CONDITIONS FOR THE ANALYTICITY OF CERTAIN SETS

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1. INTRODUCTION

In complex analysis it is not always easy to recognize an analytic function or an analytic set. The problem of recognizing an analytic function is a special case of the problem of recognizing an analytic set, since a function is analytic if and only if its graph is an analytic set.

In the first part of this paper it will be shown that a limit of a sequence $\{A_i\}$ of analytic sets of pure dimension k is an analytic set, provided that the 2k-dimensional volumes of the sets A_i are uniformly bounded. This generalizes a result of Stoll [9].

In the second part of this paper we discuss sets that are analytic except for possible singularities. The question is then whether the singularities are genuine or removable. Conditions guaranteeing the removability of singularities have been given by many authors, including Hartogs, Radó, Thullen, Remmert and Stein, Rothstein, and Stoll. Our results extend theorems of Stoll [8], [9]. We show that an analytic set A of pure dimension k defined in the complement of an analytic set B can always be continued through B, in case the 2k-dimensional volume of A is finite or $\bar{A} \cap B$ has 2k-dimensional Hausdorff outer measure 0. The first of these results was conjectured by Stoll, and it can be applied to give a simple proof of Stoll's theorem that an analytic subset of $\mathfrak{C}^n$ is algebraic if its volume of appropriate dimension doesn't grow too fast near infinity. Along the way we give simple proofs of the theorems of Radó and of Remmert and Stein, and derive some interesting properties for representing measures in certain algebras of analytic functions.

In the last section we introduce a general notion of capacity and use it to prove a very general extension of the theorem of Remmert and Stein.

2. CONVERGENCE OF ANALYTIC SETS

Rutishauser [7] gives a remarkable lower bound on the length of the curve in which an analytic subset of $\mathfrak{C}^2$ that passes through the origin intersects the unit sphere. The following result, communicated with its proof to the author by G. Stolzenberg, generalizes Rutishauser's. Rather than the precise lower bound $2\pi r$, we give here only the more easily proved lower bound r, since the precise result will be a consequence of Theorem 2.

LEMMA 1. *Let* B *be an open ball of radius* r *centered at* 0 *in* $\mathfrak{C}^n$, *and* A *a one-dimensional analytic set in some neighborhood of* $\bar{B}$, *with* $0 \in A$. *Let* S *be the boundary of* B. *Then the curve* $A \cap S$ *has length at least* r.

Proof. Let μ be any measure on $A \cap S$ that represents 0. By this we mean that μ is a nonnegative Baire measure on $A \cap S$ for which

$$f(0) = \int f\, d\mu \qquad (*)$$

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for all polynomials f in the coordinate functions $z_1, \cdots, z_n$. Take a point z^0 in the support of μ and choose coordinates so that $z_1^0 = r$, $z_i^0 = 0$ for $i \geq 2$. Let π be the projection of $\mathfrak{C}^n$ onto the complex plane $\mathfrak{C}^1$ defined by $\pi(z) = z_1$. Then $\pi(\mu) = \nu$ is a measure on $\pi(A \cap S)$ that represents 0, and $r \in$ support ν. Let L_a be the vertical line in $\mathfrak{C}^1$ passing through the point a, where $0 < a < r$. Now if L_a does not intersect $\pi(A \cap S)$, then by Runge's theorem there exists a sequence of polynomials f converging uniformly to 1 on the part R of $\pi(A \cap S)$ to the right of L_a, and to 0 on $\pi(A \cap S) - R$. Thus in equation (*) we pass to the limit and get $0 = \nu(R)$. This contradicts the fact that $r \in$ support ν. Hence each L_a intersects $\pi(A \cap S)$, so that length $\pi(A \cap S) \geq r$. Hence, length $(A \cap S) \geq r$.

LEMMA 2. *Let* B *be an open sphere of radius* r *and center* 0 *in* $\mathfrak{C}^n$. *Let* A *be a pure* k-*dimensional analytic set containing* 0 *in some neighborhood of* $\overline{B}$. *Then there exists a constant* c, *depending only on* n, *such that the* $(2k - 1)$-*dimensional volume of* $A \cap S$ *is at least* cr^{2k-1}.

Proof. There is no loss of generality in taking $r = 1$. Let P be any $(n - k + 1)$-dimensional complex linear space through 0. Then, by the previous lemma, length $(A \cap S \cap P) \geq 1$. By the techniques of integral geometry we see that the volume in question is greater than some constant c times the average of

$$\text{length } (A \cap S \cap P),$$

taken over all P. This gives the result.

LEMMA 3. *Under the hypothesis of Lemma* 2, *the* 2k-*dimensional volume of* $A \cap B$ *is at least* cr^{2k}, *where the constant* c *depends only on* n.

Proof. The result follows from Lemma 2 and Fubini's theorem.

We now establish some useful classes of sets and measures on an open set $U \subset \mathfrak{C}^n$.

Definition 1. Let U be an open set in $\mathfrak{C}^n$. A positive Baire measure μ on U will be said to have property M(b, c, k), (written $\mu \in M(b, c, k)$) if $\mu(U) \leq b$ and $\mu(B) \geq cr^k$ for any open ball $B \subset U$ of radius r whose center is in support μ. A closed subset A of U will be said to have property S(b, c, k), or $A \in S(b, c, k)$, if it supports some measure on U having property M(b, c, k). A closed subset A of U will be said to have property N(c, k) ($A \in N(c, k)$) if whenever $\mathfrak{C}^n$ is covered by disjoint square polycylinders of the form

$$\{z\colon |z_i - z_i^0| \leq r,\ 1 \leq i \leq n\},$$

of common radius r, then at most cr^{-k} of them lie in U and intersect A.

The next three lemmas are simple, and we state them without proof.

LEMMA 4. *Let* U *be a bounded open set in* $\mathfrak{C}^n$. *Let* f *be an analytic homeomorphism of some neighborhood of* $\overline{U}$ *with some open set in* $\mathfrak{C}^n$. *There exists a constant* $K > 0$ *such that if a subset* A *of* U *has property* N(c, k), *then* f(A) *has property* N(Kc, k).

LEMMA 5. *For each* n *there exists a constant* $K > 0$ *such that if* U *is an open set in* $\mathfrak{C}^n$ *and* A *has property* S(b, c, k), *then* A *has property* $N(Kbc^{-1}, k)$.

LEMMA 6. *For each value of* n *there exists a constant* δ *such that if* A *has property* N(c, k) *in some open set* U, *then the* k-*dimensional Hausdorff measure of* A *is at most* δc.

If μ is a Baire measure on an open set $U \subset \mathfrak{C}^n$ and if $\{\mu_n\}$ is a sequence of Baire measures on U, we say that $\mu_n \to \mu$ as $n \to \infty$ if $\mu_n/X \to \mu/X$ as $n \to \infty$ in the weak star topology, for each compact subset X of U. Similarly, if $\{A_n\}$ is a sequence of closed subsets of U and if A is a closed subset of U, we say that $A_n \to A$ as $n \to \infty$ if $A_n \cap X \to A \cap X$ as $n \to \infty$ in the Hausdorff metric for compact subsets of X, for every compact subset X of U.

