NONHOMOGENEOUS DIFFERENTIAL OPERATORS

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1. INTRODUCTION

Let $P = (p_{ij})$ be an n-by-n matrix whose elements are real-valued and continuous on a finite interval $[a, b]$. Differential operators of the form $LY = Y' + PY$, where Y is an n-by-n matrix, were first seriously studied by Birkhoff and Langer [1], who considered a system consisting of the differential operator $LY = Y' + PY$ and a boundary condition of the form $U(Y) = AY(a) + BY(b) = 0$, where A and B are nonsingular n-by-n matrices.

W. M. Whyburn discussed systems of the form

$$LY = 0, \qquad AY(a) + BY(b) + \int_a^b F(x)\,Y(x)\,dx = 0,$$

where F is an integrable matrix. He defined an adjoint system whose existence depends on the existence of a solution to $Z' - ZP = F$ that is nonsingular over $[a, b]$ (see [9, pages 53-54]).

R. H. Cole [3] succeeded in defining an adjoint system whose existence depends only on A and B. This system is a slight generalization of the problem discussed by Whyburn. The adjoint, however, is no longer a differential system.

The present paper generalizes the system discussed by Whyburn, but in a different direction. We shall now show that if A, B, C, D are constant matrices and K_1 and K_2 are integrable matrices, then the existence of our adjoint to the system

$$MY = LY + K_2(x)[CY(a) + DY(b)] = 0,$$

$$H(Y) = AY(a) + BY(b) + \int_a^b K_1(x)\,Y(x)\,dx = 0$$

depends only on A, B, C, and D; moreover, if the adjoint system exists, it has the same form. We shall show that if the system is incompatible, then the nonhomogeneous system $MY = F$, $H(Y) = 0$ has a solution of the form

$$Y(x) = \int_a^b G(x, t)\,F(t)\,dt,$$

where $G(x, t)$ is a formal solution of the system $MY = 0$, $H(Y) = 0$, for $x \neq t$.

If both the system $MY = 0$, $H(Y) = 0$ and its adjoint are incompatible, the Green's function for the adjoint system is $-G(t, x)$. However, if $MY = 0$, $H(Y) = 0$ is incompatible, it is possible under certain conditions for the adjoint system to be compatible, a situation that does not occur for ordinary differential systems.

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Nonhomogeneous operators such as M have recently been found to be the adjoints in Hilbert space of differential operators under general boundary conditions (see [4], [5], [6], and [7]). Nersesjan [8] has discussed nonhomogeneous operators in the case where $n = 1$.

2. GREEN'S FORMULA

The key in finding adjoint systems is Green's formula. Since Green's formula for matrices does not seem to be in the literature in the form we need, we discuss it briefly.

Let $U_1(Y) = AY(a) + BY(b)$ and $U_2(Y) = CY(a) + DY(b)$ be two boundary operators, where A, B, C, and D are constant matrices. We wish to find boundary conditions $V_1(Z)$ and $V_2(Z)$ such that Green's formula can be written

$$\int_a^b [Z(LY) + (L^* Z)Y]\,dx = V_1(Z)U_1(Y) + V_2(Z)U_2(Y).$$

If we let $V_1(A) = Z(a)E + Z(b)F$, $V_2(Z) = Z(a)G + Z(b)H$, then insertion in Green's formula yields the following four equations for E, F, G, H.

$$EA + GC = -I, \quad FB + HD = I, \quad EB + GD = 0, \quad FA + HC = 0.$$

When these can be solved for E, F, G, H, Green's formula can be written in the desired form. We state sufficient conditions for the existence of a solution, formulate the result, and leave the proof to the reader.

THEOREM 2.1. *Let* $U_1(Y) = AY(a) + BY(b)$, $U_2(Y) = CY(a) + DY(b)$.

