A THEOREM ON MAPS WITH NON-NEGATIVE JACOBIANS

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In this note we give an elementary proof of the following theorem:

THEOREM 1. *Let* M *and* N *be connected, oriented* n*-dimensional differentiable manifolds of class* C^k $(k \geq 1)$, *and let* $f: M \to N$ *be a proper* C^k *map such that the Jacobian* $J(f)$ *is non-negative at each point of* M. *Then either* $J(f) \equiv 0$ *or* f *maps* M *onto* N.

We say that f is *proper* if $f^{-1}(C)$ is compact for every compact subset C of N. We note that since M and N are oriented, the sign of $J(f)$ is well defined at each point of M.

Theorem 1 was proved for compact M in [2]. The proof of [2] uses the homological ideas of degree and local degree and requires the application of Sard's theorem on critical values. Our proof is motivated by de Rham's theorem, but makes no explicit use of the de Rham cohomology. It is self-contained, makes no reference to homology theory, and uses only elementary facts from the calculus of differential forms and, in the C^1 case, the generalized Stokes' formula.

1. AN ELEMENTARY CASE OF GREEN'S THEOREM

The material in this section is known [1], but we include it for the sake of completeness. The following lemma is a weak form of Green's theorem in R^n. We refer the reader to [1] for the standard facts concerning differential forms.

LEMMA 1. *Let* ω *be a differential* $(n-1)$*-form of class* C^1 *in* R^n *with compact support. Then* $\int_{R^n} d\omega = 0$.

Proof. In a coordinate system $(x_1, \cdots, x_n)$,

$$\omega = \sum_{i=1}^{n} a_i(x_1, \cdots, x_n)\, dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_n .$$

Thus

$$d\omega = \sum_{i=1}^{n} (-1)^{i-1} \frac{\partial a_i}{\partial x_i} dx_1 \wedge \cdots \wedge dx_n .$$

Choose a number $r > 0$ such that $\omega \equiv 0$ outside of the set

$$\{(x_1, \cdots, x_n) \mid |x_i| < r,\ i = 1, \cdots, n\} .$$

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Then $\int_{R^n} d\omega$ can be evaluated by the iterated integral

$$\int_{-r}^{r}\int_{-r}^{r}\cdots\int_{-r}^{r}\sum_{i=1}^{n}(-1)^{i-1}\frac{\partial a_i}{\partial x_i}dx_1\cdots dx_n,$$

which is obviously zero.

Let M be an oriented differentiable manifold, and let θ be a differential n-form on M with compact support V. Then $\int_M \theta$ can be defined by taking a covering of V by coordinate neighborhoods and using a partition of unity. (See [1] for details.) This allows us to define $\int_M \theta$ without appealing to triangulation theorems.

LEMMA 2. *Let* M *be an oriented,* n*-dimensional differentiable manifold, and let* ω *be a differential* (n - 1)*-form of class* C^1 *on* M *with compact support* V. *Then* $\int_M d\omega = 0$.

Proof. Let $U_1, \cdots, U_m$ be a covering of V by coordinate neighborhoods, and let $\phi_1, \cdots, \phi_m$ be a C^1 partition of unity for V relative to the covering $\{U_i\}$. Then $\omega = \Sigma_{i=1}^m \phi_i\,\omega$, and the support of ϕ_i is a compact subset of U_i. Thus, by Lemma 1,

$$\int_M d\omega = \sum_{i=1}^{m}\int_M d(\phi_i\,\omega) = \sum_{i=1}^{m}\int_{U_i} d(\phi_i\,\omega) = 0.$$

2. CONSTRUCTION

Let $0 < \varepsilon < 1$, and let h be a real-valued C^2 function on the real line such that $h(x) \geq 0$ for all x, $h(x) = 0$ for $|x| \geq \varepsilon$, and $\int_{-\infty}^{\infty} h(x)\,dx = 1$. Let

$$h_1(x) = h(x+1), \qquad h_2(x) = h(x-1), \qquad g(x) = \int_{-\infty}^{x}(h_2(t) - h_1(t))\,dt.$$

Then $g(x) = 0$ for $|x| \geq 1 + \varepsilon$. Define n-forms θ_1 and θ_2 by

$$\theta_i = h_i(x_1)\,h(x_2^2 + \cdots + x_n^2)\,dx_1 \wedge \cdots \wedge dx_n \qquad (i = 1, 2).$$

Let ω be the (n - 1)-form

$$\omega = g(x_1)\,h(x_2^2 + \cdots + x_n^2)\,dx_2 \wedge \cdots \wedge dx_n.$$

We obtain immediately

a) $d\omega = \theta_2 - \theta_1$;

b) ω, θ_1, and θ_2 have compact support;

c) $\int_{R^n} \theta_1 = \int_{R^n} \theta_2 > 0$.

Let e_1 denote the point $(1, 0, \cdots, 0)$ of R^n, and for $P \in R^n$ let $S(P, r)$ denote the open sphere of radius r with center P. Then

d) Support $\theta_1 \subset S(-e_1, 2\varepsilon^{1/2})$.