LEMMA 7. *Let* U *be an open set in* $\mathfrak{C}^n$. *Let* $\{\mu_n\}$ *be a sequence of nonnegative Baire measures on* U, *each having property* M(b, c, k). *Let* A_n = support μ_n. *Assume that* $\mu_n \to \mu$ *and* $A_n \to A$, *as* $n \to \infty$, *where* μ *is a positive Baire measure on* U *and* A *is a closed subset of* U. *Then* support μ = A.

Proof. If $x \in$ support μ and V is any neighborhood of x, then $\mu(V) \neq 0$. Therefore $\mu_n(V) \neq 0$ for all sufficiently large n, so that $A_n \cap V \neq \square$. Therefore $x \in A$.

If $x \in A$ and V is a closed neighborhood of x, let W be a neighborhood of x with $\operatorname{dist}(W, \mathfrak{C}^n - V) = d > 0$. For all sufficiently large n, we have $A_n \cap W \neq \square$. Take y in $A_n \cap W$, so that the closed sphere B of radius d about y is a subset of V. Since μ_n has property M(b, c, k) it follows that

$$\mu_n(V) \geq \mu_n(B) \geq cd^k.$$

In the limit this gives $\mu(V) \geq cd^k \neq 0$. Therefore $x \in$ support μ.

COROLLARY. *If* $\{A_n\}$ *is a sequence of closed subsets of an open set* $U \subset \mathfrak{C}^n$, *and* $A_n \to A$ *as* $n \to \infty$, *then* A *has property* $S(b_0, c, k)$ *for some* b_0 *if* A_n *has property* S(b, c, k) *for each* n.

Proof. Let μ_n be a measure in M(b, c, k) supported by A_n. By passing to a subsequence, if necessary, we may assume that μ_n converges to a measure μ on U as $n \to \infty$. By the lemma, μ = support A. Since μ has property $M(b_0, c, k)$ for some b_0 it follows that A has property $S(b_0, c, k)$.

LEMMA 8. *Let* $A \subset \mathfrak{C}^n$ *be a locally compact set of* $(2k + 1)$*-dimensional Hausdorff measure* 0, *for some* k $(0 \leq k \leq n - 1)$. *Then the set* F *of all complex* $(n - k)$*-dimensional linear subspaces* P *of* $\mathfrak{C}^n$ *passing through* 0 *and intersecting* A *in a set that is not totally disconnected is of first category.*

Proof. Without loss of generality we take A to be compact.

Let α be any real-valued real-linear function on $\mathfrak{C}^n$ that is a linear combination with rational coefficients of the real and imaginary parts $x_1, y_1, \cdots, x_n, y_n$ of the coordinate functions $z_1, \cdots, z_n$. Let c and d be rational numbers $(0 < c < d)$. Let $F(\alpha, c, d)$ consist of all P for which $[c, d] \subset \alpha(A \cap P)$. Clearly $F(\alpha, c, d)$ is closed and $F \subset \bigcup F(\alpha, c, d)$. Thus it is sufficient to show that each $F(\alpha, c, d)$ is nowhere dense. Assume there exists an interior point P_0 of $F(\alpha, c, d)$. Choose new coordinates $\tilde{z}$ on $\mathfrak{C}^n$ so that α is the real part of $\tilde{z}_{k+1}$ and the equation of P_0 is $\tilde{z}_1 = \tilde{z}_2 = \cdots = \tilde{z}_k = 0$. Then for all $a = (a_1, \cdots, a_k)$ sufficiently near to 0 (say for $|a_i| < \varepsilon$, $1 \leq i \leq k$), the subspace P_a whose equations are

$$\tilde{z}_1 + a_1 \tilde{z}_{k+1} = \cdots = \tilde{z}_k + a_k \tilde{z}_{k+1} = 0$$

belongs to $F(\alpha, c, d)$, so that α assumes all values between c and d on $A \cap P_a$. Let π be the map of $\left\{z \in \mathfrak{C}^n \colon \alpha(z) > \frac{c}{2}\right\}$ into $\mathfrak{C}^k \times \mathfrak{R}$ defined by

$$\pi(z) = (-\tilde{z}_1(\tilde{z}_{k+1})^{-1}, \cdots, -\tilde{z}_k(\tilde{z}_{k+1})^{-1}, \alpha(z)).$$

Then we see that $\pi(A)$ contains the set $\left[-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right]^{2k} \times [c, d]$, and so has a nonvoid interior. Hence the $(2k+1)$-dimensional Hausdorff measure of $\pi(A)$ is not zero, contrary to the vanishing of the $(2k+1)$-dimensional Hausdorff measure of A. This proves that $F(\alpha, c, d)$ has void interior, as desired.

THEOREM 1. *Let* $\{A_n\}$ *be a sequence of pure* 2k-*dimensional analytic sets in an open subset* U *of* $\mathfrak{C}^n$, *converging to a set* $A \subset U$. *Let the* 2k-*dimensional volumes of the* A_n *be finite and bounded by some constant* b. *Then* A *is an analytic subset of* U.

Proof. Define a measure μ_n supported on A_n by taking $\mu_n(S)$ to be the 2k-dimensional Hausdorff measure of $A_n \cap S$. Since $\mu_n(U) \leq b$, we may assume, by passing to a subsequence if necessary, that μ_n converges to some measure μ on U. By Lemma 3, $\mu_n \in M(b, c, 2k)$, so that $A_n \in S(b, c, 2k)$. By Lemma 7,

$$A \in S(b_0, c, 2k).$$

By Lemmas 5 and 6, the 2k-dimensional Hausdorff measure of A is finite. By Lemma 8, we may assume that the intersection of A with the set

$$P = \{z: z_1 = z_2 = \cdots = z_k = 0\}$$

is totally disconnected. Thus there exists an open neighborhood T of 0 in $\mathfrak{C}^{n-k}$ such that

$$\mathfrak{C}^k \times \text{bdry } T \cap (A \cap P) = \square$$

and $(\mathfrak{C}^k \times \overline{T}) \cap P \subset U$. We may thus choose an open neighborhood S of 0 in $\mathfrak{C}^k$ such that

$$(\overline{S} \times \text{bdry } T) \cap A = \square \qquad \text{and} \qquad \overline{S} \times \overline{T} \subset U.$$

Let π be the projection of $\mathfrak{C}^n$ onto $\mathfrak{C}^k$, obtained by suppression of the last $n - k$ coordinates. Since $A_0 = A \cap (S \times T)$ is a closed subset of $S \times T$ whose closure is disjoint from $S \times \text{bdry } T$, the map $\pi/A_0 = \pi_0$ of A_0 into S is proper. Also, $A_n \cap (\overline{S} \times \text{bdry } T) = \square$ for all large enough n, so that the map $\pi_n = \pi/(A_n \cap (S \times T))$ of $A_n \cap (S \times T)$ into S is also proper. From the standard theory of proper mappings of analytic varieties, as expounded for instance in [1, pp. 228-231], we see that with π_n is associated a multiplicity λ_n such that each p in S is the image under π_n of points $p_1^n, \cdots, p_{\lambda_n}^n$ in $A_n \cap (S \times T)$, and these points are distinct, except when p belongs to a proper analytic subset of S.