(i) *If* A^{-1} *and* $(CA^{-1}B - D)^{-1}$ *exist, let*

$$V_1(Z) = Z(a)\,[-A^{-1} + A^{-1}B(CA^{-1}B - D)^{-1}CA^{-1}] + Z(b)\,[(CA^{-1}B - D)^{-1}CA^{-1}],$$

$$V_2(Z) = Z(a)\,[-A^{-1}B(CA^{-1}B - D)^{-1}] + Z(b)\,[-(CA^{-1}B - D)^{-1}].$$

(ii) *If* B^{-1} *and* $(DB^{-1}A - C)^{-1}$ *exist, let*

$$V_1(Z) = Z(a)\,[-(DB^{-1}A - C)^{-1}DB^{-1}] + Z(b)\,[B^{-1} - B^{-1}A(DB^{-1}A - C)^{-1}DB^{-1}],$$

$$V_2(Z) = Z(a)\,[(DB^{-1}A - C)^{-1}] + Z(b)\,[B^{-1}A(DB^{-1}A - C)^{-1}].$$

(iii) *If* C^{-1} *and* $(AC^{-1}D - B)^{-1}$ *exist, let*

$$V_1(A) = Z(a)\,[-C^{-1}D(AC^{-1}D - B)^{-1}] + Z(b)\,[-(AC^{-1}D - B)^{-1}],$$

$$V_2(Z) = Z(a)\,[-C^{-1} + C^{-1}D(AC^{-1}D - B)^{-1}AC^{-1}] + Z(b)[(AC^{-1}D - B)^{-1}AC^{-1}].$$

(iv) *If* D^{-1} *and* $(BD^{-1}C - A)^{-1}$ *exist, let*

$$V_1(Z) = Z(a)\,[(BD^{-1}C - A)^{-1}] + Z(b)\,[D^{-1}C(BD^{-1}C - A)^{-1}],$$

$$V_2(Z) = Z(a)\,[-(BD^{-1}C - A)^{-1}BD^{-1}] + Z(b)\,[D^{-1} - D^{-1}C(BD^{-1}C - A)^{-1}BD^{-1}].$$

If the hypotheses of (i), (ii), (iii), *or* (iv) *are satisfied, then*

$$\int_a^b [Z(LY) + (L^* Z)Y]\,dx = V_1(Z)U_1(Y) + V_2(Z)U_2(Y).$$

It is easy to verify that if the hypotheses of any two of (i), (ii), (iii), and (iv) are satisfied, the expressions for $V_1(Z)$ and $V_2(Z)$ are equivalent. The usual adjoint boundary conditions are thus well defined.

3. NONHOMOGENEOUS DIFFERENTIAL OPERATORS

Throughout the remainder of this paper we shall assume that the hypothesis of at least one of (i), (ii), (iii), and (iv) of Theorem 2.1 is satisfied.

Definition. Let

$$MY = LY + K_2(x)U_2(Y), \qquad H(Y) = U_1(Y) + \int_a^b K_1(x)Y(x)\,dx,$$

where $K_1(x)$ and $K_2(x)$ denote n-by-n matrices whose elements are integrable over [a, b]. We consider the system

$$S: MY = 0, \; H(Y) = 0.$$

Definition. If

$$M^* Z = L^* Z + V_1(Z)K_1(x) \quad \text{and} \quad J(Z) = V_2(Z) + \int_a^b Z(x)K_2(x)\,dx,$$

then

$$S^*: M^* Z = 0, \; J(Z) = 0$$

is said to be adjoint to S.

We see that Green's formula now yields

$$\int_a^b [Z(MY) + (M^* Z)Y]\,dx = V_1(Z)H(Y) + J(Z)U_2(Y)$$

for all differentiable Y and Z.

Let $Y_h(x)$ satisfy $LY_h = 0$, $|Y_h(x)| \neq 0$ on [a, b]; let $Z_h(x)$ satisfy $L^* Z_h = 0$, $Z_h(x)Y_h(x) = I$ on [a, b]; and let

$$G_0(x, t) = \begin{cases} \frac{1}{2}Y_h(x)Z_h(t) & \text{for } t < x, \\ -\frac{1}{2}Y_h(x)Z_h(t) & \text{for } t > x. \end{cases}$$

It is easy to see that if C_1 and C_2 are constant matrices, then

$$Y(x) = Y_h(x) C_1 - \left(\int_a^b G_0(x, t) K_2(t)\, dt \right) U_2(Y),$$

$$Z(t) = C_2 Y_h(x) - V_1(Z) \int_a^b K_1(t) G_0(t, x)\, dt$$

formally satisfy $MY = 0$ and $M^* Z = 0$, whatever the values C_1, $U_2(Y)$, C_2, $V_1(Z)$.

