)$ $(m > n)$. It is trivial to verify that, as a function, s is a retraction. To prove that s is continuous, let A be a closed set of $(0 \times X) \cup (I \times X_n)$. For each $m > n$, $s^{-1}A$ meets $(0 \times X) \cup (I \times X_m)$ in a closed set, because the composition $s_n s_{n+1} \cdots s_{m-1}$ is continuous. Let C be a compact set in $I \times X$. Its projection C' in X is compact, hence, by 9.3, $C' \subset X_m$ for some m, and therefore $C \subset I \times X_m$. Since $s^{-1}A$ meets $I \times X_m$ in a closed set, it meets C in a closed set. Inasmuch as this holds for each compact set and $I \times X$ is in CG (see 4.3), it follows that $s^{-1}A$ is closed, hence s is continuous. Now 7.1 tells us that (X, X_n) is an NDR.

9.5. THEOREM. *Let* $f\colon X \to Y$ *be a proclusion* (see 2.5), *and let* X *be the union of an expanding sequence* $\{X_n\}$ *such that, for each* n, $f^{-1}fX_n = X_n$. *Then* $\{fX_n\}$ *is an expanding sequence of closed subspaces of* Y, *and* Y *has the topology of their union.*

Proof. Since f is proclusive and $f^{-1}fX_n = X_n$ is closed in X, it follows that fX_n is closed in Y. Suppose $B \subset Y$ meets each fX_n in a closed set. Since

$$(f^{-1}B) \cap X_n = f^{-1}B \cap f^{-1}fX_n = f^{-1}(B \cap fX_n),$$

f is proclusive, and $B \cap fX_n$ is closed, it follows that $f^{-1}B \cap X_n$ is closed for every n. Since X has the topology of the union, $f^{-1}B$ is closed in X. Since f is proclusive, it follows that B is closed in Y. This shows that Y has the topology of the union.

10. FILTERED SPACES

10.1. *Definition.* A *filtered space* X consists of a space X in CG and a sequence of closed subspaces $X_0 \subset X_1 \subset \cdots \subset X_n \subset \cdots$ such that $X = \bigcup_0^\infty X_n$ and X has the topology of the union (see 9.1). If X and Y are filtered spaces, a *mapping* $f\colon X \to Y$ is a mapping of spaces such that $fX_n \subset Y_n$ for each $n \geq 0$.

Clearly, the subspaces $\{X_n\}$ form an expanding sequence, as in 9.1, and X coincides with their union. By 9.4, any expanding sequence of NDR's has a union that is a filtered space in the above sense.

10.2. *Definition.* If $X = \{X_n\}$ and $Y = \{Y_n\}$ are filtered spaces, their *product* is the space $X \times Y$ filtered by

$$(X \times Y)_n = \bigcup_{i=0}^{n} X_i \times Y_{n-i} \qquad (n = 0, 1, 2, \cdots).$$

10.3. THEOREM. *The product space* $X \times Y$ *of filtered spaces has the topology of the union* $\{(X \times Y)_n\}$; *hence the product is a filtered space.*

Proof. We must prove that a set A in $X \times Y$ that meets each $(X \times Y)_n$ in a closed set is a closed set. Let C be any compact set of $X \times Y$. The projections

C', C'' of C in X and Y are compact. By 9.3, there are integers p and q such that $C' \subset X_p$ and $C'' \subset Y_q$; hence $C \subset (X \times Y)_{p+q}$. Since $A \cap (X \times Y)_{p+q}$ is closed, it follows that $A \cap C$ is closed. Because C is any compact set and $X \times Y$ is in CG, it follows that A is closed, and the theorem is proved.

Remark. The above definition of product is modelled on the product of two complexes filtered by skeletons. It is also the natural product to use in work with geometric resolutions over H-spaces (see [8]). It should be noted that this product is not a product in the categorical sense; for example, the diagonal mapping of X into $X \times X$ carries X_n not into $(X \times X)_n$, but into $(X \times X)_{2n}$.

10.4. *Definition.* A filtration of X is said to be *a filtration by* NDR's if X_n is an NDR in X_{n+1} for each n. By 9.4, each X_n is also an NDR in X.

10.5. THEOREM. *If* $X = \{X_n\}$ *and* $Y = \{Y_n\}$ *are filtered by* NDR's, *then* $X \times Y = \{(X \times Y)_n\}$ *is also a filtration by* NDR's.

