PACKING INEQUALITIES FOR CIRCLES

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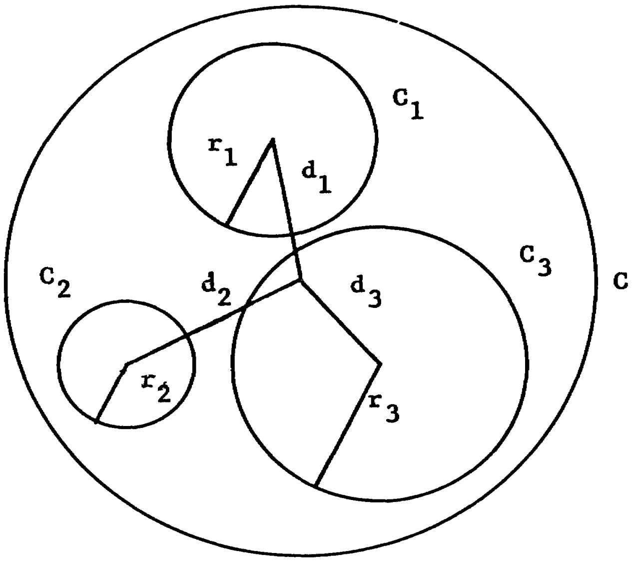
1. INTRODUCTION

Let the three non-overlapping discs C_1, C_2, C_3 lie inside the unit disc C: $|z| \leq 1$. Let r_i $(i = 1, 2, 3)$ designate the radius of C_i, and let d_i designate the distance from the center of C_i to the origin. Then

$$d_1 d_2 d_3 + r_1^2 + r_2^2 + r_3^2 \leq 1. \tag{1}$$



In this paper, we shall prove (1) and similar inequalities for nonoverlapping discs C_i contained in the unit disc C. Let C_i designate the open disc

$$(x - x_i)^2 + (y - y_i)^2 < r_i^2 \qquad (i = 1, 2, \cdots, n).$$

Our goal is to find simple inequalities relating the quantities x_i, y_i, r_i. As for example, from geometry, we see that a necessary and sufficient condition that C_i and C_j do not overlap is that $(x_i - x_j)^2 + (y_i - y_j)^2 \geq (r_i + r_j)^2$.

2. INEQUALITIES DERIVED FROM REAL VARIABLE THEORY

Note that if $f(x, y)$ is a non-negative integrable function defined on C, then

$$\sum_{i=1}^{n} \int\int_{C_i} f(x, y)\, dx\, dy \leq \int\int_{C} f(x, y)\, dx\, dy. \tag{2}$$

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If f is positive, equality in (2) holds if and only if $n = 1$ and $C_1 = C$. For a fixed f, each integral in the left hand member of (2) can be expressed in terms of x_i, y_i, r_i. Simple selections of f are: $f(x, y) = 1$; $f(x, y) = 1 + Ax + By$ with $A^2 + B^2 \leq 1$; $f(x, y) = x^2 + y^2$; $f(x, y) = xy + 1/2$. These selections lead to the inequalities

$$\sum_{i=1}^{n} r_i^2 \leq 1 \qquad \text{(Area Inequality)} \tag{3}$$

$$\sum_{i=1}^{n} (Ax_i + By_i + 1)r_i^2 \leq 1 \qquad \text{(Moment Inequality)} \tag{4}$$

$$\sum_{i=1}^{n} \{r_i^4 + 2r_i^2(x_i^2 + y_i^2)\} \leq 1 \qquad \text{(Moment of Inertia Inequality)} \tag{5}$$

$$\sum_{i=1}^{n} (1 + 2x_i y_i)r_i^2 \leq 1 \qquad \text{(Product of Inertia Inequality)} \tag{6}$$

Equality holds if and only if $n = 1$, $x_1 = y_1 = 0$, $r_1 = 1$. Similar inequalities can be obtained in n dimensions by employing n-fold integrals.

3. INEQUALITIES DERIVED FROM COMPLEX VARIABLE THEORY

We shall use the mean value theorem in the following form.

