ENTIRE FUNCTIONS ON BANACH ALGEBRAS

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1. INTRODUCTION

If f is an entire function and A is a (complex) Banach algebra with unit, then for any x in A the element f(x) is well defined so that f may be regarded in a natural way as a mapping of A into itself. The question "When does f map A *onto* itself?" was raised by Kurepa, who gave [2] a necessary and sufficient condition on f in order that it should map the algebra of all $n \times n$ matrices onto itself. This condition may be formulated as follows:

(K) For every complex number α, there exists a ζ such that $f(\zeta) = \alpha$ and $f'(\zeta) \neq 0$.

Thus, for instance, the equation $X^2 = Y$ cannot be solved for X in $n \times n$ matrices (in general), while the equation $X^3 - X = Y$ can be. It should be noted that (K) implies the trivially necessary condition

(N) For every α, there exists a ζ such that $f(\zeta) = \alpha$; that is, f omits no finite value,

which must be satisfied if the range of f is even going to contain the scalars.

In this note we take this question up for certain other algebras, notably for the algebra of bounded operators on (infinite dimensional) Hilbert space. In the sequel, D_r is used consistently to denote the open disc in the plane with center 0 and radius r, except that, for simplicity, we write D instead of D_1 for the unit disc.

2. MULTIPLES OF THE SHIFT

Outstanding among the operators Y for which the equation $f(X) = Y$ is hard to solve is the unilateral shift, and all our computations will be focussed on that fact. Consequently, in order to avoid minor difficulties, it will be convenient to begin with the separable case.

Let H be a Hilbert space with a fixed orthonormal basis $\{e_k\}_{k=0}^{\infty}$, and denote by V the shift operator defined by $Ve_k = e_{k+1}$ $(k = 0, 1, \cdots)$. If A is any bounded operator commuting with V and if we write

$$Ae_0 = a = \sum \alpha_k e_k ,$$

then $Ae_k = AV^k e_0 = V^k a$ so that the matrix of A has the form

Received September 12, 1962.

Research on this work was supported, in part, by the U.S. Air Force Office of Scientific Research.

$$\begin{pmatrix} \alpha_0 & 0 & 0 & \ddots \\ \alpha_1 & \alpha_0 & 0 & \ddots \\ \alpha_2 & \alpha_1 & \alpha_0 & \ddots \\ \ddots & \ddots & & \ddots \end{pmatrix}.$$

We express this relation by writing, formally, $A \sim \Sigma\alpha_k V^k$.

Now let $f(\zeta) = \Sigma\beta_k \zeta^k$ be the given entire function, and suppose that for some bounded operator X,

$$f(X) = rV,$$

where r is a fixed positive real number. Then, of course, X commutes with V. Let $X \sim \Sigma\, \xi_k V^k$. Since $\Sigma\, |\xi_k|^2 < \infty$ the series $\Sigma\, \xi_k \zeta^k$ converges and defines a regular function g in D. The function $h = f \circ g$ is then also analytic in D and has a Taylor expansion

$$h(\zeta) = \sum \sigma_k \zeta^k.$$

But now, as is not hard to see, $f(X) \sim \Sigma\, \sigma_k V^k$; and, since by hypothesis $f(X) = rV$, it must be that $h(\zeta) = f(g(\zeta)) = r\zeta$ identically in D.

The claim $f(X) \sim \Sigma\, \sigma_k V^k$ may be established as follows. Direct computation shows the result holds if f is a power of ζ or, more generally, a polynomial. Let $P_n(\zeta) = \Sigma_{k=0}^{n} \beta_k \zeta^k$ and write

$$h_n(\zeta) = P_n(g(\zeta)) = \sum \sigma_{k,n} \zeta^k.$$

Then, as noted, $P_n(X) \sim \Sigma\, \sigma_{k,n} V^k$; and since $P_n(X) \to f(X)$ uniformly, the matrix entries $\sigma_{k,n}$ certainly tend, as $n \to \infty$, to those of f(X). But also $h_n \to h$ uniformly on compact sets in D so that $\lim_{n\to\infty} \sigma_{k,n} = \sigma_k$ for each k.