Proof. By 6.3, $(X_i, X_{i-1}) \times (Y_{n-i}, Y_{n-i-1})$ is an NDR, hence by 7.1 there is a retraction

$$r_i\colon I \times X_i \times Y_{n-i} \to 0 \times X_i \times Y_{n-i} \cup I \times (X_i \times Y_{n-i-1} \cup X_{i-1} \times Y_{n-i})$$

for $i = 0, 1, \cdots, n$. (Interpret X_{-1} as $\emptyset$.) Now

$$I \times (X \times Y)_n - (0 \times (X \times Y)_n \cup I \times (X \times Y)_{n-1})$$

is the union of the disjoint open sets

$$(I - 0) \times (X_i - X_{i-1}) \times (Y_{n-i} - Y_{n-i-1}) \qquad (i = 0, 1, \cdots, n).$$

We define a retraction

$$r\colon I \times (X \times Y)_n \to 0 \times (X \times Y)_n \cup I \times (X \times Y)_{n-1}$$

by setting r equal to the identity on the subspace, and equal to r_i on the i^{th} open set for $i = 0, 1, \cdots, n$. Then r coincides with r_i on $I \times X_i \times Y_{n-i}$ for each i; since each r_i is continuous and $I \times (X \times Y)_n$ is the union of these closed sets, it follows that r is continuous. By 7.1, $(X \times Y)_{n-1}$ is an NDR in $(X \times Y)_n$, and the theorem is proved.

11. COMPARISONS WITH OTHER CATEGORIES

Spanier has developed the notion of a quasi-topological space, for the purpose of solving some of the difficulties outlined in our introduction [13]. A quasi-topology on a set X is a rule assigning to each compact Hausdorff space C a family of functions from C to X that satisfy four conditions. In case X is a topological space, all continuous maps $C \to X$ satisfy these conditions, hence X has an associated quasi-topology. This gives a functor $T \to QT$ from the category T of topological spaces and continuous maps to the category of quasi-topological spaces and quasi-continuous maps.

Simce QT is very large, it satisfies the first two of the three conditions for a convenient category. As to the five parts of the third condition, it seems likely that they also hold; in fact, Spanier's Propositions 3.2 and 4.1 are parts (3) and (2), respectively. In evaluating these propositions, we must keep in mind that the functor

$T \to QT$ is many-to-one; if the initial spaces are in T, and the constructions appearing on the two sides of an equality are carried out in T, then the conclusion is not that the resulting topological spaces are the same, but that they have the same associated quasi-topologies.

It would therefore appear that our results in the category CG are definitely sharper than Spanier's. This is not the case, however, in view of the fact that the functor $T \to QT$ restricted to $CG \to QT$ is injective and imbeds CG in QT as a full subcategory. (The proof of this is an easy exercise.) Thus our result 5.6 on the exponential law follows from Spanier's Theorem 4.1. However, our result 4.4 on a product of proclusions does not follow from Spanier's 3.2, because the definitions of a proclusion in CG and QT do not agree under the functor (see Spanier's Note on page 4).

Recently, Michael [8] has defined and studied a category of spaces he calls $\aleph_0$-spaces. It is a subcategory of the category of regular Lindelöf (paracompact) spaces. It includes separable metric spaces and countable CW-complexes. It neither includes CG nor is included in CG. Michael shows that it is closed under the formation of countable cartesian products and of function spaces with the CO-topology (see [8, Propositions 6.1, 9.3]). It is also closed under the formation of unions of expanding sequences, because any compact set of the union lies in some space of the sequence. As for decomposition spaces, Michael shows that a regular decomposition space of a separable metric space is in the category (see 2.1), and he gives a counterexample to a stronger statement.

Thus Michael's category satisfies rather well our first two criteria for a convenient category. But Michael gives no results to indicate that it satisfies any part of our third criterion. In fact, it seems to us that parts (3) and (4) cannot hold if product spaces are given the cartesian topology, and part (2) cannot hold if function spaces have the CO-topology.

In 1959, Milnor [9] called attention to the advantages of the category W (W_0) of spaces having the homotopy types of (countable) CW-complexes. Its chief advantage is that a mapping of the category that induces an isomorphism of homotopy groups is a homotopy equivalence. The disadvantages are, first, that it does not contain certain simple spaces (for example, Cantor sets), and, second, that it is difficult to show that W is closed under certain standard operations. A main part of Milnor's paper is devoted to showing that Y^X is in W when Y is in W and X is compact metric.

Milnor's W should not be regarded as a competitor of our category CG. Since the objective in using W is to have a semieffective theory of homotopy type, its disadvantages are acceptable and probably necessary. Those who use W would have no objection to restricting themselves to the intersection of W and CG. Indeed, this might be advantageous, since it is not clear that W has a product.

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