LEMMA 1. *Let* D *designate the disc* $|z - z_0| < r$. *If* f *is regular in* $\overline{D}$, *then*

$$\iint_D f(z)\, dx\, dy = \pi r^2 f(z_0). \tag{7}$$

Proof. The expansion $f(z) = \Sigma_{n=0}^{\infty} (n!)^{-1} f^{(n)}(z_0)(z - z_0)^n$ converges uniformly in $\overline{D}$. Hence,

$$\iint_D f(z)\, dx\, dy = \sum_{n=0}^{\infty} (n!)^{-1} f^{(n)}(z_0) \iint_D (z - z_0)^n\, dx\, dy.$$

Set $z - z_0 = \rho e^{i\theta}$. Then

$$\iint_D (z - z_0)^n\, dx\, dy = \int_0^{2\pi} \int_0^r \rho^{n+1} e^{in\theta}\, d\rho\, d\theta.$$

This last integral equals πr^2 if $n = 0$ and equals 0 if $n = 1, 2, \cdots$. Substituting these values in the infinite series yields (7).

THEOREM 2. *Let* C_i $(i = 1, 2, \cdots, n)$ *designate* n *discs that lie in* S: $|z| \leq R$ *without overlapping. Then, if* $z_k = x_k + iy_k$,

$$R^2 \left| \frac{z_1 z_2 \cdots z_n}{R^n} \right| + \sum_{i=1}^{n} r_i^2 \leq R^2. \tag{8}$$

In particular, if S *is the unit disc* C,

$$|z_1 z_2 \cdots z_n| + \sum_{i=1}^{n} r_i^2 \leq 1, \tag{8'}$$

and, if further n = 3, *we obtain the inequality* (1).

Proof. Set $B = S - \bigcup_{i=1}^{n} C_i$. Let f be a function that is regular in $\bar{S}$. Then

$$\iint_S f(z)\,dx\,dy = \sum_{i=1}^{n} \iint_{C_i} f(z)\,dx\,dy + \iint_B f(z)\,dx\,dy. \tag{9}$$

Using the Lemma, we see that

$$\pi R^2 f(0) = \sum_{i=1}^{n} \pi r_i^2 f(z_i) + \iint_B f(z)\,dx\,dy. \tag{10}$$

Now select

$$f(z) = \prod_{k=1}^{n} \frac{R(z - z_k)}{R^2 - z\bar{z}_k}.$$

It is easily verified that f is regular in $\bar{S}$, that $f(z_k) = 0$ $(k = 1, 2, \cdots, n)$, that

$$f(0) = (-1)^n \prod_{k=1}^{n} (z_k/R),$$

and that $|f(z)| = 1$ for $|z| = R$. By the maximum principle, it follows that $|f(z)| \leq 1$ in S. Inserting this information in (10), we find

$$\pi R^2 (-1)^n \prod_{k=1}^{n} (z_k/R) = \iint_B f(z)\,dx\,dy. \tag{11}$$

Hence, taking absolute values, we conclude

$$R^2 \prod_{k=1}^{n} (|z_k|/R) \leq \frac{1}{\pi} \iint_B dx\,dy = R^2 - \sum_{i=1}^{n} r_i^2. \tag{12}$$

This completes the proof.

Several remarks are in order. In the first place, the upper bound in (8) is attained if $n = 1$, $z_1 - 0$, $r_1 = R$. In the second place, the quantity

$$\pi R^2 \prod_{k=1}^{n} (|z_k|/R)$$

emerges as a simple lower bound for the area of the multiply connected region B. If $z_1, \cdots, z_n$ are all located near $\{|z| = R\}$, then this quantity will be near πR^2, and the areas of the discs C_i are therefore small—as indeed they should be. If one of the z_i is zero, this term contributes nothing, and (8) reduces to the area inequality (3). This degeneration can be avoided by the following simple device.