It now follows that g is a *schlicht* mapping of D onto some domain U in the complex plane and that f, in turn, is a *schlicht* map of U back onto D_r. We have proved the following lemma.

LEMMA 1. *If* f *is entire and if* $f(X) = rV$ *possesses a bounded operator solution, then there exists in the plane a domain* U *that is mapped by* f *conformally and homeomorphically onto the disc* D_r.

THEOREM 2. *Let* f *be an entire function, and let* $\mathfrak{B}$ *denote the algebra of all bounded operators on* H. *Then the following are equivalent conditions on* f.

(i) $f(\mathfrak{B}) = \mathfrak{B}$,

(ii) $f(\mathfrak{B})$ *contains all multiples* rV *of the shift* V,

(iii) f *satisfies the condition*

(U) *There exist a sequence* $\{W_n\}$ *of domains in the plane and a sequence* $\{r_n\}$ *of positive numbers tending to* $+\infty$ *such that, for each* n, f *is a* schlicht *map of* W_n *onto* D_{r_n}.

Moreover, unless f *is linear, the domains* W_n *in* (iii) *may be chosen to be disjoint.*

Note. Condition (U), which clearly says only that f maps certain domains one-to-one and conformally onto arbitrarily large discs, has been stated here in this fashion for the convenience of the proof. It is worth remarking that the possibility of choosing the W_n disjoint shows that if f is nonlinear and satisfies (U) then its Riemann surface has, over every finite point, an infinite number of *schlicht* sheets.

Proof. (ii) $\Rightarrow$ (iii). Suppose f is not linear. Let $r = 1, 2, \cdots$, successively, in the lemma, and choose $U_1, U_2, \cdots$ accordingly. If $m < n$ and if $U_m \cap U_n = W \neq 0$, then the regular functions $(f|U_m)^{-1}$ and $(f|U_n)^{-1}$ agree on the open set $f(W) \subset D_m$ and must therefore agree throughout D_m, whence $U_m \subset U_n$. In other words, either $U_m \subset U_n$ or else the domains are disjoint. Moreover, there can be no infinite nested sequence of U_n's. Indeed, suppose

$$U_{n_1} \subset U_{n_2} \subset \cdots \subset U_{n_k} \subset \cdots$$

with $n_1 < n_2 < \cdots$. Let

$$U = \bigcup_k U_{n_k}.$$

Then f maps U homeomorphically and conformally onto the whole plane, so that $(f|U)^{-1}$ is entire and also one-to-one on the whole plane. Such a function is necessarily linear. Therefore, f agrees on U with a linear function and must be linear itself, contrary to assumption. It follows that, given any finite collection $n_1 < n_2 < \cdots < n_k$ of integers, there exists an $n > n_k$ such that U_n is disjoint from each of $U_{n_1}, \cdots, U_{n_k}$, and we proceed by induction. Let $W_1 = U_1$, $r_1 = 1$, and if $W_i = U_{n_i}$ and $r_i = n_i$ have already been chosen for $i = 1, \cdots, k$, let n_{k+1} be the first $n > n_k$ for which U_n is disjoint from $W_1, \cdots, W_k$ and define $W_{k+1} = U_{n_{k+1}}$, $r_{k+1} = n_{k+1}$.

(iii) $\Rightarrow$ (i). For any fixed Y in $\mathfrak{B}$ select n to be such that $r_n > \|Y\|$, and let

$$g = (f|W_n)^{-1}$$

so that g is analytic on D_{r_n}. If $X = g(Y)$, then clearly $Y = f(X)$.

