SOME REMARKS ON UNICITY AND CONTINUITY THEOREMS FOR ORDINARY DIFFERENTIAL EQUATIONS

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1. INTRODUCTION

In the recent text of Birkhoff and Rota on differential equations [1] proofs of unicity and continuity theorems are referred to properties of functions f that satisfy a differential inequality such as

$$f' \leq Kf, \tag{1.1}$$

where K is a constant. This book considers ordinary differential equations for the case of finite dimensional vector spaces. The same approach is to be found also in [5] for the treatment of unicity theorems. The object of the present note is to make and justify the observation that properties of continuous functions satisfying a simple extension of (1.1) permit us to treat in a straightforward manner unicity and continuity questions for first order ordinary differential equations with solutions taking values in a given normed vector space. It will be seen that the present approach treats unicity and continuity questions where the Lipschitz condition is replaced by a condition of Osgood type (see [4, pp. 146-148]) and the criterion is of the "almost everywhere" kind envisaged by Carathéodory [2, p. 673]. Results of the same nature—more restricted but, by way of compensation, tractable by elementary means—are also obtained when the Lebesgue aspects of the criterion are suppressed and the function g is taken to be the constant one. For a general study of unicity and continuity theorems reference is made to [3].

I am greatly indebted to Professor Cesari and the referee for their valuable suggestions.

2. A BASIC THEOREM

We consider a continuous real-valued function f on a bounded closed interval $\{a \leq x \leq b\}$. We denote the lower right derivate of f at x by $\rho f(x)$ and the lower left derivate of f at x by $\lambda f(x)$. We suppose that g is an integrable function on $\{a \leq x \leq b\}$ which takes values in the extended real line and that

$$G(x) = \int_a^x g(t)\, dt. \tag{2.1}$$

Let ϕ denote a function taking finite real values, the domain of which consists of the finite reals, and which satisfies:

(1) $\phi'(x)$ exists and is nonnegative, $x \neq 0$;

(2) $\lim_{x \downarrow 0} \phi(x) = -\infty$, $\lim_{x \uparrow 0} \phi(x) = +\infty$;

(3) $\phi(0) = 0$.

Received October 8, 1962.

The following theorem will be seen to serve as a base for unicity and continuity studies.

THEOREM 1. *Suppose that* $\rho f(x) < +\infty$ *when* $f(x) \neq 0$ $(a < x < b)$ *and that for almost all* x *for which* $f(x) \neq 0$,

$$\rho\phi \circ f(x) \leq g(x). \tag{2.2}$$

Then each of the sets $\{f(x) > 0\}$, $\{f(x) = 0\}$, $\{f(x) < 0\}$ *is connected. Further* $f(x) > 0 = f(y)$ *implies* $x < y$, *and* $0 = f(y) > f(z)$ *implies* $y < z$. *On the first (third) set* $\phi \circ f - G$ *is nonincreasing.*

The theorem remains valid if "ρ*" is replaced by "*λ*".*

Proof. We start by showing that $\phi \circ f - G$ is nonincreasing on each component of $\{f(x) > 0\}$. Suppose for the moment that $\inf g > -\infty$. We introduce a nonnegative, lower semicontinuous, integrable function h on $\{a \leq x \leq b\}$ which takes the value $+\infty$ at each point of the set of measure zero consisting of the x for which $f(x) > 0$ and either (2.2) fails to hold or $g(x)$ is not the derivative (finite or not) of G at x. The function h may be constructed in the well-known manner as the restriction to $\{a \leq x \leq b\}$ of the sum of the characteristic functions of open sets O_n $(n = 1, 2, \cdots)$ each of which contains the set of zero measure in question and which are such that the sum of their measures is finite. We also introduce a positive number ε and let

$$G_\varepsilon(x) = \int_a^x (g + \varepsilon h)\, dt. \tag{2.3}$$

It is immediate that the lower right derivate of $\phi \circ f - G_\varepsilon$ is nonpositive at each point of $\{f(x) > 0\}$ different from a. Consequently, $\phi \circ f - G_\varepsilon$ is nonincreasing on each component of $\{f(x) > 0\}$; see [3, p. 534]. On letting $\varepsilon \to 0$, we see that the same conclusion holds for $\phi \circ f - G$. The fact that $\phi \circ f - G$ is nonincreasing on each component of $\{f(x) > 0\}$ when $\inf g = -\infty$ follows on noting that, for each finite constant A, the inequality (2.2) holds for almost all x of the set $\{f(x) > 0\}$ when g is replaced by $\max\{g, A\}$.

Of course, the same result holds if "ρ" is replaced by "λ".

It is now easy to see that $\phi \circ f - G$ is nonincreasing on each component of $\{f(x) < 0\}$. It suffices to introduce $\phi_1(x) = -\phi(-x)$ and $f_1(x) = -f(a + b - x)$ and to observe that

$$\lambda\phi_1 \circ f_1(x) = \rho\phi \circ f(a + b - x). \tag{2.4}$$

The argument is concluded by noting that the map

$$x \to \phi_1 \circ f_1(x) + G(a + b - x)$$

is nonincreasing on each component of $\{f_1(x) > 0\}$ and thereupon drawing the consequence that $\phi \circ f - G$ is nonincreasing on each component of $\{f(x) < 0\}$.