Now the 2k-dimensional volume of $A_n \cap (S \times T)$ is at least λ_n times the 2k-dimensional volume of S. Hence the λ_n are uniformly bounded, so that by passing to a subsequence if necessary, we may take them to be a fixed constant λ. For each z in $S \times T$ we write $p = \pi(z)$. Let z^0 be a fixed point in $(S \times T) - A$, and set $p^0 = \pi(z^0)$. By passing to another subsequence if necessary, we may assume that for each i $(1 \leq i \leq \lambda)$ the points p_i^{0n} of $S \times T$ converge to points p_i^0 of $S \times \overline{T}$ as $n \to \infty$. Clearly $p_i^0 \in A$, so that in fact $p_i^0 \in S \times T$ $(1 \leq i \leq \lambda)$. Now let g be any analytic function on $\mathfrak{C}^n$ with $g(z^0) \neq g(p_i^0)$ $(1 \leq i \leq \lambda)$. For each n define the analytic function f_n on $S \times T$ by

$$f_n(z) = \prod_{i=1}^{\lambda} (g(z) - g(p_i^n)).$$

Since the f_n are uniformly bounded on $S \times T$, we may assume that as $n \to \infty$ they converge to an analytic function f on $S \times T$, uniformly on compact subsets of $S \times T$. Then f vanishes on A because $A_n \to A$ and f_n vanishes on A_n. Also,

$$f(z^0) = \prod_{i=1}^{\lambda} (g(z^0) - g(p_i^0) \neq 0.$$

Thus A is an analytic set.

3. CONDITIONS FOR REMOVABLE SINGULARITIES

The following proof of Radó's theorem is an improvement of a proof of Wermer, given by Glicksberg [4].

RADÓ'S THEOREM. *Let* f *be continuous on* $D = \{z: |z| \leq 1\}$ *and analytic at all those points of* int D *at which it does not vanish. Then* f *is analytic on* int D.

Proof. Let $E = \{z: f(z) = 0\}$, $F = \text{bdry } E$, $B = \text{bdry } D$. Let $\mathscr{A}$ be the closed sub-algebra of $C(D)$ generated by the two functions f and z, and σ its Šilov boundary. Since all functions in $\mathscr{A}$ are analytic on $D - F - B$, $\sigma \subset F \cup B$.

Consider a point z_0 of $D - E$. By [1], there exists a Jensen measure μ_0 for z_0 on σ. Hence

$$-\infty < \log |f(z_0)| \leq \int \log |f(z)| \, d\mu_0(z).$$

From this it follows that $\mu_0(E) = 0$, so that μ_0 is a measure on B. Thus $|g(z_0)| \leq \|g\|_B$ for each g in $\mathscr{A}$. Since the points of $D - E$ are dense in D - int E, we see that $|g(z_0)| \leq \|g\|_B$ for all z_0 in F. Thus $\|g\| \leq \|g\|_B$, or $\sigma \subset B$. The same argument shows that if G is any closed subdisk of D and H is its boundary, then $\|g\|_G = \|g\|_H$ for all g in $\mathscr{A}$. Thus by the maximality theorem of Rudin [6] or of Wermer [10], $\mathscr{A}$ consists of functions analytic on int D, as desired.

For later use we remark that Radó's theorem can be extended to several variables. In other words, it remains true when D is replaced by its n-fold product D^n. The proof is a trivial reduction to the case $n = 1$.

At this point we can't resist giving a simple proof of part of the Remmert-Stein theorem, which will serve as a prototype for the proof of the stronger theorems to follow. The essential idea is that of [2].

REMMERT-STEIN THEOREM. *Let* U *be an open subset of* $\mathfrak{C}^n$, B *an analytic subset of* U, *and* A *an analytic subset of* $U - B$. *If* B *is of dimension at most* $k - 1$ *and* A *is of pure dimension* k, *then* $\overline{A} \cap U$ *is an analytic subset of* U.

Proof. Let z^0 be any point of $\overline{A} \cap U$, say $z^0 = 0$. There exists a complex linear $(n - 1)$-dimensional subvariety P_1 of $\mathfrak{C}^n$ that contains no analytic component of either A or B. Thus $B \cap P_1$ is an analytic subset of $U \cap P_1$ of dimension at most $k - 2$, and $A \cap P_1$ is a pure $(k - 1)$-dimensional analytic subset of $(U - B) \cap P_1$.

Working now inside P_1, we can find an $(n-2)$-dimensional complex linear subspace P_2 of P_1 such that $B \cap P_2$ is an analytic subset of dimension at most $k-3$ of $U \cap P_2$, and $A \cap P_2$ is a pure $(k-2)$-dimensional analytic subset of $(U-B) \cap P_2$. Continuing by induction, we finally obtain an $(n-k)$-dimensional linear subspace P_{n-k} of $\mathfrak{C}^n$ such that $B \cap P$ and $A \cap P_{n-k}$ are countable, so that $(A \cup B) \cap P_{n-k}$ is totally disconnected. We may take P_{n-k} to have the equations $z_1 = \cdots = z_k = 0$. As in the proof of Theorem 1 above, there exist open neighborhoods S of 0 in $\mathfrak{C}^k$ and T of 0 in $\mathfrak{C}^{n-k}$ such that the projection π of $\mathfrak{C}^n$ into $\mathfrak{C}^k$, obtained by discarding the last $n-k$ coordinates, gives a proper map of $(A \cup B) \cap (S \times T)$ into S. Since B is an analytic set, this implies that $\pi^{-1}(p) \cap B \cap (S \times T)$ is finite for all p in S. Hence $\pi^{-1}(p) \cap A \cap (S \times T)$ is countable. Write $S_0 = S - \pi(B \cap (S \times T))$. Then S_0 is a connected dense open subset of S. Now π maps $A \cap (S_0 \times T)$ properly into S_0, and $A \cap (S_0 \times T)$ is an analytic subset of $S_0 \times T$. With this map is associated a multiplicity λ such that for each p in S_0 there exist points $p_1, \cdots, p_\lambda$ in $A \cap (S_0 \times T)$ with $\pi(p_i) = p$.

