A NEW TREATMENT OF THE HAAR INTEGRAL

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We shall present a new proof of the existence and uniqueness of a right invariant integral on a locally compact topological group. We believe that the present approach has at least an intuitive advantage over the classical one (see [1]). The possibility of constructing the Haar integral by the present method was suggested to us by Professor A. M. Gleason.

The reader need know no more than the definition and some simple properties of locally compact topological groups, but a previous acquaintance with the Haar integral would ease the task of reading this note.

Let G be a locally compact group. If f is a real valued function on G, then the closure of the set of points $x \in G$ such that $f(x) \neq 0$ is called the support of f and is denoted by spt(f). Let L be the set of continuous real valued functions on G with compact support, and let L^+ be the subset of L consisting of non-negative functions. For any function f on G and any element $y \in G$ we let $R_y(f)$ denote the function with $R_y(f)(x) = f(xy)$. Note that $R_y(R_z(f)) = R_{yz}(f)$.

DEFINITION 1. *A right invariant integral on* G *is a real valued function* I *defined on* L *such that:*

(1) $I(f + g) = I(f) + I(g)$;

(2) $I(af) = aI(f)$, *where* a *is a real number;*

(3) $f \in L^+$ *and* $f \not\equiv 0$ *imply that* $I(f) > 0$;

(4) $I(R_y(f)) = I(f)$ *for all* $y \in G$ *and* $f \in L$.

We note that if I is defined only on L^+ satisfying these conditions (with a positive in (2)), then I can be extended to a right invariant integral in one and only one way. In the remainder of this note we shall restrict our attention to functions in L^+. All summations in this note are finite.

We shall briefly describe the ideas of our construction of the integral before entering into the details of the exposition. Let $g \in L^+$, $g \not\equiv 0$ be fixed. For a function $f \in L^+$ ($f \not\equiv 0$) we say that g dominates f (written $g \gtrsim f$) if g can be cut up and the pieces right translated and added together in such a way that the new function dominates f pointwise. (That is, g dominates f if we can write $g = \Sigma g_i$ in L^+ and find points x_i in G so that $\Sigma R_{x_i}(g_i) \geq f$ pointwise. The definition we actually adopt is slightly different but equivalent. See Definitions 2 and 3 and statement (2) below Definition 3.) We then consider the sets $\{s \mid sg \lesssim f\}$ and $\{s \mid sg \gtrsim f\}$ of positive real numbers, the first of which is seen to contain all sufficiently small numbers; the second, all sufficiently large numbers. It is shown that these sets do not intersect in more than one point (essentially Theorem A) and that there is no gap between them, except for possibly one point (essentially Theorem B). These facts, although intuitively plausible, are by no means easy to show, and it might come as a surprise to those who are unacquainted with such matters that the first mentioned fact would not necessarily be true if both right and left translation were allowed (see the

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remark below Theorem A). In fact, if both right and left translation are allowed and if G does not possess an integral which is both right and left invariant, then each of the sets $\{s \mid sg \lesssim f\}$ and $\{s \mid sg \gtrsim f\}$ consists of all positive real numbers. The reader is invited to inspect the multiplicative group of all matrices of the form

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \qquad (a > 0)$$

with regard to this effect.

Once the above facts are established, it follows that the closures of the sets $\{s \mid sg \lesssim f\}$ and $\{s \mid sg \gtrsim f\}$ intersect in exactly one point. This point is defined to be the integral of f. The integral, of course, depends on the choice of g, since the integral of g is clearly unity. The facts that this actually defines an integral and that this integral is essentially unique (up to a multiplicative constant) are then easily established.

It might be of interest to note that our treatment makes no use of the axiom of choice, nor would the use of the axiom of choice seem to aid matters at all. Also, as a byproduct of our proof, we obtain an interesting, though not unknown, approximation theorem (Corollary 2 of Lemma 5). This result can be obtained more easily once the existence of an integral is established, and, in fact, it is used implicitly in at least one of the uniqueness proofs in the literature.

We shall now proceed with the details of our constructions.

DEFINITION 2. *If* f *and* g *are in* L^+, *we write* $f \sim g$ *provided there exist functions* $f_1, \cdots, f_n$ *in* L^+ *and elements* $x_1, \cdots, x_n$ *in* G *such that* $f = \Sigma f_i$ *and* $g = \Sigma R_{x_i}(f_i)$.