LEMMA 3.1.' *If* $|U_2(Y_h)| \neq 0$, *then for arbitrary* $U_2(Y)$

$$Y(x) = \left[Y_h(x) U_2(Y_h)^{-1} \left\{ I + U_2 \left(\int_a^b G_0(x, t) K_2(t)\, dt \right) \right\} - \int_a^b G_0(x, t) K_2(t)\, dt \right] U_2(Y).$$

Proof. Apply U_2 to $Y(x)$ and solve for C_1. Note that $U_2(Y)$ can be chosen arbitrary. By $Y_0(x)$ we shall denote the solution of $MY = 0$ satisfying the condition $U_2(Y_0) = I$.

THEOREM 3.2. *If* $|U_2(Y_h)| \neq 0$, *then* S *is compatible if and only if* $|H(Y_0)| = 0$.

Proof. If S is compatible, we see from Lemma 3.1 that $H(Y) = H(Y_0) U_2(Y) = 0$. A necessary and sufficient condition that this be true without $U_2(Y) = 0$ is that $|H(Y_0)| = 0$.

COROLLARY. *If* $K_2(x) = 0$, *then* S *is compatible if and only if* $|H(Y_h)| = 0$.

It is easy to see that if $K_2(x) = 0$, then U_2 can be chosen so that $|U_2(Y_h)| \neq 0$.

LEMMA 3.3. *If* $|V_1(Z_h)| \neq 0$, *then for arbitrary* $V_1(Z)$,

$$Z(x) = V_1(Z) \left[\left\{ I + V_1 \left(\int_a^b K_1(t) G_0(t, x)\, dt \right) \right\} V_1(Z_h)^{-1} Z_h(x) - \int_a^b K_1(t) G_0(t, x)\, dt \right].$$

Proof. Apply V_1 to $Z(x)$ and solve for C_2. By $Z_0(x)$ we denote the solution satisfying the condition $V_1(Z_0) = I$.

THEOREM 3.4. *If* $|V_1(Z_h)| \neq 0$, *then* S^* *is compatible if and only if* $|J(Z_0)| = 0$.

The proof is similar to that of Theorem 3.2.

COROLLARY. *If* $K_1(x) = 0$, *then* S^* *is compatible if and only if* $|J(Z_h)| = 0$.

Again it is easy to see that V_1 can be chosen so that $|V_1(Z_h)| \neq 0$.

THEOREM 3.5. *If* $|U_2(Y_h)| \neq 0$ *and* $|V_1(Z_h)| \neq 0$, *then* S *is compatible if and only if* S^* *is compatible.*

Proof. If S is compatible, then Green's formula applied to $Y_0(x)$ and $Z_0(x)$ gives $H(Y_0) + J(Z_0) = 0$. Thus $0 = |H(Y_0)| = -|J(Z_0)|$, and S^* is compatible. Symmetry completes the argument.

4. THE NONHOMOGENEOUS GREEN'S MATRICES

In the event S, S* are incompatible, it is possible to find solutions for systems

$$\text{NH:} \qquad MY = F, \qquad H(Y) = 0,$$

$$\text{NH}^*: \qquad M^* Z = K, \quad J(Z) = 0.$$

THEOREM 4.1. *If* S *is incompatible and* $|U_2(Y_h)| \neq 0$, *then* NH *is compatible. If*

$$G(x, t) = Y_0(x) H(Y_0)^{-1} H(Y_h) U_2(Y_h)^{-1} U_2(G_0(x, t))$$
$$- Y_0(x) H(Y_0)^{-1} H(G_0(x, t)) - Y_h(x) U_2(Y_h)^{-1} U_2(G_0(x, t)) + G_0(x, t),$$

where $H(G_0(x, t))$ *and* $U_2(G_0(x, t))$ *are computed with* t *fixed, then*

$$Y(x) = \int_a^b G(x, t) F(t)\, dt$$

satisfies NH.