Let h be an analytic function on U that vanishes on B but does not vanish on any irreducible analytic component of A. Define the analytic function $\tilde{h}$ on S_0 by

$$\tilde{h}(p) = \prod_{i=1}^{\lambda} h(p_i).$$

Then $\tilde{h}(p) \to 0$ whenever $p \to S - S_0$, since then one of the p_1 goes to B. Thus, if we set $\tilde{h}(p) = 0$ for p in $S - S_0$, then $\tilde{h}$ is continuous on S and analytic where it does not vanish. It is therefore analytic on S, so that $S - S_0$ is contained in a proper analytic subset F of S. Now let z^0 be any point of $S \times T - \bar{A}$. Let $\{p^n\}$ be a sequence of points of S_0 converging to $p^0 = \pi(z^0)$, such that, for $1 \leq i \leq \lambda$, $\{p_i^n\}$ converges to a point p_i^0 in $\bar{A}$. Take a function f analytic on $\mathfrak{C}^n$ with $f(z^0) \neq f(p_i^0)$ $(1 \leq i \leq n)$. Define the bounded analytic function g on $(S_0 \times T)$ by

$$g(z) = \prod_{i=1}^{\lambda} (f(z) - f(p_i)),$$

where $p = \pi(z)$. Since g is a polynomial in f,

$$g(z) = \sum_{i=0}^{\lambda} a_i(p) f(z)^i,$$

with coefficients a_i that are bounded analytic functions on S_0, it can be continued analytically on $S \times T$. By construction, g vanishes on $\bar{A}$. Also,

$$g(z^0) = \prod_{i=1}^{\lambda} (f(z^0) - f(p_i^0)) \neq 0.$$

Thus $\bar{A} \cap (S \times T)$ is an analytic set, as was to be proved.

LEMMA 9. *Let* U *be an open subset of* $\mathfrak{C}^n$, *and* B *a proper analytic subset of* U. *Let* A *be an analytic subset of* $U - B$, *of pure dimension* k, *and such that* $\bar{A} \cap B$ *has* $2k$-*dimensional Hausdorff measure* 0. *Then* $\bar{A} \cap U$ *is analytic.*

Proof. Assume that $0 \in \overline{A} \cap U$, and that there exists an analytic function h on U vanishing on B but not vanishing on any component of U. Since the $(2k+1)$-dimensional Hausdorff measure of $\overline{A}$ is zero, Lemma 8 implies that there exist coordinates such that $\overline{A} \cap \{z\colon z_1 = \cdots = z_k = 0\}$ is totally disconnected and the functions $z_1, \cdots, z_k$ have rank k at at least one point of every irreducible analytic component of A. As we saw in the proof of Theorem 1 above, this implies that there exist open neighborhoods $S \subset \mathfrak{C}^k$ and $T \subset \mathfrak{C}^{n-k}$ of 0 such that the projection π of $S \times T$ onto S is proper on $\overline{A} \cap (S \times T)$. The closed subset $F = \pi((S \times T) \cap (\overline{A} \cap B))$ of S has 2k-dimensional Hausdorff measure 0, and so is nowhere dense in S.

Let K be any component of $S - F$. Then π maps $A \cap \pi^{-1}(K) \cap (S \times T)$ properly onto K. We have thus an associated multiplicity λ, such that to each p in K correspond points $p_1, \cdots, p_\lambda$ in $A \cap (S \times T)$ with $\pi(p_i) = p$. Define an analytic function f on S by the rule

$$f(p) = \begin{cases} 0 & (p \in S - K), \\ \prod_{i=1}^{\lambda} h(p_i) & (p \in K). \end{cases}$$

To show that f is analytic on S, it suffices, by Radó's theorem, to show it is continuous, in other words, to show that if a sequence $\{p^n\}$ of points in K converges to a point p^0 in $S - K$, then $\{p_i^n\}$ converges to the set B for some i. By passing to a subsequence if necessary, we may assume that for each i, $\{p_i^n\}$ converges to a point p_i^0 of $S \times T$. Assume none of the p_i^0 is in B, so that $p_i^0 \in A$ $(1 \leq i \leq \lambda)$. Then there exist a neighborhood U of p^0 in S and neighborhoods $U_1, \cdots, U_\lambda$ of $p_1^0, \cdots, p_\lambda^0$ in A such that π projects each U_i properly onto U.

Write $X = \overline{A} \cap (\overline{S} \times \overline{T}) - (U_1 \cup \cdots \cup U_\lambda)$. Then X is compact and the set

$$W = X - (B \cup ((\text{bdry } S) \times \overline{T}) \cup \text{bdry } U_1 \cup \cdots \cup \text{bdry } U_\lambda)$$

is analytic. Let S_0 be the intersection of a 1-dimensional complex linear subspace P of $\mathfrak{C}^k$ with S, chosen to contain a point q_1 of $\pi(W)$ and a point q_2 of $K \cap U$ with

$$\text{dist}(q_1, q_2) < \text{dist}(q_2, \text{bdry } U)$$

and with $h(z^0) \neq 0$ for all z^0 in $A \cap \pi^{-1}(q_1)$. Let

$$X_0 = \overline{A} \cap (\overline{S}_0 \times \overline{T}) - (U_1 \cup \cdots \cup U_\lambda).$$

Then X_0 is compact, and the set

$$W_0 = X_0 - (B \cup ((\text{bdry } S_0) \times \overline{T}) \cup \text{bdry } U_1 \cup \cdots \cup \text{bdry } U_\lambda)$$

is analytic. Let π_0 be the projection of X_0 into $\overline{S}_0$.

Let λ be a complex-linear functional on $\mathfrak{C}^k$ with $\lambda(q_2) = 0$, and such that

$$|\lambda(q_1)| < \inf \{\lambda(p)\colon p \in (\text{bdry } S \cup \text{bdry } U) \cap P\}.$$

Let $\mathscr{A}_0$ consist of all functions in $C(X_0)$ that are analytic on W_0. Then $(\lambda \circ \pi_0)^{-1} \in \mathscr{A}_0$, and this function is larger at every point z^0 of $\pi_0^{-1}(q_1) \cap (X \cap A)$

than it is anywhere on $\pi_0^{-1}(\text{bdry } S_0 \cup \text{bdry } U)$. Thus z^0 does not have a Jensen measure on $\pi_0^{-1}(\text{bdry } S_0 \cup \text{bdry } U)$. On the other hand, since all functions in $\mathscr{A}_0$ are analytic on W_0, z^0 has a Jensen measure μ on

$$X_0 - W_0 = (B \cup \text{bdry } U_1 \cup \cdots \cup \text{bdry } U_\lambda \cup \pi_0^{-1}(\text{bdry } S_0)) \cap X_0 .$$

Since h vanishes on B and $h(z_0) \neq 0$, it follows that $\mu(B \cap X_0) = 0$. Hence μ is a measure on

$$(\text{bdry } U_1 \cup \cdots \cup \text{bdry } U_\lambda \cup \pi_0^{-1}(\text{bdry } S_0)) \cap X_0 = \pi_0^{-1}(\text{bdry } S_0 \cup \text{bdry } U) .$$

This contradiction shows that some p_i^0 is in B. Therefore f is continuous on S. Therefore it is analytic. Therefore the set $S - K = F$ is contained in an analytic subset of S. Now π does not map any irreducible analytic component of $A \cap (S \times T)$ into F, for if it did, the functions $z_1, \cdots, z_k$ would not have rank k at any point of that component. Hence $A \cap (K \times T)$ is dense in $A \cap (S \times T)$. It now follows as in the proof of the Remmert-Stein theorem above that $\overline{A} \cap (S \times T)$ is an analytic subset of $S \times T$, as desired.