LEMMA 1. *If* $f, g \in L^+$ *and* $g = \Sigma g_i$ (*in* L^+), *then* $f \sim g$ *iff there are functions* $f_i \in L^+$ *with* $f = \Sigma f_i$ *and* $f_i \sim g_i$.

Proof. Suppose that $f \sim g$, so that there are functions g_j' with $g = \Sigma g_j'$ and $f = \Sigma R_{x_j}(g_j')$ for some points $x_j \in G$. The function $g_{i,j}$ defined by

$$g_{i,j}(x) = \begin{cases} \dfrac{g_i(x)\, g_j'(x)}{g(x)}, & \text{if } g(x) > 0 \\ 0 & , \text{ if } g(x) = 0 \end{cases}$$

is continuous, since $g_{i,j}(x) \leq g(x)$; and, moreover, it follows that $g_i = \Sigma_j g_{i,j}$ and $g_j' = \Sigma_i g_{i,j}$. Define $f_i = \Sigma_j R_{x_j}(g_{i,j}) \sim g_i$. Then we see that

$$\sum_i f_i = \sum_{i,j} R_{x_j}(g_{i,j}) = \sum_j R_{x_j}(g_j') = f.$$

The converse is easy and is left to the reader to prove.

COROLLARY. $\sim$ *is an equivalence relation.*

Proof. Symmetry and reflexivity are trivial. For transitivity suppose that $f \sim g$ and $g \sim h$. Then there exist functions g_i and points y_i with $g = \Sigma g_i$ and $h = \Sigma R_{y_i}(g_i)$. By the Lemma, $f = \Sigma f_i$, where $f_i \sim g_i$ and hence $f_i \sim R_{y_i}(g_i)$. Then, again by the Lemma, $f = \Sigma f_i \sim \Sigma R_{y_i}(g_i) = h$.

LEMMA 2. *Let* $f \in L^+$, *and let* U *be any open subset of* G. *Then there exists a function* $\phi \in L^+$ *with* $\mathrm{spt}(\phi) \subset U$ *such that* $f \sim \phi$.

Proof. It suffices to write $f = \Sigma f_i$ where f_i is such that $\mathrm{spt}(f_i)$ is contained in some right translate of U. Let V be open with $\overline{V}$ compact and $\overline{V} \subset U$. Let $\bigcup_i Vx_i$ be a finite covering of $\mathrm{spt}(f)$. Let $h_i \in L^+$ be such that $h_i(x) = 1$ for $x \in Vx_i$ and such that $\mathrm{spt}(h_i) \subset Ux_i$, and let $h = \Sigma h_i$. Then $h(x) \geq 1$ for $x \in \mathrm{spt}(f)$. Let $f_i = fh_i/h$ (zero outside $\mathrm{spt}(f)$). Then $\Sigma f_i = fh/h = f$ and $\mathrm{spt}(f_i) \subset \mathrm{spt}(h_i) \subset Ux_i$, as was to be shown.

DEFINITION 3. *Let* $f, g \in L^+$. *We write* $f \gtrsim g$ *if there exist functions* $f' \sim f$ *and* $g' \sim g$ *such that* $f'(x) \geq g'(x)$ *for all* x.

Some easily verified properties of this relation are listed below. The parenthetical remarks are intended to aid the reader with some of the less obvious portions of the proofs.

(1) $f \gtrsim g$ iff $f = f_1 + f_2$ (in L^+) with $f_1 \sim g$.

(With the notation of the definition, if $f \gtrsim g$, then $f \sim f' = g' + h$ for some $h \in L^+$. By Lemma 1, $f = f_1 + f_2$, where $f_1 \sim g' \sim g$ and $f_2 \sim h$.)

(2) $f \gtrsim g$ iff there exists an $f' \sim f$ with $f'(x) \geq g(x)$ for all x.

(Let $f' = g + f_2$, where f_2 is as in (1).)

(3) $f \gtrsim g$ and $g \gtrsim h$ imply that $f \gtrsim h$.

($g \gtrsim h$ implies $g = g_1 + g_2$ with $g_1 \sim h$. By (2), $f \gtrsim g_1$, so that $f = f_1 + f_2$ with $f_1 \sim g_1 \sim h$. Thus $f \gtrsim h$ by (1).)

(4) For any $f, g \in L^+$ with $g \not\equiv 0$ there exists a number s such that $sg \gtrsim f$.

(Apply Lemma 2 to the set $U = \{x \mid g(x) > 0\}$.)

(5) $f_i \gtrsim g_i$ $(i = 1, 2)$ imply that $f_1 + f_2 \gtrsim g_1 + g_2$.