Proof. As before, the function

$$Y(x) = Y_h(x) C_1 - \int_a^b G_0(x, t) K_2(t)\, dt\, U_2(Y) + \int_a^b G_0(x, t) F(t)\, dt$$

formally satisfies $MY = F$. To find C_1 and $U_2(Y)$, we note that

$$U_2(Y) = U_2(Y_h) C_1 - U_2\left(\int_a^b G_0(x, t) K_2(t)\, dt\right) U_2(Y) + U_2\left(\int_a^b G_0(x, t) F(t)\, dt\right).$$

Thus, if $|U_2(Y_h)| \neq 0$, then

$$C_1 = U_2(Y_h)^{-1}\left[I + U_2\left(\int_a^b G_0(x, t) K_2(t)\, dt\right)\right] U_2(Y)$$
$$- U_2(Y_h)^{-1} U_2\left(\int_a^b G_0(x, t) F(t)\, dt\right)$$

and

$$Y(x) = Y_0(x) U_2(Y) - Y_h(x) U_2(Y_h)^{-1} U_2\left(\int_a^b G_0(x, t) F(t)\, dt\right) + \int_a^b G_0(x, t) F(t)\, dt.$$

To find $U_2(Y)$, we use the boundary condition $H(Y) = 0$. It implies that

$$0 = H(Y_0)U_2(Y) - H(Y_h)U_2(Y_h)^{-1}U_2\left(\int_a^b G_0(x, t)F(t)\,dt\right) + H\left(\int_a^b G_0(x, t)F(t)\,dt\right).$$

Since S is incompatible. $|H(Y_0)| \neq 0$,

$$U_2(Y) = H(Y_0)^{-1}H(Y_h)U_2(Y_h)^{-1}U_2\left(\int_a^b G_0(x, t)F(t)\,dt\right)$$

$$- H(Y_0)^{-1}H\left(\int_a^b G_0(x, t)F(t)\,dt\right),$$

and

$$Y(x) = Y_0(x)H(Y_0)^{-1}H(Y_h)U_2(Y_h)^{-1}U_2\left(\int_a^b G_0(x, t)F(t)\,dt\right)$$

$$- Y_0(x)H(Y_0)^{-1}H\left(\int_a^b G_0(x, t)F(t)\,dt\right)$$

$$- Y_h(x)U_2(Y_h)^{-1}U_2\left(\int_a^b G_0(x, t)F(t)\,dt\right) + \int_a^b G_0(x, t)F(t)\,dt\,.$$

If we write $Y(x) = \int_a^b G(x, t)F(t)\,dt$, then $G(x, t)$ has the form written in the statement of Theorem 4.1.

The Green's function for NH^* is found in a similar manner, provided S^* is incompatible and $|V_1(Z_h)| \neq 0$. We find that the function $Z(x) = \int_a^b K(t)G^*(x, t)\,dt$ satisfies the system NH^* when

$$G^*(x, t) = V_1(G_0(t, x))V_1(Z_h)^{-1}J(Z_h)J(Z_0)^{-1}Z_0(x)$$

$$- J(G_0(t, x))J(Z_0)^{-1}Z_0(x) - V_1(G_0(t, x))V_1(Z_h)^{-1}Z_h(x) + G_0(t, x)\,.$$

Again, $V_1(G_0(t, x))$ and $J(G_0(t, x))$ are computed with t fixed.

It is easy to show that $G(x, t)$ has the following properties.

(i) For all x, t $(a \leq x \leq b,\ a \leq t \leq b)$ except $x = t$, $G(x, t)$ is continuous in both x and t. Further, $G(x, x - 0) - G(x, x + 0) = I$.

(ii) $MG(x, \bar{t}) = 0$ for fixed $\bar{t}$ except when $x = \bar{t}$.

(iii) $H(G(x, \bar{t})) = 0$ for fixed $\bar{t}$.

(iv) $G(x, t)$ is unique and is completely determined by (i), (ii), and (iii).

THEOREM 4.2. *If* $|U_2(Y_h)| \neq 0$, $|V_1(Z_h)| \neq 0$, *and* S *and* S^* *are incompatible, then* $G(x, t)$ *and* $G^*(x, t)$, *associated with* NH *and* NH^*, *respectively, satisfy* $G(x, t) + G^*(t, x) = 0$.

Proof. For any continuous functions F and K, write

$$Y(x) = \int_a^b G(x, t) F(t) dt \quad \text{and} \quad Z(x) = \int_a^b K(t) G^*(x, t) dt;$$

then $MY = F$, $H(Y) = 0$ and $M^* Z = K$, $J(Z) = 0$. Inserting these in Green's formula, we have the relation

$$\int_a^b \int_a^b K(x) [G^*(t, x) + G(x, t)] F(t) dt dx = 0$$

for all K and F. Since K and F can be any continuous functions, the result follows. For a more detailed argument of this last step, see Birkhoff and Langer [1, page 69].

