A CLASS OF WEIGHT FUNCTIONS THAT ADMIT TCHEBYCHEFF QUADRATURE

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By a weight function $W(x)$ we mean a real-valued nonnegative function on $[-1, 1]$ for which the proper or improper Riemann integral exists and has the value 1. If the system of equations

$$\frac{1}{n}\sum_{i=1}^{n} x_{i,n}^{k} = \int_{-1}^{1} x^k W(x)\,dx \qquad (k = 1, \cdots, n) \tag{1}$$

has real solutions for all positive integers n, we say that $W(x)$ *admits Tchebycheff quadrature.*

Hermite proved that the function $W(x) = 1/\pi\sqrt{1 - x^2}$ admits Techbycheff quadrature (see [1]). As far as the author knows, the literature lists no other examples.

THEOREM. *If* $-1/4 \leq a \leq 1/4$, *then the function*

$$W(x) = \frac{1}{\pi\sqrt{1 - x^2}}\,\frac{1 + 2ax}{1 + 4a^2 + 4ax} \tag{2}$$

is a weight function and admits Tchebycheff quadrature.

This theorem establishes the existence of an infinite one-parameter family of what could properly be called Tchebycheff weight functions. It thus becomes reasonable to pose the problem of characterizing all Tchebycheff weight functions.

Proof of the theorem. In Lemmas 1 and 2 we develop a method for investigating the solutions of equations (1). We then apply this method to the weight function (2), in Lemmas 3 and 4, to complete the proof.

LEMMA 1. *Let* $W(x)$ *be a weight function, and let*

$$m_k = \int_{-1}^{1} x^k W(x)\,dx \qquad (k = 0, 1, \cdots).$$

The function

$$f(z) = z \exp\left(-\sum_{k=1}^{\infty} \frac{m_k}{kz^k}\right) \qquad (|z| > 1) \tag{3}$$

has a simple pole at infinity, and $\lim_{z\to\infty} f(z)/z = 1$. *For each positive integer* n, *the terms with nonnegative powers of* z *in the Laurent expansion of* $(f(z))^n$ *about infinity form a monic polynomial* $F_n(z)$ *of degree* n.

To prove this lemma, we observe that $|m_k| \leq 1$ for all k, that the function defined by

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$$\sum_{k=1}^{\infty} \frac{m_k}{kz^k}$$

is zero at infinity, and that we can obtain the Laurent expansion about infinity of $(f(z))^n$ by formally raising the Laurent expansion of $f(z)$ to the nth power.

LEMMA 2. *Let* $W(z)$, $\{m_k\}$ $(k = 0, 1, \cdots)$, $f(z)$, *and* $F_n(z)$ *be defined as in Lemma* 1, *and let*

$$F_n(z) = \prod_{i=1}^{n} (z - z_{i,n}).$$

Then

$$\frac{1}{n} \sum_{i=1}^{n} z_{i,n}^k = m_k \qquad (k = 1, \cdots, n). \tag{4}$$

Proof. On the one hand,

$$\frac{F_n'(z)}{n F_n(z)} = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{z - z_{i,n}} = \frac{1}{n} \sum_{i=1}^{n} \sum_{k=0}^{\infty} \frac{z_{i,n}^k}{z^{k+1}} = \sum_{k=0}^{\infty} \frac{\frac{1}{n} \sum_{i=1}^{n} z_{i,n}^k}{z^{k+1}}. \tag{5}$$

On the other hand, there exists a function $G(z)$, analytic in $|z| > 1$ and vanishing at infinity, such that

$$F_n(z) = f^n(z) + G(z) = f^n(z)\left(1 + \frac{G(z)}{f^n(z)}\right) = f^n(z)(1 + H_n(z)),$$

where $H_n(z)$ is analytic in some neighborhood of infinity and has a zero of multiplicity at least $n + 1$ at infinity. It follows that

$$\frac{F_n'(z)}{n F_n(z)} = \frac{f'(z)}{f(z)} + \frac{H_n'(z)}{n(1 + H_n(z))} = \frac{f'(z)}{f(z)} + \sum_{k=n+1}^{\infty} \frac{c_{n,k}}{z^{k+1}}.$$

By (3),

$$\frac{f'(z)}{f(z)} = \frac{1}{z} + \sum_{1}^{\infty} \frac{m_k}{z^{k+1}} = \sum_{0}^{\infty} \frac{m_k}{z^{k+1}},$$

and therefore

$$\frac{F_n'(z)}{n F_n(z)} = \sum_{k=0}^{n} \frac{m_k}{z^{k+1}} + \sum_{k=n+1}^{\infty} \frac{m_k + c_{n,k}}{z^{k+1}}. \tag{6}$$

The lemma now follows if we compare the coefficients in (5) and (6).