(6) $f \gtrsim g$ and $s > 0$ imply that $sf \gtrsim sg$.

THEOREM A. *If* $f \in L^+$, $f \not\equiv 0$, *then* $f \gtrsim sf$ *implies that* $s \leq 1$.

The proof will be given later.

Remark. If one allows both left and right translation in all our definitions, then Theorem A would be false in general. In fact it is easily seen that Theorem A would hold if and only if there is an integral on G which is both left and right invariant.

The following corollary is of interest, but will not be used.

COROLLARY. $f \gtrsim g$ *and* $g \gtrsim f$ *imply that* $f \sim g$.

Proof. If $f \sim g$ is false, then $g \gtrsim f$ implies that $g = g_1 + g_2$ with $g_1 \sim f$ and $g_2 \not\equiv 0$. Then $g_2 \gtrsim \varepsilon f$ for some $\varepsilon > 0$ by (4), and then

$$f \gtrsim g = g_1 + g_2 \gtrsim f + \varepsilon f = (1 + \varepsilon)f,$$

contrary to Theorem A.

Theorem A implies immediately that if $f, g \in L^+$ with $g \not\equiv 0$, then

$$\sup\{s \mid sg \lesssim f\} \leq \inf\{s \mid sg \gtrsim f\} .$$

We claim that these quantities are, in fact, equal, and we will define the integral of f to be their common value (with g fixed).

THEOREM B. *If* $f, g \in L^+$, $g \not\equiv 0$, *and* $\varepsilon > 0$, *then there exists a number* t *such that* $f \lesssim tg \lesssim (1+\varepsilon)f$.

The proof of this theorem will be given later.

COROLLARY. $\sup\{s \mid sg \lesssim f\} = \inf\{s \mid sg \gtrsim f\}$.

Proof. If t is as in Theorem B, then

$$\inf\{s \mid sg \gtrsim f\} \leq t \leq \sup\{s \mid sg \lesssim (1+\varepsilon)f\} = (1+\varepsilon)\sup\{s \mid sg \lesssim f\},$$

and this, together with the inequality resulting from Theorem A, yields the equality.

DEFINITION 4. *Let* $g \in L^+$, $g \not\equiv 0$. *For* $f \in L^+$ *let*

$$I_g(f) = \sup\{s \mid sg \lesssim f\} = \inf\{s \mid sg \gtrsim f\}.$$

THEOREM (Existence). I_g *is a right invariant integral.*

Proof. We need only show additivity. Using the definition $I_g(f) = \inf\{s \mid sg \gtrsim f\}$, we see that if $sg \gtrsim f_1$ and $tg \gtrsim f_2$, then $(s+t)g \gtrsim f_1 + f_2$, and hence

$$s + t \geq I_g(f_1 + f_2).$$

It follows that $I_g(f_1) + I_g(f_2) \geq I_g(f_1 + f_2)$. The opposite inequality is proved in a similar manner using the definition $I_g(f) = \sup\{s \mid sg \lesssim f\}$.

THEOREM (Uniqueness). *If* I *is any right invariant integral and if* $g \in L^+$ *with* $g \not\equiv 0$, *then* $I = I(g)\, I_g$.

Proof. We may assume for convenience that $I(g) = 1$. Then we must show that $I(f) = I_g(f)$ for all $f \in L^+$. Clearly, $f \sim h$ implies that $I(f) = I(h)$. Also, $f(x) \leq h(x)$ for all x implies that $I(f) \leq I(h)$. Thus we see that $f \lesssim h$ implies that $I(f) \leq I(h)$. Now, if $sg \gtrsim f$, then $s = sI(g) \geq I(f)$. Thus

$$I_g(f) = \inf\{s \mid sg \gtrsim f\} \geq I(f).$$

But, also, $tg \lesssim f$ implies that $t = tI(g) \leq I(f)$, and thus

$$I_g(f) = \sup\{t \mid tg \lesssim f\} \leq I(f).$$

Thus $I_g(f) = I(f)$, as was to be shown.