When $K_2 \equiv 0$, the choice of $U_2(Y)$ is more or less arbitrary. That is to say, when $|A| \neq 0$; the choice $C = 0$, $D = I$ gives a suitable adjoint system. When $|B| \neq 0$; the choice $C = -I$, $D = 0$ gives a suitable adjoint system. In either case, any adjoint system compatible with Section 2 is equivalent to these when they are defined.

5. INDETERMINATE CASES

In the preceding two sections, the fundamental assumptions $|U_2(Y_h)| \neq 0$, $|V_1(Z_h)| \neq 0$ were usually made. We now show that frequently if one of these determinants vanishes, the other does also.

THEOREM 5.1. *If one of the conditions*

(i) A^{-1}, C^{-1}, *and* $(CA^{-1}B - D)^{-1}$ *exist,*

(ii) B^{-1}, D^{-1}, *and* $(DB^{-1}A - C)^{-1}$ *exist,*

(iii) C^{-1}, $(AC^{-1}D - B)^{-1}$ *exist,*

(iv) D^{-1}, $(BD^{-1}C - A)^{-1}$ *exist*

holds, then $|U_2(Y_h)| \neq 0$ *if and only if* $|V_1(Z_h)| \neq 0$.

We note from Green's formula (Section 2) that

$$\text{in case (i)}, \; CY_h(a) V_1(Z_h) AC^{-1} (CA^{-1}B - D) Y_h(b) = U_2(Y_h),$$

$$\text{in case (ii)}, \; DY_h(b) V_1(Z_h) BD^{-1} (DB^{-1}A - C) Y_h(a) = -U_2(Y_h),$$

$$\text{in case (iii)}, \; CY_h(a) V_1(Z_h) (AC^{-1}D - B) Y_h(b) = -U_2(Y_h),$$

$$\text{in case (iv)}, \; DY_h(b) V_1(Z_h) (BD^{-1}C - A) Y_h(a) = U_2(Y_h).$$

The trouble that arises when these determinants vanish is indeed fundamental in the preceeding sections. It is impossible to find C_1 in $Y(x)$ and C_2 in $Z(x)$ by the prescribed method. However, under various circumstances it is possible to proceed. If

(i) $|H(Y_h)| \neq 0$,

(ii) $\left| I + U_2 \left(\int_a^b G_0(x, t) K_2(t)\, dt \right) \right| \neq 0$, or

(iii) $\left| H \left(\int_a^b G_0(x, t) K_2(t)\, dt \right) \right| \neq 0$,

it is possible to determine whether or not S is compatible, and (in the incompatible case) to determine the Green's function.

Similarly, if

(iv) $|J(Z_h)| \neq 0$,

(v) $\left| I + V_1 \left(\int_a^b K_1(t) G_0(t, x)\, dt \right) \right| \neq 0$, or

(vi) $\left| J \left(\int_a^b K_1(t) G_0(t, x)\, dt \right) \right| \neq 0$,

it is possible to do the same for S^*. These conditions seem to be independent, unlike the previous situation.

If $|H(Y_h)| \neq 0$, we solve for C_1 by applying H to both sides of the equation preceding Lemma 3.1 and then applying U_2.

If $\left| I + U_2 \left(\int_a^b G_0(x, t) K_2(t)\, dt \right) \right| \neq 0$, we apply U_2 and solve for $U_2(Y)$ first. Then H is applied.

If $\left| H \left(\int_a^b G_0(x, t) K_2(t)\, dt \right) \right| \neq 0$, we apply H and solve for $U_2(Y)$. Then U_2 is applied.

The cases (iv), (v), and (vi) can be handled similarly.

In each of these cases, if the systems are incompatible, then the corresponding nonhomogeneous systems can be solved by the same procedure. In each case a Green's function can be found.

When it exists, the Green's function for NH satisfies conditions (i), (ii), (iii), and (iv) listed for $G(x, t)$.

If both NH and NH* are compatible, then the corresponding Green's functions satisfy the property listed in Theorem 4.2.

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