We denote by E_1 the z-plane with the interval $[-1, 1]$ deleted, and by $\sqrt{z^2 - 1}$ the branch of the function in E_1 that is positive on the ray $z > 1$.

LEMMA 3. *For* $-1/2 < a < 1/2$,

$$\frac{z+\sqrt{z^2-1}}{2}+a = z\exp\left(-\sum_{k=1}^{\infty}\frac{m_k}{kz^k}\right), \tag{7}$$

where the function (2) *is used in the definition of* m_k.

Proof. The function $z - \sqrt{z^2-1}$ is analytic in E_1 and tends to 1 in modulus, as z tends to any point of $[-1, 1]$ from E_1. Since the function is regular at infinity, the maximum modulus principle yields the inequality $|z - \sqrt{z^2-1}| < 1$ for $z \in E_1$. Since

$$|(z+\sqrt{z^2-1})(z-\sqrt{z^2-1})| = 1,$$

it follows that $|z+\sqrt{z^2-1}| > 1$ for $z \in E_1$. For $-1/2 < a < 1/2$, the function

$$f(z) = \frac{z+\sqrt{z^2-1}}{2} + a \tag{8}$$

has no zero in E_1, and therefore

$$\frac{f'(z)}{f(z)} = \frac{1}{\sqrt{z^2-1}}\,\frac{z+\sqrt{z^2-1}}{z+\sqrt{z^2-1}+2a} \tag{9}$$

is analytic in E_1 and has a simple zero at infinity. If C is a simple closed curve containing $[-1, 1]$ and if z is exterior to C, then by Cauchy's formula

$$\frac{f'(z)}{f(z)} = \frac{1}{2\pi i}\oint \frac{f'(\zeta)}{f(\zeta)}\,\frac{d\zeta}{\zeta - z}, \tag{10}$$

because the integrand has residue zero at infinity. For x in the interval $[-1, 1]$, we define $(\sqrt{x^2-1})^+$, $(f(x))^+$, and $(f'(x))^+$ as the limiting values of $\sqrt{z^2-1}$, $f(z)$, and $f'(z)$, respectively, as z approaches x through values with positive imaginary parts. In an analogous manner, we define $(\sqrt{x^2-1})^-$, $(f(x))^-$, $(f'(x))^-$.

We deform the path of integration and write (10) in the form

$$\begin{aligned}\frac{f'(z)}{f(z)} = \frac{1}{2\pi i}\Bigg[&\int_{-1+\varepsilon}^{1-\varepsilon}\left(\frac{f'(x)}{f(x)}\right)^{+}\frac{dx}{x-z} + \oint_{|\zeta-1|=\varepsilon}\frac{f'(\zeta)}{f(\zeta)}\,\frac{d\zeta}{\zeta-z} \\ &+ \int_{1-\varepsilon}^{-1+\varepsilon}\left(\frac{f'(x)}{f(x)}\right)^{-}\frac{dx}{x-z} + \oint_{|\zeta+1|=\varepsilon}\frac{f'(\zeta)}{f(\zeta)}\,\frac{d\zeta}{\zeta-z}\Bigg],\end{aligned} \tag{11}$$

where $0 < \varepsilon < 1$. It is readily verified that as ε tends to zero, the second and fourth integrals on the right-hand side of (11) tend to zero.