The remainder of this note will be devoted to the proofs of Theorems A and B. To motivate the procedure we note that Theorem A is trivial if a right invariant integral exists. Thus we shall replace the argument in which we use an integral by one in which we use the operation of adding the values of a function at "almost right equally spaced points." To prove Theorem B, it will be seen to be sufficient to approximate the function f uniformly by a linear combination $\Sigma\, c_i R_{x_i}(\phi)$, where $\phi \sim g$, since $\Sigma\, c_i R_{x_i}(\phi) \sim (\Sigma\, c_i)\phi \sim (\Sigma\, c_i)g$. Again we try to "approximate" the argument tha if there were to exist a *left* invariant integral, then, if $\mathrm{spt}(\phi)$ were contained in a sufficiently small neighborhood of e, f(x) would be close to

$$\frac{\int f(y^{-1})\,\phi(xy)\,dy}{\int \phi(y)\,dy}.$$

On the other hand, the latter is approximately equal to

$$\frac{\sum f(y_i^{-1})\,\phi(xy_i)}{\sum \phi(y_i)},$$

where the y_i are "almost *left* equally spaced points" sufficiently close together. Alternatively one could prove these theorems using the approximation to an integral employed in the usual treatment of the Haar integral. Such an approximation is invariant and almost additive, whereas ours is additive and almost invariant. We now begin the details of the proof. The intuitive notion of "almost left equally spaced points" is made precise by the next lemma.

LEMMA 3. *Let* C *be a compact subset of* G *and* N *be a compact neighborhood of the identity element* $e \in G$. *Let* $U \subset N$ *be a symmetric open neighborhood of* e, *and let* $\{x_i U\}_{i \in I}$ *be a covering of* CN *by left translates of* U *with a minimal number of elements. For any* $x \in G$ *let* $J = J(x) = \{j \in I \mid xx_j \in C\}$. *Then there exists a one to one mapping* $\sigma\colon J \to I$ *(into) such that* $x_{\sigma(j)} \in xx_j U^2$ *for all* $j \in J$.

Proof. Let $j_1, \cdots, j_n$ be distinct elements of J, and assume that

$$xx_{j_1} U \cup \cdots \cup xx_{j_n} U$$

meets the sets $x_{i_1} U, x_{i_2} U, \cdots, x_{i_m} U$ and no others. Then

$$\bigcup_{k=1}^{n} xx_{j_k} U \subset \bigcup_{k=1}^{m} x_{i_k} U$$

since $xx_j U \subset CU \subset CN$ for $j \in J$. Thus $n \leq m$, for otherwise one could replace the sets $x_{j_k} U$ $(k = 1, \cdots, n)$ by the sets $x^{-1} x_{i_k} U$ $(k = 1, \cdots, m)$ and obtain a smaller covering of CN.

By the "marriage problem" lemma below, we can find a one to one mapping $\sigma\colon J \to I$ such that $xx_j U \cap x_{\sigma(j)} U \neq \emptyset$. Thus, since U is symmetric, $x_{\sigma(j)} \in xx_j U^2$, as was to be shown.

LEMMA 4 (The marriage problem). *For a certain finite collection of boys and girls say that every subset of* k *boys* $(k = 1, 2, \cdots)$ *know among them at least* k *girls. Then it is possible to marry each boy to a girl that he knows (not violating the law against bigamy and assuming that everyone is willing).*

The precise mathematical formulation and the proof of this well-known lemma are left to the reader.

LEMMA 5. *Let* K *be a compact subset of* G, *let* $\varepsilon > 0$, *and let* $g \in L^+$ *with* $g \not\equiv 0$. *There exist points* $x_i \in G$ *such that*

$$\left| \frac{\sum g(xx_i)}{\sum g(x_i)} - 1 \right| < \varepsilon$$

for all $x \in K$. *Moreover the* x_i *can be chosen so that the above inequality is simultaneously true for any finite number of given functions* g.

Proof. Since the last statement follows easily from the proof, we shall restrict attention to the case of one function g.

For any two sets U, V with non-empty interior and compact closure we let $[U, V]$ be the least number of left translates of U needed to cover V. Clearly, $[U, W] \leq [U, V][V, W]$.

Let $C = \mathrm{spt}(g) \cup K^{-1}(\mathrm{spt}(g))$. Let $a = \sup\{g(x) \mid x \in G\}$, and put

$$V = \{x \mid g(x) \geq 2a/3\}$$

and $W = \{x \mid g(x) \geq a/3\}$. Let N be a compact neighborhood of e in G such that $VN \subset W$. Choose $\delta > 0$ such that $\delta < (a\varepsilon)/(3[V, CN])$. We can find a symmetric neighborhood U of e with $U \subset N$ such that $x^{-1}y \in U^2$ implies that $|g(x) - g(y)| < \delta$.

With these choices let $\{x_i U\}_{i \in I}$ be a minimal covering of CN.