THEOREM 3. *Let the* C_i *be* n *non-overlapping discs that lie in the annulus* A: $0 < \rho < |z| < R$. *Then,*

$$(R^2 - \rho^2) \prod_{k=1}^{n} (|z_k|/R) + \rho^2 + \sum_{i=1}^{n} r_i^2 \leq R^2. \tag{13}$$

Proof. Write $B = A - \bigcup_{i=1}^{n} C_i$. Then,

$$\pi(R^2 - \rho^2) f(0) = \iint_A f(z)\,dx\,dy = \sum_{i=1}^{n} \pi r_i^2 f(z_i) + \iint_B f(z)\,dx\,dy.$$

By selecting f as before, there is obtained the inequality

$$(R^2 - \rho^2) \prod_{i=1}^{n} (|z_i|/R) \leq \frac{1}{\pi} \iint_B dx\,dy = (R^2 - \rho^2) - \sum_{i=1}^{n} r_i^2.$$

A geometrical interpretation of (8) or (13) would be interesting. For the cases n = 2 and n = 3, O. Shisha, in a written communication, has given an algebraic proof of (8') (with a sharp inequality sign) and has strengthened (1) to the sharp inequality

$$d_1 d_2 d_3 + r_1 r_2 r_3 + r_1^2 + r_2^2 + r_3^2 < 1.$$

4. RELATED INEQUALITIES

THEOREM 4. *Let* f *be regular in* C: $|z| \leq 1$ *and real on the real axis, and let* $\Re f(z) \geq 0$ *in* C. *Then,*

$$(u - 1)^2 f(u) + u^2 f(u - 1) \leq f(0) \quad (0 \leq u \leq 1). \tag{14}$$

Equality holds for $u = 0$ *and* $u = 1$.

Proof. Select $z_1 = u$, $r_1 = 1 - u$; $z_2 = u - 1$, $r_2 = u$, $0 \leq u \leq 1$. Then the discs C_1 and C_2 are contained in C, do not overlap, and are tangent at $z = 2u - 1$. Hence from (10),

$$f(0) - (u - 1)^2 f(u) - u^2 f(u - 1) = \frac{1}{\pi} \iint_B f(z)\,dx\,dy = \frac{1}{\pi} \iint_B \Re f(z)\,dx\,dy \geq 0.$$

Example. $(u - 1)^2 e^u + u^2 e^{u-1} \leq 1 \quad (0 \leq u \leq 1)$.

Similar inequalities can be found by taking several points on the x-axis.

The hypothesis of regularity on $\{|z| = 1\}$ may be dropped, and (14) (with $0 < u < 1$) can be compared with the following familiar "one point" inequality [3; p. 169]: if f is regular in $\{|z| < 1\}$ and $\Re f(z) \geq 0$ there, and if $f(0) = 1$, then $|f(z)| \leq (1 + |z|)/(1 - |z|)$.

THEOREM 5. *Let* $f(x, y)$ *be non-negative and harmonic in* C: $|z| \leq 1$. *Let* T *designate the set of points* (x, y) *with* $f(x, y) \geq \sigma$. *Then, if the* $C_1, \cdots, C_n$ *are non-overlapping discs lying in* C *whose centers* (x_i, y_i) *lie in* T,

$$\sum_{i=1}^{n} r_i^2 \leq f(0, 0)/\sigma. \tag{15}$$

Proof. From (2) and the mean value theorem, we see that

$$\sigma \sum_{i=1}^{n} r_i^2 \leq \sum_{i=1}^{n} f(x_i, y_i) r_i^2 \leq f(0, 0).$$

Similar theorems can be derived in n dimensions.

Example. $f(x, y) = x + 1$ is non-negative and harmonic in C. Select $\sigma = 5/4$. Then T is the half-plane $x \geq 1/4$. Hence, at most $4/5$ of C can be covered by non-overlapping discs lying in C whose centers lie in T. The exact upper bound would be interesting to know.