Since $(\sqrt{x^2-1})^+ = i\sqrt{1-x^2}$ and $(\sqrt{x^2-1})^- = -i\sqrt{1-x^2}$, we see from (9) that $\left(\frac{f'(x)}{f(x)}\right)^+$ is the complex conjugate of $\left(\frac{f'(x)}{f(x)}\right)^-$, and the first and third integrals combine to yield in the limit

$$\frac{1}{2\pi i}\int_{-1}^{1}\left[\left(\frac{f'(x)}{f(x)}\right)^{+}-\left(\frac{f'(x)}{f(x)}\right)^{-}\right]\frac{dx}{x-z}=\frac{1}{\pi}\int_{-1}^{1}\Im\left(\frac{f'(x)}{f(x)}\right)^{+}\frac{dx}{x-z}. \tag{12}$$

Since

$$\left(\frac{f'(x)}{f(x)}\right)^{+}=\frac{1}{i\sqrt{1-x^2}}\,\frac{x+i\sqrt{1-x^2}}{x+i\sqrt{1-x^2}-2a}$$

a computation shows that

$$\Im\left(\frac{f'(x)}{f(x)}\right)^{+}=-\frac{1}{\sqrt{1-x^2}}\,\frac{1+2ax}{1+4a^2+4ax}.$$

Thus from (11) and (12) we finally arrive at the formula

$$\frac{f'(z)}{f(z)}=\int_{-1}^{1}\frac{W(x)}{z-x}\,dx, \tag{13}$$

where $W(x)$ is given by (2).

Recall that z was originally chosen in the exterior of the curve C. Since the integral in (13) is analytic for all z in E_1, it represents the analytic function $\frac{f'(z)}{f(z)}$ throughout E_1.

We find that for $|z|>1$, $\frac{f'(z)}{f(z)}=\sum_0^\infty \frac{m_k}{z^{k+1}}$, where

$$m_k=\int_{-1}^{1}x^kW(x)\,dx \qquad (k=0,1,\cdots). \tag{14}$$

If we let $g(z)=f(z)/z$ for $|z|>1$, then $g(\infty)=1$ and $g(z)$ does not vanish for $|z|>1$; therefore there is a branch of $\log g(z)$, defined in the domain $|z|>1$, such that $\log g(\infty)=0$. Both the function $\log g(z)$ and the function defined by the series

$$-\sum_{k=1}^{\infty}\frac{m_k}{kz^k}$$

have the same derivative, namely $\sum_{k=1}^\infty \frac{m_k}{z^{k+1}}$, so that

$$\log g(z)=-\sum_{k=1}^{\infty}\frac{m_k}{kz^k}$$

for $|z|>1$, the constant of integration being determined by comparing values at infinity. Thus

$$\frac{f(z)}{z} = \exp\left(-\sum_{k=1}^{\infty}\frac{m_k}{kz^k}\right) \qquad (|z| > 1).$$

By the definition of $f(z)$ and m_k (see (8), (14), and (2)), the relation (7) is now established; this completes the proof of the lemma.

LEMMA 4. *If* $-1/4 \le a \le 1/4$, $W(x)$ *is defined by* (2), *and*

$$m_k = \int_{-1}^{1} x^k W(x)\,dx \qquad (k = 0, 1, \cdots),$$

then the system of equations

$$\frac{1}{n}\sum_{i=1}^{n} x_{i,n}^k = m_k \qquad (k = 1, 2, \cdots, n) \tag{15}$$

has real solutions for all positive integers n.

Proof. We first note that for the range of a in the lemma, $W(x)$ is nonnegative and admits an improper Riemann integral. From (9), (13), and (14) it can be deduced that $m_0 = 1$, so that $W(x)$ is a weight function. Thus the proof of this lemma will complete the proof of the theorem.

To find solutions of (15) for a fixed value of n, we proceed according to Lemmas 1 and 2 and form the expression

$$z \exp\left(-\sum_{k=1}^{\infty}\frac{m_k}{kz^k}\right),$$

which defines an analytic function $f(z)$ for $|z| > 1$. Our next step is to find the polynomial part of the Laurent expansion of $(f(z))^n$ about infinity, and to investigate its zeros. In the particular case under investigation, we know by Lemma 3 that for $|z| > 1$,

$$f(z) = z \exp\left(-\sum_{1}^{\infty}\frac{m_k}{kz^k}\right) = \frac{z + \sqrt{z^2 - 1}}{2} + a,$$

and we shall use this to show that the expression

$$T_n^{(a)}(z) = \left(\frac{z + \sqrt{z^2 - 1}}{2} + a\right)^n + \left(\frac{z - \sqrt{z^2 - 1}}{2} + a\right)^n - a^n \tag{16}$$

is the polynomial we seek. To begin with, $T_n^{(a)}(z)$ is analytic in E_1. If we regard each of the first two terms on the right-hand side of (16) as a sum of three terms raised to the nth power, and expand it by the multinomial expansion, we observe that whenever $(\sqrt{z^2 - 1})^k$ appears in a term in the first expansion, then $(-1)^k(\sqrt{z^2 - 1})^k$ appears at the corresponding place in the second expansion. Thus $T_n^{(a)}(z)$ is a polynomial in E_1, and since it is continuous in the plane, it is a polynomial.