COROLLARY. *The only polynomials* P *such that* $P(\mathfrak{B})$ *contains all multiples of* V *are the linear polynomials.*

It is a simple matter to dispose of the assumption that H is separable. Indeed, let $H = H_0 \oplus H_1$ with H_0 separable and consider the operator $\tilde{V} = V \oplus 0$. If $\tilde{A}$ is an operator on H that commutes with $\tilde{V}$, then, writing $\tilde{A}$ as a 2×2 matrix,

$$\tilde{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

we obtain the relations $AV = VA$ and $VB = CV = 0$ so that, in particular, $B = 0$. But then, for two such operators

$$\tilde{A}_1 = \begin{pmatrix} A_1 & 0 \\ C_1 & D_1 \end{pmatrix}, \quad \tilde{A}_2 = \begin{pmatrix} A_2 & 0 \\ C_2 & D_2 \end{pmatrix},$$

$$\tilde{A}_1 A_2 = \begin{pmatrix} A_1 A_2 & 0 \\ * & * \end{pmatrix};$$

and the proof can be formulated as before.

The simplest example of a nonlinear entire function satisfying (U) seems to be

$$f(\zeta) = e^{\zeta} \sin \zeta .$$

(I am indebted for this example to John Wermer and Richard Askey.) That the function does, in fact, have the desired property may be seen as follows. Since $f'(0) = 1$, there certainly exists some domain $U = U_1$ about 0 that is mapped *schlicht* onto a small disc D_s. But then, since

$$f(\zeta + 2n\pi) = e^{2n\pi} f(\zeta),$$

the domain $U_{n+1} = U_1 + 2n\pi$ maps in the same fashion onto the disc of radius $e^{2n\pi} s$.

3. ADEQUATE ALGEBRAS

The above proof that (iii) $\Rightarrow$ (i) is clearly perfectly general. Thus if f has property (U) it is universally onto in the sense that it maps every Banach algebra onto itself. Let us, for the nonce, call a Banach algebra *adequate* if it has the property that the only entire functions mapping it onto itself satisfy (U) and consequently do the same to every Banach algebra. Thus, the foregoing shows that the algebra $\mathscr{B}$ of bounded operators on a Hilbert space is adequate if H is infinite dimensional but is inadequate otherwise. In the balance of this note we consider some further examples of adequate and inadequate algebras.

Although the proof of Theorem 2 was so phrased as to make no use of the fact, it is well known that the correspondence indicated there by "$A \sim \Sigma \alpha_k V^k$" is, in effect, an isomorphism between the commutant of V in $\mathscr{B}$ and the algebra H^∞ consisting of the functions bounded and holomorphic in D. Thus the same argument serves to show that H^∞ is also adequate. Indeed, the argument is even easier when we start with an explicitly given function algebra: f acts by composition, and we simply express the functions $r\zeta$ in the form f(x). The same trick shows that the algebra A_0 of the continuous functions on $\overline{D}$ that are analytic in D is adequate, while only a slight refinement of the same idea is needed to prove the next result.

THEOREM 3. *Any Dirichlet algebra is adequate if* (1) *it is generated by a single element and* (2) *its maximal ideal space contains at least one analytic disc.* (See [3].)

There are also plenty of adequate commutative algebras that do not possess a single generator.

THEOREM 4. *If* X *is any compact subset of the plane having non-empty interior, then* C(X) *is adequate.*

Proof. Choose r and σ so that $y(\zeta) = r\zeta + \sigma$ maps X onto a large disc, and then solve $f(x) = y$.

Somewhat more interesting is the following.

THEOREM 5. *The algebra* C_0 *of all functions continuous on the unit circle is adequate.*

Proof. Consider, once again, the function $y \in C_0$ defined by $y(\zeta) = r\zeta$, $|\zeta| = 1$, and solve the equation $f(x) = y$. The range of x is a simple Jordan loop which is mapped by f in one-to-one fashion onto the circle with center 0 and radius r. But then, according to the argument principle (see, for example, [1, p. 187]), f is a *schlicht* mapping of the interior domain of the loop onto D_r.