5. INEQUALITIES DERIVED FROM FUNCTIONAL ANALYSIS

Let S be a region of the complex plane that possesses a Bergman kernel function $K_S(z, \overline{w})$. (See [1], [2], or [3]). The related Hilbert Space $L^2(S)$ consists of all functions that are single-valued and regular in S and such that

$$\|f\|^2 = \iint_S |f(z)|^2 \, dx\, dy < \infty \quad \text{with} \quad (f, g) = \iint_S f \bar{g} \, dx\, dy.$$

THEOREM 6. *Let* C_i $(i = 1, \cdots, n)$ *be non-overlapping discs contained in* S *and possessing centers* z_i *and radii* r_i. *Then, if* S *has finite area,*

$$n^2/\text{area}(S) \leq \sum_{j,k=1}^{n} K_S(z_j, \bar{z}_k) \leq \pi^{-1}(r_1^{-2} + r_2^{-2} + \cdots r_n^{-2}). \tag{16}$$

In particular, if S *is the unit disc,* (16) *becomes*

$$n^2 \leq \sum_{j,k=1}^{n} (1 - z_j \bar{z}_k)^{-2} \leq r_1^{-2} + r_2^{-2} + \cdots + r_n^{-2}. \tag{17}$$

Proof. Let $a_1, \cdots, a_n$ be arbitrary constants, and set (for $f \in L^2(S)$)

$$L(f) = \sum_{i=1}^{n} a_k \iint_{C_k} f(z) \, dx\, dy = \pi \sum_{i=1}^{n} a_k r_k^2 f(z_k). \tag{18}$$

We can also write L as follows. Let $E = \bigcup_{i=1}^{n} C_i$, and define a complex-valued function a to be equal to a_i on C_i and zero elsewhere on S. Then

$$L(f) = \iint_E a(z)f(z)\,dx\,dy\,. \tag{19}$$

By the Schwarz inequality,

$$\begin{aligned} |L(f)|^2 &\leq \left(\iint_E |a(z)|^2\,dx\,dy\right)\left(\iint_E |f|^2\,dx\,dy\right) \\ &\leq \pi(|a_1|^2 r_1^2 + \cdots + |a_n|^2 r_n^2) \iint_S |f(z)|^2\,dx\,dy\,. \end{aligned} \tag{20}$$

It follows therefore that

$$\|L\|^2 \leq \pi(|a_1|^2\, r_1^2 + \cdots + |a_n|^2\, r_n^2)\,. \tag{21}$$

On the other hand, $\|L\|^2 = L_z\, L_{\bar{w}}\, K_S(z, \bar{w})$ (where L_z means L applied to the z variable and $L_{\bar{w}}(f(\bar{w}))$ means $\overline{L_w(\overline{f(\bar{w})})}$. (See, for example, [2].) Hence, by (18),

$$\|L\|^2 = \pi^2 \sum_{j,k=1}^{n} a_j\, \bar{a}_k\, r_j^2\, r_k^2\, K_S(z_j, \bar{z}_k)\,. \tag{22}$$

Therefore,

$$\pi \sum_{j,k=1}^{n} a_j\, \bar{a}_k\, r_j^2\, r_k^2\, K_S(z_j, \bar{z}_k) \leq |a_1|^2\, r_1^2 + \cdots + |a_n|^2 r_n^2. \tag{23}$$

If we now set $a_j = r_j^{-2}$ $(j = 1, \cdots, n)$, we obtain the right-hand inequality of (16).

To obtain the left-hand inequality in (16), we observe that $|L(f)|^2 \leq \|L\|^2\, \|f\|^2$. Hence,

$$\left|\pi \sum_{i=1}^{n} a_k\, r_k^2\, f(z_k)\right|^2 \leq \|L\|^2\ \|f\|^2.$$

Setting $a_k = r_k^{-2}$ and using (22), we obtain the inequality

$$\left|\sum_{i=1}^{n} f(z_k)\right|^2 / \|f\|^2 \leq \sum_{j,k=1}^{n} K_S(z_j, \bar{z}_k)\,. \tag{24}$$

The special selection $f = 1$ yields (16). The kernel function for the unit circle is $\pi^{-1}(1 - z\bar{w})^{-2}$, and this yields (17).

It should be noted that the special case of (16), $K_S(z_1, \bar{z}_1) < \pi^{-1} r_1^{-2}$, has been stressed by Bergman [1] in his work on the kernel function.

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