3. PROOF IN THE C^2 CASE

We need the following elementary result.

3.1. *Let* Q_1 *and* Q_2 *be points of the connected* n-*dimensional differentiable manifold* N. *Then there exists an open set* U *of* N *that contains* Q_1 *and* Q_2 *and is diffeomorphic to* R^n.

Proof. We write $Q_1 \sim Q_2$ if there exists an open set U as described above. We claim that $\sim$ is an equivalence relation. The symmetry and reflexivity of $\sim$ are obvious. To prove transitivity, assume we are given open sets U_1, U_2 diffeomorphic to R^n, and points Q_1, Q_2, Q_3 such that $Q_1, Q_2 \in U_1$ and $Q_2, Q_3 \in U_2$. Let h be a diffeomorphism of N such that $h(Q_i) = Q_i$ $(i = 1, 2)$ and $h(Q_3) \in U_1 \cap U_2$. Then $Q_i \in h^{-1}(U_1)$ $(i = 1, 2, 3)$. Thus $Q_1 \sim Q_3$, and the relation $\sim$ is transitive. Each equivalence class is open and closed. Since N is connected, this proves 3.1.

The proof of Theorem 1 is by contradiction. Assume that f is not onto and that $J(f) \not\equiv 0$. Since f is proper, $N - f(M)$ is an open set. Let $Q_1 \in (N - f(M))$. Choose $P \in M$ such that $J(f)(P) \neq 0$, and let $Q_2 = f(P)$. Let U be an open set containing Q_1 and Q_2 as in 3.1, and let H be a diffeomorphism of U onto R^n. We may choose H so that $H(Q_1) = -e_1$, $H(Q_2) = e_1$. Choose an $\varepsilon > 0$ such that

$$S(-e_1, 2\varepsilon^{1/2}) \subset H(N - f(M)).$$

Let θ_1, θ_2, ω be differential forms on R^n defined as in Section 2, and let θ_1' (respectively, θ_2', ω') be defined to be $H^*\theta_1$ (respectively, $H^*\theta_2$, $H^*\omega$) on U and zero on $N - U$. Then θ_1', θ_2', ω' are C^1 forms on N with compact support, and $d\omega' = \theta_2' - \theta_1'$. Since f is proper, $f^*\theta_1'$, $f^*\theta_2'$, and $f^*\omega'$ have compact support. By the choice of Q_1 and Q_2,

$$\int_M f^* \theta_1' = 0, \qquad \int_M f^* \theta_2' > 0.$$

Thus $\int_M f^*(\theta_2' - \theta_1') > 0$. But, by Lemma 2,

$$\int_M f^*(\theta_2' - \theta_1') = \int_M f^*(d\omega') = \int_M d(f^*\omega') = 0.$$

This gives a contradiction.

4. THE C^1 CASE

If M and N are C^1 manifolds, then we can only speak of differential forms of class C^0, and hence d is no longer defined. In this case we can use Stokes' formula as a definition of the exterior derivative. More precisely, given a k-form θ and a (k - 1)-form ω of class C^0 on a C^1 manifold W, we say that $\theta = d\omega$ if, for every C^1 singular k-simplex σ in W, we have $\int_\sigma \theta = \int_{\partial\sigma} \omega$. If $f: V \to W$ is a C^1 map, then for every C^1 singular k-simplex τ in V,

$$\int_\tau f^*\theta = \int_{f_*\tau} \theta = \int_{\partial(f_*\tau)} \omega = \int_{f_*(\partial\tau)} \omega = \int_{\partial\tau} f^*\omega .$$

Thus $d(f^*\omega) = f^*\theta$. With these definitions and under the assumption that $d\omega$ is defined, Lemma 1 is valid for C^0 forms (essentially by definition), and Lemma 2 follows easily.

We return to the proof of Theorem 1, using the notation of Section 3. Let θ_1, θ_2 and ω be defined as before. It follows easily from the standard Stokes' formula that $d\omega = \theta_2 - \theta_1$ (considered as a relation between C^0 forms on R^n). If we denote by g the map $H \circ f: f^{-1}(U) \to R^n$, it follows that $d(g^*\omega) = g^*\theta_2 - g^*\theta_1$. As before,

$$\int_{f^{-1}(U)} g^*\theta_2 > 0 \quad\text{and}\quad \int_{f^{-1}(U)} g^*\theta_1 = 0 .$$

However, by Lemma 2,

$$\int_{f^{-1}(U)} g^*(\theta_2 - \theta_1) = \int_{f^{-1}(U)} d(g^*\omega) = 0 .$$

5. APPLICATIONS

The argument used in proving Theorem 1 gives yet another elementary proof of the the fundamental theorem of algebra, which uses only the elementary form of Green's theorem given in Lemma 1 and the fact that a complex polynomial defines a proper map of R^2 into R^2. If M and N are complex analytic manifolds with the induced orientations and f is a proper holomorphic map, then J(f) is automatically nonnegative, and the conclusion of Theorem 1 holds.

REFERENCES

1. G. de Rham, *Variétés Différentiables,* Hermann et Cie, Paris, 1955.

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