First, using only the fact that $C \supset \mathrm{spt}(g)$, we see that if $g(xx_i) \neq 0$, then $xx_i \in C$, and hence $i \in J = J(x)$ (x fixed, but arbitrary). Thus $x_{\sigma(i)} \in xx_i U^2$ which implies that $|g(x_{\sigma(i)}) - g(xx_i)| < \delta$. Thus it follows that

$$\sum_{i \in I} g(xx_i) \leq \sum_{i \in J} (g(x_{\sigma(i)}) + \delta) \leq \sum_{i \in I} (g(x_i) + \delta) = \sum_{i \in I} g(x_i) + \delta[U, CN].$$

However, notice that if $x_i U \cap V \neq \emptyset$, then $x_i \in VU \subset VN \subset W$, and hence

$$\sum g(x_i) \geq [U, V]a/3 .$$

Thus

$$\frac{\delta[U, CN]}{\sum g(x_i)} \leq \frac{3\delta[U, CN]}{a[U, V]} \leq \delta \frac{3[V, CN]}{a} < \varepsilon .$$

We now see that

$$\frac{\sum g(xx_i)}{\sum g(x_i)} < 1 + \varepsilon ,$$

using only the fact that $C \supset \mathrm{spt}(g)$.

Second, choose $x \in K$, and put

$$C' = xC = x\,\mathrm{spt}(g) \cup xK^{-1}\,\mathrm{spt}(g) \supset \mathrm{spt}(g) .$$

The sets $xx_i U$ give a minimal covering of $xCN = C'N$. Put $y_i = xx_i$. Then, as above, we see that

$$\sum g(x^{-1} y_i) \leq \sum g(y_i) + \delta[U, CN]$$

(since $[U, CN] = [U, C'N]$). Thus

$$\sum g(xx_i) \geq \sum g(x_i) - \delta[U, CN],$$

and it follows that

$$\frac{\sum g(xx_i)}{\sum g(x_i)} > 1 - \varepsilon .$$

COROLLARY 1. *If* $f \sim g$, *then, given* $\varepsilon > 0$, *there exist points* $y_j \in G$ *such that*

$$\left| \frac{\sum g(y_j)}{\sum f(y_j)} - 1 \right| < \varepsilon .$$

Proof. Let $f = \Sigma f_i$ and $g(x) = \Sigma f_i(xx_i)$ for all x. We shall use Lemma 5 in its equivalent "other sided" form for the functions f_i and $K = \{x_i\}$ to obtain points $y_j \in G$ such that

$$\left| \frac{\sum_j f_i(y_j x_i)}{\sum_j f_i(y_j)} - 1 \right| < \varepsilon \quad \text{for all } i .$$

It follows that

$$\left| \frac{\sum_j g(y_j)}{\sum_j f(y_j)} - 1 \right| = \left| \frac{\sum_i \sum_j f_i(y_j x_i)}{\sum_i \sum_j f_i(y_j)} - 1 \right| < \varepsilon$$

since, for positive a_i and b_i, $|(a_i/b_i) - 1| < \varepsilon$ for all i, implies that

$$|\sum a_i - \sum b_i| \leq \sum |a_i - b_i| < \varepsilon \sum b_i$$

and hence that $|(\Sigma a_i/\Sigma b_i) - 1| < \varepsilon$.

Proof of Theorem A. Suppose $f \gtrsim sf$, that is, $f \sim g$ with $g(x) \geq sf(x)$ for all x. Given $\varepsilon > 0$, Corollary 1 shows that there are points y_i such that

$$1 + \varepsilon \geq \frac{\sum g(y_i)}{\sum f(y_i)} \geq \frac{s\sum f(y_i)}{\sum f(y_i)} = s .$$

This must be true for all $\varepsilon > 0$, and thus $s \leq 1$ as claimed.

COROLLARY 2. *Given* $f \in L^+$ *and* $\varepsilon > 0$, *there exists a symmetric neighborhood* U *of* e *such that for any* $g \in L^+$ $(g \neq 0)$ *with* $\operatorname{spt}(g) \subset U$ *we can find constants* $c_i \geq 0$ *and points* $y_i \in G$ *such that*

$$|f(x) - \sum c_i g(xy_i)| < \varepsilon$$

for all $x \in G$. *Also it may be assumed that* $\operatorname{spt}(\Sigma c_i R_{y_i}(g)) \subset U^2 \operatorname{spt}(f)$.