The following theorem is the last step before the proof of our result on removability of singularities for analytic sets with finite volume (Theorem 3). It is also of interest in itself, because of the strong information it gives for representing measures on the intersection of 1-dimensional analytic sets with spheres.

THEOREM 2. *Let* $B \subset \mathfrak{C}^n$ *be an open ball of radius* R *about* 0, U *a neighborhood of* $\overline{B}$, P *a complex subvariety of* U, *and* A *an analytic subset of pure dimension* 1 *in* $U - P$, *with* $0 \in A$. *Let* $S = \text{bdry } B$, *so that* $S \cap A$ *has the structure of an analytic arc, except perhaps at a countable set* H *of points. For each* z^0 *in* $S \cap A - H$ *let* ψ *be the angle between* $S \cap A$ *and the intersection of* S *with the* 1-*dimensional complex linear variety* L *through* 0 *and* z^0. *Then some positive Baire measure* μ *on* $A \cap S$ *represents* 0, *in other words, satisfies the condition* $\int F\, d\mu = F(0)$ *for each analytic function* F *on* $\mathfrak{C}^n$. *If* ds *denotes the arc-length measure on* $S \cap A$, *then* μ *is absolutely continuous with respect to* ds, *and for the Radon-Nikodym derivative* $\frac{d\mu}{ds}$ *we have the inequality*

$$\frac{d\mu}{ds} \leq \frac{|\cos \psi|}{2\pi R} .$$

Proof. Let $\overline{A} \cap \overline{B} = X$. Let $\mathscr{A}$ be the closed subalgebra of C(X) consisting of all uniform limits on X of analytic functions on U. The Šilov boundary σ of $\mathscr{A}$ is a subset of $(\overline{B} \cap P) \cup (A \cap S)$. Let μ be any Jensen measure on σ for 0. If λ is an analytic function on U vanishing on P and not at 0, then

$$-\infty < \log |\lambda(0)| \leq \int \log |\lambda|\, d\mu ,$$

so that $\mu(\sigma \cap \overline{B} \cap P) = 0$. Thus μ is a measure on $A \cap S$, as desired.

Consider now any measure μ on $A \cap S$ representing 0. Consider a point z^0 in $A \cap S - H$. Choose orthonormal complex coordinates $z_1, \cdots, z_n$ on $\mathfrak{C}^n$ with $z_1^0 = R$ and $z_i^0 = 0$ $(2 \leq i \leq n)$. Then $z_2 = \cdots = z_n = 0$ are the equations of L, so that if we let $z = z(s)$ be a function of the arc length s on $S \cap A$ measured from the point z^0, the projection of $z(s)$ on L has coordinates $(z_1(s), 0, \cdots, 0)$. Thus

$$|\cos \psi(z^0)| = \left|\frac{dz_1(s)}{ds}\right|.$$

Now since $z_1^0 = R$ and this is the maximum of $|z_1|$ on S, an appropriate choice of signs actually gives

$$\frac{dz_1(s)}{ds} = i \cos \psi(z^0).$$

Let π be the projection of $\mathfrak{C}^n$ onto $\mathfrak{C}^1$ defined by $\pi(z) = R^{-1} z_1$. The measure $\nu = \pi(\mu)$ on $\mathfrak{C}^1$ represents 0, and it is a measure on $\gamma = \pi(A \cap S)$. Assume that $\cos \psi(z^0) \neq 0$, so that $1 = \pi(z^0) \in \gamma$ and γ is an analytic arc tangent to

$$E = \{z: |z| = 1\}$$

at the point 1. Then $\theta(1 - r)^{-1} \to \infty$ as $\zeta = re^{i\theta} \to 1$ along γ. We may therefore choose $\lambda = \lambda(\theta)$ so that $(\theta - \lambda)(1 - r)^{-1} = h(\zeta) \to \infty$ and $\frac{\lambda}{\theta} \to 1$ as $\zeta \to 1$ along γ.

Now the Poisson kernel for the unit disk is

$$D(\zeta, \phi) = \frac{1}{2\pi} \frac{1 - r^2}{1 - 2r \cos(\theta - \phi) + r^2},$$

where $\zeta = re^{i\theta}$. For all sufficiently small $\varepsilon > 0$ we have the estimate

$$\frac{1}{2\pi}\int_{-\varepsilon}^{\varepsilon} \frac{(1 - r^2)d\phi}{1 - 2r\cos\phi + r^2} \geq \frac{1 - r^2}{\pi}\int_0^{\varepsilon}\frac{d\phi}{(1 - r)^2 + r\phi^2}$$

$$= \frac{1 - r^2}{\pi r}\int_0^{\varepsilon}\frac{d\phi}{\phi^2 + r^{-1}(1 - r)^2} = \frac{1 + r}{\pi\sqrt{r}}\arctan\frac{\varepsilon\sqrt{r}}{1 - r}.$$

If we transfer the measure ν to the unit circle E by use of the kernel D, the transferred measure will be $(2\pi)^{-1} d\phi$, the only measure on E representing 0. Therefore

$$(2\pi)^{-1} d\phi = \left[\int D(\zeta, \phi)\, d\nu(\zeta)\right] d\phi.$$

Let now γ_t be the set of all $\zeta = re^{i\theta}$ in γ with $-t \leq \theta \leq t$. Integrating the last equation from $\phi = -t$ to $\phi = t$, we get

$$\pi^{-1} t = \int_{-t}^{t}\int D(\zeta, \phi)\, d\nu(\zeta)\, d\phi \geq \int_{\gamma_t}\left[\int_{-t}^{t} D(\zeta, \phi)\, d\phi\right] d\nu(\zeta)$$

$$\geq (2\pi)^{-1}\int_{\gamma_t}\left[\int_{-t+|\theta|}^{t-|\theta|}\frac{1 - r^2}{1 - 2r\cos\phi + r^2}\, d\phi\right] d\nu(\zeta)$$

$$\geq \int_{\gamma_t} \frac{1+r}{\pi\sqrt{r}} \arctan \frac{(t - |\theta|)\sqrt{r}}{1 - r}\, d\nu(\zeta) \geq \int_{\gamma_\lambda} \frac{1+r}{\pi\sqrt{r}} \arctan \frac{(t - \lambda)\sqrt{r}}{1 - r}\, d\nu(\zeta),$$

where $\lambda = \lambda(t)$. From this it follows that

$$\limsup_{t \to 0} \nu(\gamma_\lambda)\pi t^{-1} \leq 1.$$

Hence

$$\limsup_{t \to 0} \nu(\gamma_\lambda)\lambda^{-1} \leq 1,$$

or simply

$$\limsup_{t \to 0} \nu(\gamma_t)\pi t^{-1} \leq 1.$$

If we let A_s be the part of $A \cap S$ extending for length s on both sides of z^0, then since $\left|\frac{d\pi}{ds}\right| = \frac{|\cos \psi(z^0)|}{R}$ at z^0, it follows that

$$\limsup_{s \to 0} \frac{\mu(A_s)}{s} \leq \frac{|\cos \psi(z^0)|}{R} \limsup_{\delta \to 0} \frac{\nu_\delta}{\delta} \leq \frac{|\cos \psi(z^0)|}{\pi R}.$$