4. TWO INADEQUATE ALGEBRAS

If X is totally disconnected then C(X) is certainly not adequate. The facts concerning this special situation are essentially well known, but we write down the details nonetheless.

THEOREM 6. *Let* X *be a totally disconnected compact Hausdorff space, and let* f *be an analytic function on some domain. Then the equation* $f(x) = y$ *can be solved in* C(X) *if only the range of* y *is contained in that of* f. *In particular, if* f *is entire, then* f *maps* C(X) *onto itself if and only if it satisfies* (N).

Proof. Our space has a base of compact-open sets, whence it follows that it suffices to solve the problem locally. On the other hand, if $f(\zeta_0) = a_0$, then in a suitably chosen neighborhood U of ζ_0,

$$f(\zeta) = (g(\zeta))^n + \alpha_0 ,$$

where g is a *schlicht* map of U onto some disc D_s and n is the index of the first nonzero Taylor coefficient (after α_0) in the expansion of f about ζ_0. The result is thus reduced to the following special case.

LEMMA 7. *The equation* $y = x^n$ *always has a solution in* C(X) *if* X *is totally disconnected.*

So far as I know, this fact was first observed by Kaplansky.

Proof. Let $Q = \{\xi \in X | y(\xi) = 0\}$, and suppose there is given a continuous function $\tilde{x}$ defined on X - Q and satisfying the equation $\tilde{x}(\xi)^n = y(\xi)$ there. Then the function x obtained by extending $\tilde{x}$ to be 0 on Q is continuous, since

$$\lim_{\tau \to +0} \tau^{1/n} = 0 ,$$

and therefore provides a solution of the equation. But now, Q is a compact G_δ set in X and it follows at once that X - Q can be written as the union of an ascending sequence of compact-open sets. Finally, taking the differences of successive pairs of these sets enables us to specialize to the case of a function $y \in C(X)$ which is nowhere zero. But then the equation is locally solved by means of analytic functions.

Another interesting special case is the algebra C_1 of all functions continuous on the unit interval [0, 1]. The following lemma was called to my attention in a conversation with Carl Pearcy and Burton Randol.

LEMMA 8. *The equation* $y = x^n$ *always has a solution in* C_1.

Proof. Just as before, it suffices to define a solution on X - Q, and in this case we deal with the complementary open intervals by straightforward analytic continuation.

Now suppose, once again, that f is entire, and let y be an element of C_1, that is, a continuous arc in the plane. If $0 \leq \tau_0 \leq 1$ and $f(\zeta_0) = x(\tau_0)$, then, applying the lemma, fore and aft as required, we see that the equation $y = f(x)$ can still be solved locally: there exists a function x, defined and continuous on some subinterval of [0, 1] comprising a relative neighborhood of τ_0, and satisfying the equation $f(x(\tau)) = y(\tau)$ there. Suppose now that f satisfies (N) and choose a maximal element in the partially ordered set of such local solutions. If α, β are the left and right end-points of the interval on which x_0 is defined, then an easy and familiar argument shows that either x_0 is defined at β, in which case $\beta = 1$, or else

$$\lim_{\tau \to \beta - 0} |x_0(\tau)| = +\infty.$$

Similarly, either α is in the domain of definition of x_0, in which case $\alpha = 0$, or else $\lim_{\tau \to \alpha + 0} |x_0(\tau)| = +\infty$. Thus, if it should be that $y \notin f(C_1)$, we would be in the following bizarre situation: no matter what initial point τ_0 is chosen, and no matter how our local solution is continued from τ_0, it retreats down an asymptotic path of f. In any case, the following result obtains.

THEOREM 9. *If* f *is entire, satisfies* (N), *and has no finite asymptotic values, then* f *maps* C_1 *onto itself.*

We offer a final observation which we found to be amusing. We have seen that C_0 is adequate, while C_1 is not. Thus it is possible for an inadequate algebra to possess an adequate subalgebra of co-dimension 1.

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