Proof. Let U be symmetric and such that $xy^{-1} \in U$ implies that

$$|f(x) - f(y)| < \varepsilon'$$

(ε' is to be chosen later). Take K in Lemma 5 to contain $U^2\,\mathrm{spt}(f)$. Let g have support in U, and apply Lemma 5 to obtain points $y_i \in G$ such that

$$\left|\frac{\sum g(xy_i)}{\sum g(y_i)} - 1\right| < \varepsilon'' \quad \text{for all } x \in K.$$

Then, for $x \in K$,

$$\left| f(x) - \frac{\sum f(x)\, g(xy_i)}{\sum g(y_i)} \right| \leq f(x)\varepsilon''.$$

Also, noting that $g(xy_i) \neq 0$ implies that $xy_i \in U$ and hence that

$$|f(x) - f(y_i^{-1})| < \varepsilon',$$

we conclude

$$\left| \frac{\sum f(x)\, g(xy_i)}{\sum g(y_i)} - \frac{\sum f(y_i^{-1})\, g(xy_i)}{\sum g(y_i)} \right| \leq \frac{\sum |f(x) - f(y_i^{-1})|\, g(xy_i)}{\sum g(y_i)}$$

$$\leq \varepsilon' \frac{\sum g(xy_i)}{\sum g(y_i)} \leq \varepsilon'(1 + \varepsilon'').$$

Thus

$$\left| f(x) - \frac{\sum f(y_i^{-1})\, g(xy_i)}{\sum g(y_i)} \right| \leq f(x)\varepsilon'' + \varepsilon'(1 + \varepsilon'') < \varepsilon$$

if ε' and ε'' are suitably chosen. We define $c_i = f(y_i^{-1})/\Sigma\, g(y_j)$.

Now, if $x \notin U^2\,\mathrm{spt}(f)$ but $x \in \mathrm{spt}(R_{y_i}(g)) \subset Uy_i^{-1}$, then $y_i \in x^{-1}U$; hence $\mathrm{spt}(R_{y_i}(g)) \subset U^2 x$. Thus $\mathrm{spt}(R_{y_i}(g)) \cap \mathrm{spt}(f) = \emptyset$, and we may discard the function $R_{y_i}(g)$. With this understanding we see that the inequality $|f(x) - \Sigma\, c_i\, g(xy_i)| < \varepsilon$ is trivial for $x \notin U^2\,\mathrm{spt}(f)$ and hence for $x \notin K$.

LEMMA 6. *If* $f, g \in L^+$, $f(x) \geq g(x)$ *for all* x, *and* $f \not\equiv g$, *then there exists a function* $f' \sim f$ *such that* $f'(x) > g(x)$ *for all* $x \in \mathrm{spt}(g)$.

Proof. $h = f - g \in L^+$, and h is positive on some open set U. Let

$$\mathrm{spt}(g) \subset \bigcup_{i=1}^{n} Ux_i^{-1},$$

and put $f' = g + \Sigma\, R_{x_i}(h)/n$.

Proof of Theorem B. By Lemma 6, there are functions $h \sim (1 + \varepsilon/2)f$ and $k \sim (1 + \varepsilon)f$ such that $h(x) > f(x)$ for $x \in \mathrm{spt}(f)$ and $k(x) > h(x)$ for $x \in \mathrm{spt}(h)$. Let C be a compact set with $\mathrm{spt}(h) \subset$ interior (C) and $C \subset$ interior $(\mathrm{spt}(k))$, and let U be a symmetric neighborhood of e such that $U^2\,\mathrm{spt}(h) \subset C$. Let $\phi \sim g$ be such that $\mathrm{spt}(\phi) \subset U$.

Let $\delta > 0$ be smaller than both

$$\inf\{h(x) - f(x) \mid x \in \operatorname{spt}(f)\} \text{ and } \inf\{k(x) - h(x) \mid x \in C\}.$$

By Corollary 2 of Lemma 5, there exist constants $c_i \geq 0$ and points y_i such that

$$\left|h(x) - \sum c_i \phi(xy_i)\right| < \delta \text{ and } \operatorname{spt}\left(\sum c_i R_{y_i}(\phi)\right) \subset U^2 \operatorname{spt}(h) \subset C.$$

Then we see that $f(x) \leq \Sigma\, c_i\, \phi(xy_i) \leq k(x)$ for all $x \in G$ so that $f \lesssim tg \lesssim (1 + \varepsilon)f$ for $t = \Sigma\, c_i$.

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