Therefore μ is absolutely continuous with respect to arc length, and

$$\left|\frac{d\mu}{ds}\right| \leq \frac{|\cos \psi(z^0)|}{2\pi R},$$

as desired.

It remains to consider the case where $z^0 \in A \cap S - H$ and $\cos \psi(z^0) = 0$. Here let $\nu_\delta = \nu(\{\zeta : |\zeta - 1| \leq \delta\})$, for each $\delta > 0$. The function

$$h(\zeta) = \Re \frac{1}{\zeta - (1+\delta)} = \frac{x - (1+\delta)}{(x - (1+\delta))^2 + y^2}$$

is harmonic for $\zeta \neq 1 + \delta$, and negative for $|\zeta| \leq 1$. Thus

$$-1 \leq \frac{-1}{1+\delta} = h(0) = \int h(\zeta)\, d\nu(\zeta)$$

$$\leq \nu_\delta \max\{h(\zeta) : |\zeta - 1| \leq \delta,\ |\zeta| \leq 1\} \leq \frac{-\nu_\delta}{4\delta},$$

or

$$\nu_\delta \leq 4\delta.$$

Combined with the above expression for $\left|\frac{d\pi}{ds}\right|$, this gives $\left|\frac{d\mu}{ds}\right| = 0$ at z^0, as desired.

COROLLARY 1. *Under the hypothesis of Theorem* 2, length $(A \cap S) \geq 2\pi R$.

COROLLARY 2. *There exists a constant* $c > 0$, *depending only on* n, *such that if* B *is an open ball of radius* R *about* 0 *in* $\mathfrak{C}^n$, *and if* P *is an analytic subset of* B, *and* A *a pure* k-*dimensional analytic subset of* B - P, *with* $0 \in A$, *then the* 2k-*dimensional Hausdorff measure of* A *is at least* cR^{2k}.

Proof. This follows from Corollary 1 just as Lemma 3 followed from Lemma 1.

THEOREM 3. *Let* U *be an open subset of* $\mathfrak{C}^n$, *and* P *an analytic subset of* U. *Let* A *be an analytic subset of* U - P *of pure dimension* k *and finite* 2k-*dimensional volume. Then* $\overline{A} \cap U$ *is an analytic subset of* U.

Proof. Let V be any open subset of U containing P. Define a measure μ_V by taking $\mu_V(S)$ to be the 2k-dimensional volume of $A \cap V \cap S$. By Corollary 2 to Theorem 2, $\mu_V \in M(\|\mu_V\|, c, 2k)$ on V. Thus support $\mu_V \in S(\|\mu_V\|, c, 2k)$. By Lemma 5,

$$\text{support } \mu_V \in N(K\|\mu_V\|c^{-1}, 2k).$$

Now $\overline{A} \cap P \subset$ support μ_V for all V. Also, $\|\mu_V\|$ can be made arbitrarily small. Thus $\overline{A} \cap P$ has 2k-dimensional Hausdorff measure 0. By Lemma 9, $\overline{A} \cap U$ is an analytic set, as desired.

Stoll [9] has shown that Theorem 3 implies that if A is a pure k-dimensional analytic set in $\mathfrak{C}^n$, such that $\nu_r/2^k$ is bounded as $r \to \infty$, where ν_r is the 2k-dimensional volume of $A \cap \{z: |z| \leq r\}$, then A is algebraic. Stoll [9] was not able to demonstrate Theorem 3 for general n and k, but got this condition for the algebraic character of A by other methods.

4. CAPACITY

For an optimal generalization of the Remmert-Stein theorem, some notion of capacity in $\mathfrak{C}^n$ seems necessary. See Rothstein [5] for a Remmert-Stein theorem based on capacity in $\mathfrak{C}^1$. After introducing the appropriate notions, we shall prove a Remmert-Stein type theorem (Theorem 4 below) which contains Rothstein's result and Lemma 9 above as special cases.

Definition 2. Let X be a compact Hausdorff space and $\mathscr{A}$ a subalgebra of C(X). A Baire subset B of X will be said to have capacity 0 for $\mathscr{A}$ if the set of all points z in X - B that admit no Jensen measure μ with $\mu(B) \neq 0$ is dense in X - B.

Definition 3. Let U be a bounded open set in $\mathfrak{C}^n$, and B a Baire subset of bdry U. Let $\mathscr{A}$ be the algebra of all bounded analytic functions on U, and σ the spectrum of $\mathscr{A}$. Let σ_U be the closure of U as a subset of σ, so that $\sigma_U \subset \sigma$ and σ_U contains the Šilov boundary of $\mathscr{A}$. Let $\tilde{B}$ consist of all points x in σ_U that lie over some point p of B, so that f(x) = f(p) for all functions f analytic in a neighborhood of $\overline{U}$. We say that B has capacity 0 relative to U if the set Ω consisting of all points in U that have no Jensen measure μ on σ_U with $\mu(\tilde{B}) \neq 0$ is dense in U.

LEMMA 10. *Let* U *be a bounded open set in* $\mathfrak{C}^n$, *and let* V *be an open subset of* U *such that* B = (bdry V) $\cap$ U *has capacity* 0 *relative to* V. *Then* V *is dense in* U, *and every bounded analytic function on* V *extends to* U.

Proof. Without loss of generality we take V to be connected. Consider first the case n = 1. If B is not totally disconnected, there exists a disk

$$D = \{z: |z - z_0| \leq r\}$$

about some point z_0 of V such that the component C of $V \cap \operatorname{int} D$ containing z_0 does not contain all of the set $E = \operatorname{bdry} D$ in its boundary. We may assume that $z_0 \in \Omega$. Let $F = \operatorname{bdry} C$, and let μ_0 be a Jensen measure for z_0 on $\tilde{F}$. Since μ_0 vanishes on $\tilde{B}$, we see that μ_0 is actually a measure on $\tilde{E} \cap \tilde{F}$. Thus the projection of μ_0 onto the complex plane is a Jensen measure ν_0 for z_0 on $E \cap F$. Since $E \cap F$ is a proper closed subset of E, this is impossible. Therefore B is totally disconnected. Hence V is dense in U.