We next show that the set $\{f(x) = 0\}$ is connected. If this were not the case, there would exist a component of $\{f(x) > 0\}$ or of $\{f(x) < 0\}$ at the endpoints of which f vanished. However such a component cannot exist by virtue of the monotone character of $\phi \circ f - G$ and the condition (2) imposed on ϕ. Further, if $\{f(x) = 0\} \neq \emptyset$,

$$\sup\{f(x) > 0\} \leq \min\{f(x) = 0\} \leq \max\{f(x) = 0\}$$
$$\leq \inf\{f(x) < 0\} .$$

All the assertions of the theorem follow.

3. UNICITY AND CONTINUITY THEOREMS

We consider the differential equation

$$y' = F(x, y), \tag{3.1}$$

where F is a function taking values in a normed vector space B and the domain D of F is a subset of the Cartesian product $\{a \leq x \leq b\} \times B$, the projection of which under $(x, y) \to x$ is the interval $\{a \leq x \leq b\}$ itself. We suppose that $a \leq c \leq b$.

Unicity. The following unicity theorem subsumes the usual unicity theorems involving Lipschitz conditions ($\phi'(x) = x^{-1}$, $x > 0$) or Osgood conditions

$$(\phi'(x) = [\psi(x)]^{-1}, \quad x > 0 ,$$

where ψ is a continuous function defined on the nonnegative reals satisfying $\psi(0) = 0$, $\psi(x) > 0$ for $x > 0$, and $\int_0^1 [\psi(x)]^{-1} dx = +\infty$).

THEOREM 2. *Let* y *and* z *denote solutions of* (3.1) *satisfying* $y(c) = z(c)$. *Suppose that there exist a nonnegative integrable function* g *on* $\{a \leq x \leq b\}$ *and a function* ϕ *satisfying the conditions stated in Section* 2 *such that for almost all* x $(a \leq x \leq b)$ *either* (1) $y(x) = z(x)$, *or* (2) $y(x) \neq z(x)$ *and*

$$\phi'[|y(x) - z(x)|] \cdot |F[x, y(x)] - F[x, z(x)]| \leq g(x). \tag{3.2}$$

Then $y = z$.

Proof. Setting $f = |y - z|$, we observe that f satisfies almost everywhere the inequality

$$\rho\phi \circ f(x) = \phi'[f(x)]\rho f(x) \leq g(x)$$

when $f(x) > 0$. Further setting $f_2(x) = f(a + b - x)$, we see that f_2 satisfies almost everywhere the inequality

$$\rho\phi \circ f_2(x) \leq g(a + b - x)$$

when $f(a + b - x) > 0$. The theorem now follows from Theorem 1.

Continuity. Let $\{z\}$ denote a family of solutions of (3.1) which satisfy (3.2) with respect to a given solution y of (3.1). Application of Theorem 1 yields the inequality

$$\phi[|z(x) - y(x)|] \leq \phi[|z(c) - y(c)|] + |G(x) - G(c)|, \qquad a \leq x \leq b .$$

It follows that

$$||z - y|| = \max_{a \leq x \leq b} |z(x) - y(x)|$$

tends to zero with $|z(c) - y(c)|$. This observation leads at once to standard continuity theorems concerning the dependence of a solution on its initial value since

$$\begin{aligned} |z(x_2) - y(x_1)| &\leq |z(x_2) - y(x_2)| + |y(x_2) - y(x_1)| \\ &\leq \|z - y\| + |y(x_2) - y(x_1)|. \end{aligned}$$

The use of derivate inequalities is also available for the study of continuity questions involving approximate solutions when hypotheses of the Osgood type hold. By way of illustration, we consider the following situation without insisting on great generality (see [1, pp. 105-107]). Let ψ be defined and continuous on $\{0 \leq x \leq +\infty\}$, where $\psi(0) = 0$, $\psi(x) > 0$ for $x > 0$, and the Osgood divergence condition

$$\int_0^1 [\psi(x)]^{-1}\,dx = +\infty \tag{3.3}$$

is fulfilled. We suppose that $a \leq c \leq b$, that $\varepsilon > 0$, that y is a solution of (3.1), that z is a differentiable function on $\{a \leq x \leq b\}$ such that

$$\{(x, z(x)) \mid a \leq x \leq b\} \subset D$$

and

$$|z'(x) - F[x, z(x))]| < \varepsilon, \qquad a \leq x \leq b,$$

and, finally, that

$$|F[x, y(x)] - F[x, z(x)]| \leq \psi[f(x)], \qquad a \leq x \leq b,$$

where $f = |y - z|$. Then

$$\int_1^{f(x)} \frac{dy}{\psi(y) + \varepsilon} \leq |x - c| + \int_1^{f(c)} \frac{dy}{\psi(y) + \varepsilon}, \qquad a \leq x \leq b. \tag{3.4}$$

That the inequality (3.4) holds may be seen by noting that

$$\begin{aligned} \rho f(x) &\leq |F[x, y(x)] - F[x, z(x)]| + |F[x, z(x)] - z'(x)| \\ &\leq \psi[f(x)] + \varepsilon, \qquad a < x < b, \end{aligned}$$

and that a corresponding inequality holds for the map $x \to f(a + b - x)$, and that, as a consequence, the lower right derivates of

$$x \to \int_1^{f(x)} \frac{dy}{\psi(y) + \varepsilon}$$

and

$$x \to \int_1^{f(a+b-x)} \frac{dy}{\psi(y) + \varepsilon}$$

do not exceed one on $\{a < x < b\}$.

From (3.4) we conclude by an elementary argument: given $\eta > 0$, there exists a $\delta > 0$ such that if $\max\{\varepsilon, f(c)\} < \delta$, then $f(x) < \eta$, $a \leq x \leq b$.

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