We are interested in the polynomial part of the Laurent expansion about infinity of the first term in the right-hand side of (16). The Laurent expansion about infinity of the second term begins with a^n and is followed by negative powers of z, since the function is regular at infinity and has the value a^n at infinity. The third term cancels the a^n. Thus the polynomial part of the Laurent expansion about infinity of the first term and of the sum of the three terms is the same, and is equal to the left-hand side of (16).

To complete the proof we shall use Lemma 2 and show that the zeros of $T_n^{(a)}(z)$ are real for $-1/4 \le a \le 1/4$.

We first note that for $a = 0$, $T_n^{(a)}(z)$ is precisely the Tchebycheff polynomial of degree n, normalized to be monic. The zeros are known to be real. Thus the result of Hermite mentioned in the introduction is proved.

In the general case our procedure will be to show that $T_n^{(a)}(\cos\theta)$ vanishes for n distinct values of θ in the interval $[0, \pi]$. Since to each of these values there corresponds a different value of $\cos\theta$, it will follow that $T_n^{(a)}(z)$ has n distinct zeros, and since the polynomial is of degree n, that all its zeros are real.

From (16) we see that

$$T_n^{(a)}(\cos\theta) = \left(\frac{w}{2}+a\right)^n + \left(\frac{\bar{w}}{2}+a\right)^n - a^n, \tag{17}$$

where $w = \cos\theta + i\sin\theta$. Thus

$$T_n^{(a)}(\cos\theta) = 2\Re\left(\frac{w}{2}+a\right)^n - a^n = 2a^n(\Re(\lambda w+1)^n - 1/2), \tag{18}$$

where $\lambda = 1/2a$.

The case $a = 0$ has already been considered. Thus we must deal with the cases $\lambda \ge 2$ and $\lambda \le 2$. Suppose first that $\lambda \ge 2$. As θ goes from 0 to π, the point $\lambda w + 1$ describes the upper half of the circle with center 1 and radius λ. Since

$$|\lambda w + 1| \ge \lambda - 1 \ge 1,$$

this semicircle lies entirely in the domain $|z| \ge 1$, $\Im z > 0$, except for its endpoints. Let $z(\theta) = \lambda w + 1$. Then the principal argument $\operatorname{Arg} z(\theta)$ is a continuous function of θ on $[0, \pi]$, and $\operatorname{Arg} z(0) = 0$, $\operatorname{Arg} z(\pi) = \pi$. Thus there exist points $\{\theta_i\}$ $(i = 1, \cdots, n)$ in $[0, \pi]$ such that $0 = \theta_0 < \theta_1 < \cdots < \theta_n = \pi$ and such that

$$\operatorname{Arg} z(\theta_k) = k\pi/n \qquad (k = 0, 1, \cdots, n).$$

Since $|z(\theta)| \ge 1$ for θ in $[0, \pi]$, it follows that

$$(z(\theta_k))^n = \gamma_k(-1)^k \qquad (k = 0, 1, \cdots, n),$$

where γ_k is real, positive, and not less than one. Since $\Re z(\theta)$ is a continuous function, there are values $\{\theta_k^*\}$ $(k = 1, \cdots, n)$ such that

$$\theta_{k-1} < \theta_k^* < \theta_k \quad \text{and} \quad \Re z(\theta_k^*) = 1/2 \qquad (k = 1, \cdots, n).$$

From (18) we see, on recalling the definition of $z(\theta)$, that $T_n^{(a)}(\cos\theta_k^*) = 0$. Thus $T_n^{(a)}(z)$ has n real zeros $\{\cos\theta_k^*\}$ $(k = 1, \cdots, n)$, and by Lemma 2, these are solutions of equations (15). Entirely similar considerations hold for $-1/4 \leq a < 0$, which corresponds to the case $\lambda \leq -2$. Thus the proof of Lemma 4 is complete.

REFERENCE

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