Since B is totally disconnected, if z^0 is any point in V there exists a simple closed curve γ in V surrounding z^0. Let $\mathscr{A}_0$ be the subalgebra of $C(\gamma)$ obtained by restricting the functions in $\mathscr{A}$ to γ, and let $\mathscr{A}_1$ be its closure in $C(\gamma)$. Now z^0 has a Jensen measure for $\mathscr{A}$ on $\tilde{e}$, where $e = \gamma \cup (B \cap \operatorname{int} \gamma)$, and therefore on $\tilde{\gamma} = \gamma$. Thus evaluation at z^0 is a point of the spectrum of $\mathscr{A}_1$. By Wermer's maximality theorem [10], it follows that all functions in $\mathscr{A}_1$ are boundary values of analytic functions on $\operatorname{int} \gamma$. Hence all functions in $\mathscr{A}$ can be extended analytically to $\operatorname{int} \gamma$, as desired.

Consider now the general case $n \geq 1$. Let Ω be the dense G_δ-subset of V consisting of the points having no Jensen measure μ on σ_V with $\mu(\tilde{B}) > 0$. Let P be any 1-dimensional linear variety in $\mathfrak{C}^n$, intersecting Ω in a dense subset of $V \cap P$, and let U_0 be any component of $U \cap P$ containing a point of V. Let $V \cap U_0 = V_0$ and $B_0 = (\operatorname{bdry} V_0) \cap U_0$. Now if B_0 were not of capacity zero relative to V_0 (where V_0 is considered as an open subset of P and P is identified with $\mathfrak{C}^1$), there would exist a Jensen measure μ_z^0 for each z belonging to an open subset Γ of V_0, relative to the algebra $\mathscr{A}_0$ of all bounded analytic functions on V_0, with $\mu_z^0(\tilde{B}_0) > 0$. Let $\omega: \mathscr{A} \to \mathscr{A}_0$ be the restriction map that takes bounded analytic functions on V into bounded analytic functions on V_0. The adjoint map ω^* takes σ_{V_0} into σ_V. Thus $\omega^*(\mu_z^0) = \mu_z$ is a Jensen measure for z relative to the algebra $\mathscr{A}$, and $\mu_z(\tilde{B}) \geq \mu_z^0(\tilde{B}_0) > 0$. Hence $\Gamma \subset V - \Omega$, contrary to the fact that $\Omega \cap P$ is dense in $V \cap P$. Thus B_0 is of capacity 0 relative to V_0. By the case $n = 1$ already considered, B_0 is totally disconnected, V_0 is dense in U_0, and every bounded analytic function on V_0 extends to U_0.

Assume now that V is not dense in U. Then there exists a variety P as above, with the additional property that U_0 contains a point of $\operatorname{int}(U - V)$. This contradicts the fact that V_0 is dense in U_0. Hence V is dense in U.

Consider a bounded analytic function f on V. Let z^0 be any point of B. Choose P as above, and so that in addition it passes through z^0 and intersects V in a point of the component U_0 of $U \cap P$ containing z^0. Since $B \cap U_0$ is totally disconnected, there exists a simple closed curve γ in V_0 surrounding z^0. Take P to have equations $z_2 = \cdots = z_n = 0$. There exists a neighborhood S of 0 in $\mathfrak{C}^{n-1}$ such that

$$\{z: (z_1, 0, \cdots, 0) \in \gamma, (z_2, \cdots, z_n) \in S\}$$

is a subset of V. Thus f extends from V to

$$W = \{z: (z_1, 0, \cdots, 0) \in \operatorname{int} \gamma, (z_2, \cdots, z_n) \in S\}$$

to be an analytic function of the variable z_1 in W and analytic in all variables in $V \cap W$. Thus f is analytic in W, as was to be proved.

THEOREM 4. *Let* U *be a bounded open set in* $\mathfrak{C}^n$, B *a closed subset of* U, A *an analytic subset of* U - B *of pure dimension* k *such that* $B \subset \bar{A}$. *Let* B *be of capacity* 0 *relative to the algebra* $\mathscr{A}$ *of all continuous functions on* $\bar{A}$ *that are analytic on* A. *Let there exist an analytic map* π *of* U *onto a connected open subset* S *of* $\mathfrak{C}^k$ *that is proper on* B, *with* $\pi(B) \neq S$. *Then* $\bar{A} \cap U$ *is an analytic subset of* U.

Proof. We first reduce the problem to the case in which π is proper on $\bar{A}$ and has countable level sets on A. To this end, replace B by the set of those points in $\bar{A}$ at which $\bar{A}$ is not analytic. Then we may assume that $\bar{A}$ is analytic at *no* point of B. If B is void, there is nothing to prove; we therefore assume B is not void, and consider $z^0 \in B$ with

$$\pi(z^0) \in S \cap \text{bdry}\, \pi(B).$$

Since $\pi^{-1}(\pi(z_0)) \cap B$ is compact, there exist a relatively compact open subset U_1 of U and a relatively compact open subset S_1 of S such that

$$\pi(U_1) = S_1 \qquad \text{and} \qquad \pi(B \cap \text{bdry}\, U_1) \subset \text{bdry}\, S_1.$$

By [1, Theorem 1], we can find a mapping π_1 of U into $\mathfrak{C}^k$, uniformly near to π on any compact subset of U, whose level sets on A are all countable. If S_2 is any connected relatively compact subset of S_1 that contains $\pi(z^0)$, we may take π_1 so near to π that

$$\pi_1(B \cap \text{bdry}\, U_1) \cap S_2 = \Box$$

and

$$\pi_1(B) \cap S_2 \neq S_2.$$

Let $U_2 = U_1 \cap \pi_1^{-1}(S_2)$. Then π_1 maps U_2 onto S_2 and

$$\pi_1(B \cap \text{bdry}\, U_2) \cap S_2 = \Box,$$

so that π_1 maps $B \cap U_2$ properly into S_2. Take

$$p \in S_2 \cap \text{bdry}\, \pi_1(B \cap U_2).$$

Now $\Gamma = \pi_1^{-1}(p) \cap (B \cap U_2)$ is compact, and $\pi_1^{-1}(p) \cap (A \cap U_2)$ is countable. There therefore exists a relatively compact open subset U_3 of U_2 such that $\Gamma \subset U_3$ and

$$(\text{bdry}\, U_3) \cap \pi_1^{-1}(p) \cap (A \cup B) = \Box.$$

There thus exists a connected open neighborhood S_3 of p in $\mathfrak{C}^k$ such that

$$\pi_1^{-1}(S_3) \cap (\text{bdry}\, U_3) \cap (A \cup B) = \Box.$$

Write

$$U_4 = \pi_1^{-1}(S_3) \cap U_3.$$

Then π_1 maps U_4 onto S_3 and maps $(A \cup B) \cap U_4$ properly into S_3. Also, $B \cap U_4 \neq \Box$, $\pi_1(B \cap U_4) \neq S_3$. Therefore, if we can prove our theorem for the case where the map π is proper on $\overline{A} \cap U = A \cup B$ and has countable level sets on A, it will follow in the general case.

Assume therefore that π is proper on $\overline{A} \cap U$ and has countable level sets on A. Let K be any component of $S - \pi(B)$, and let $L = S \cap \text{bdry } K$. Then π maps $A \cap \pi^{-1}(K)$ properly onto K. Associated with this map there is a multiplicity λ, such that for each p in K there exist points $p_1, \cdots, p_\lambda$ in A, counted with multiplicities, such that $\pi(p_i) = p$ $(1 \leq i \leq \lambda)$. As in the proof of Lemma 9 above, we see that if a sequence $\{p^n\}$ of points of K converges to a point p in L, then at least one of the points p_i^n converges to B.

Since the set Ω of points in $\overline{A} - B$ not having Jensen measures μ on $\overline{A}$ with $\mu(B) > 0$ is a dense G_δ-set in $\overline{A} - B$, there exists a dense set of points p^0 in K such that $p_1^0, \cdots, p_\lambda^0$ are distinct and such that $p_i^0 \in \Omega$ $(1 \leq i \leq \lambda)$. For $0 < \theta < 1$, for every compact subset D of B, and for $1 \leq i \leq \lambda$ there exists, by Lemma 3 of [1], f_i in $\mathscr{A}$ with

$$1 = f_i(p_i^0) > \|f_i\|^{\theta} \|f_i\|_D^{1-\theta} .$$

Choose D so that $D = B \cap \pi^{-1}(\pi(D))$. Let g_i be any analytic function on $\mathfrak{C}^n$ with

$$g_i(p_i^0) = 1, \qquad g_i(p_j^0) = 0 \quad \text{for } j \neq i .$$

There exist constants $r < 1$, $c_1, \cdots, c_\lambda$ with

$$\|f_i\|^{\theta} < c_i, \qquad \|f_i\|_D^{1-\theta} < rc_i^{-1} .$$

Let ε be any positive number. There exist a constant c and positive integers $N_1, \cdots, N_\lambda$, all depending on ε, such that

$$\|f_i^{N_i}\|^{\theta} \leq c, \qquad \|f_i^{N_i}\|_D^{1-\theta} \leq \varepsilon c^{-1}$$

for $1 \leq i \leq \lambda$. Define

$$g = \sum_{i=1}^{\lambda} g_i f_i^{N_i},$$

and let $\gamma = \max\{\|g_i\| : 1 \leq i \leq \lambda\}$. Then

$$\|g\| < \lambda\gamma c^{\frac{1}{\theta}}, \qquad \|g\|_D \leq \lambda\gamma(\varepsilon^{-1})^{\frac{1}{1-\theta}} .$$

Also, $g(p_i^0) = 1$ $(1 \leq i \leq \lambda)$. Define the analytic function f on K by

$$f(p) = \prod_{i=1}^{\lambda} g(p_i) .$$

Then f is bounded, $f(p^0) = 1$,

(*) $$\|f\| \leq \|g\|^{\lambda},$$

and, since at least one of the points p_i converges to B, as $p \to L$,

$$(\ddagger) \qquad \|f\|_{\tilde{M}} \leq \|g\|^{\lambda-1} \|g\|_D, \qquad \text{where } M = L \cap \pi(D).$$

Write $\alpha = 1 - \lambda(1 - \theta)$. Then $\alpha < 1$, and by taking θ near enough to 1 we can make α arbitrarily near to 1. We compute, from (*) and ($\ddagger$):

$$\begin{aligned} \|f\|^{\alpha} \|f\|_{\tilde{M}}^{1-\alpha} &\leq \|g\|^{\alpha\lambda+(1-\alpha)(\lambda-1)} \|g\|_D^{1-\alpha} \\ &\leq \left(\frac{1}{\lambda\gamma c^{\frac{1}{\theta}}}\right)^{\alpha+\lambda-1} \left(\frac{1}{\lambda\gamma(\varepsilon c^{-1})^{\frac{1}{1-\theta}}}\right)^{1-\alpha} = \tau\varepsilon^{\lambda}, \end{aligned}$$

where τ is a constant. Thus, for ε small enough,

$$\|f\|^{\alpha} \|f\|_{\tilde{M}}^{1-\alpha} < 1 = f(p^0).$$

This shows that p^0 has no Jensen measure μ on σ_K with $\mu(\tilde{M}) \geq 1 - \alpha$. Since M can be an arbitrarily large compact subset of L, and since α can be arbitrary, p^0 does not have a Jensen measure on $\tilde{L}$ with $\mu(\tilde{L}) > 0$. Since p^0 can belong to a dense subset of K, it follows that L is of capacity 0 relative to K. By Lemma 10, K is dense in S and every bounded analytic function in K extends to S. It follows as in the proof of the Remmert-Stein theorem that $\bar{A} \cap U$ is an analytic subset of U, as was to be proved.

Actually, the hypothesis that B is of capacity 0 relative to $\mathscr{A}$ could be replaced by the following slightly weaker assumption, as the proof of Theorem 4 shows. Let $\mathscr{A}_0$ be the algebra of all bounded analytic functions on A, and σ_A the closure of A as a subset of the spectrum σ of $\mathscr{A}_0$. Let $\tilde{B}_0$ consist of all points p in σ_A that lie over a point x in B, in the sense that some net (or filter) in A converges to x in the topology of $A \cup B$ and converges to p in the topology of σ_A. The weaker assumption is then that the set of points of A having no Jensen measure μ for the algebra $\mathscr{A}_0$ with $\mu(\tilde{B}_0) > 0$ is dense in A.

Here is a justification of the definition of capacity given in Definitions 2 and 3 above. The proof is simple but tedious, and we omit it.

Justification. Let Γ be a compact subset of the complex plane, and γ a Jordan curve surrounding Γ, so $\Gamma \subset \operatorname{int} \gamma$. Let $\mathscr{A}$ be the algebra on $X = \gamma \cup \operatorname{int} \gamma$ obtained by taking the closure of all polynomials. Let p be any point of $\operatorname{int} \gamma - \Gamma$. Then the following statements are equivalent.

(a) Γ has capacity 0 in the usual sense,

(b) if μ is any Jensen measure for p on X, relative to the algebra $\mathscr{A}$, then $\mu(\Gamma) = 0$,

(c) Γ has capacity 0 relative to $\mathscr{A}$, in the sense of Definition 2,

(d) Γ has capacity 0 relative to the open set $\operatorname{int} \gamma - \Gamma$, in the sense of Definition 3.

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