PARTIALLY FREE SUBSETS OF EUCLIDEAN n-SPACE

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In an earlier paper [7] I have studied subcontinua of euclidean n-space E^n ($n \geq 2$) that are what might be called *partially free*: for each positive number ε, they admit ε-transformations (continuous mappings) into sets that either do not meet the original set at all (such sets are called *free*), or meet it in a set having prescribed dimensional limitations. Continua that are free were the subject of earlier papers [2], [4], but as a suitable modification of the well-known Alexander Horned Sphere shows, even such a simple continuum as the 2-sphere in E^3 is not always free. However, it was recently shown by Bing [1] that a 2-sphere S in E^3 is always partially free, in that it can be subjected to ε-transformations into sets which meet S only in a 0-dimensional set ("Cantor set"). And in the paper first cited above, I showed that a converse of Bing's theorem holds (thus furnishing a new positional characterization of the 2-sphere in E^3), and I found an analogous theorem for the generalized manifold in E^n.

These results have the following noteworthy features: let C denote the original set, f the transformation, and T the closed subset of C such that

$$C' = f(C) \subset (E^n - C) \cup T;$$

then

(1) in the case of Bing's result, f is a homeomorphism on C - T,

(2) if U is an arbitrary component of E^n - C, one can always assume that $C' \subset U \cup T$, and

(3) the set T depends on ε.

It is easy to show that in (1) one may assume that $f(C - T) \subset E^n - C$ (although this does not necessarily imply that $f(T) \subset T$). However, the results found in [7] did not require (1) at all, the most general type of continuous mapping being sufficient for the converse theorem. Consequently one might search for a set of sufficient conditions that would either incorporate (1) in significant fashion or modify it so that only certain types of "monotoneity" conditions are imposed upon the mapping. Also, one may ask that condition (2)—that one can "push" C - T into *either* complementary domain—be deleted, no assumption being made as to where in E^n - C the set C' - T falls (in the case of the Alexander Horned Sphere, the sphere is free relative to one complementary domain and only partially free relative to the other); and that the effect of making T independent of ε be considered. Each of these possibilities is incorporated in at least one of the theorems stated below; the first two appear in all the main theorems, and the third in Theorems 1, 3 and 4.

We begin with a Lemma that extends Theorem 1 of [2] to partially free sets (here and elsewhere in this paper, the symbol rel stands for "relative to").

LEMMA 1. *In* E^n, *let* C *be a continuum such that for each* $\varepsilon > 0$ *there exist a subset* T *of* C *that is a frontier set* rel C *not disconnecting either* C *or* E^n, *and an*

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ε-transformation $f(C) = C'$ *into* $(E^n - C) \cup T$ *that carries* $C - T$ *into* $E^n - C$. *Then* C *is a frontier set* rel E^n; *and if* C *cuts* E^n, *then* $E^n - C$ *consists of exactly two domains of which* C *is the common boundary.*

Proof. Suppose $E^n - C$ has at least three components U, V, and W, and let $u \in U$, $v \in V$, $w \in W$. Then there exists an $\varepsilon > 0$ such that under every ε-transformation of C, the image separates each of the point pairs (u, v), (u, w), (v, w) [2, p. 157]. Let T, f, and C' be as described in the hypothesis of the Lemma. Also let

$$C_1 = (C - T) \cup f^{-1}(f(T) - T)$$

and $C'' = f(C_1)$.

Since T is a frontier set rel C and C - T is connected, the set C_1, and hence also C'', is connected. And since $C'' = f(C) - C$, the set C'' must lie in a single component of $E^n - C$ and contain all points of C' not in T. It must therefore fail to meet two of the domains U, V, W. Suppose it meets neither V nor W. Since T does not cut E^n, there exist $v_1 \in F(V) - T$ and $w_1 \in F(W) - T$. Hence the set $K = V \cup (C - T) \cup W$ is connected, inasmuch as V, C - T, and W are connected and C - T contains the limit points v_1 and w_1 of V and W, respectively. However, $K \cap C' = \emptyset$, and this contradicts the fact that by the choice of ε, C' separates v and w in E^n. We conclude that E^n has at most two components.

Every point of C is a frontier point rel E^n. For otherwise C would contain some $S(x, 2\varepsilon)$ with $x \in C$ and $\varepsilon > 0$. Again, let T and f be as described in the hypothesis. Since T is a frontier set rel C, there would exist a point

$$y \in (C - T) \cap S(x, \varepsilon).$$

But evidently $f(y) \in S(x, 2\varepsilon)$ and therefore it could not lie in $E^n - C$.

Suppose C cuts E^n. As already shown, $E^n - C$ has then exactly two components, U and V. Let $B = F(U)$, and suppose $C - B \neq \emptyset$. Let $p' \in C - B$ and $q \in U$. Let $\varepsilon > 0$ be such that, under every ε-transformation of B, the image of B separates p' and q, and $2\varepsilon < d(p', \overline{U})$. Let T and f be as described in the hypothesis of the Lemma. Then, since T is a frontier set rel C, there exists $p \in (C - T) \cap S(p', \varepsilon)$. It is easily shown that no ε-mapping can throw a point of B into $S(p', \varepsilon)$, and consequently f(B) separates p and q. The set C'' defined above must lie in V, since $f(p) \in C''$ and $f(p) \notin U$. And since T fails to separate E^n, $B - T \neq \emptyset$, so that $U \cup (C - T)$ is a connected subset of $E^n - f(C) \subset E^n - f(B)$ containing both p and q; this contradicts the fact that f(B) separates p and q. We conclude that $C = F(U) = F(V)$.

COROLLARY. *In* E^n, *let a compact set* C *be the common boundary of (at least) two domains* U *and* V; *and for each* $\varepsilon > 0$ *let there exist a closed set* T *such that* $H_{n-2}(T) = 0$ *and* $\dim T \leq n - 2$, *and such that some* ε*-transformation of* C *into* $(E^n - C) \cup T$ *carries* $C - T$ *into* $E^n - C$. *Then* $E^n - C = U \cup V$.

Proof. Since C is a common boundary of two domains, and $\dim T \leq n - 2$, T is a frontier set rel C. And since $H_{n-2}(T) = 0$, the set C - T is connected [3, Theorem 4].

Remark. The necessity of the condition in Lemma 1 that T do not disconnect C is shown by simple examples; for instance, let C be a 2-sphere in E^3, together with a radius, and let T consist only of the point where that radius meets the 2-sphere. The same example, but with T consisting of all points on the radius mentioned, shows the necessity for the assumption that T be a frontier set rel C. Finally, to show the necessity of the assumption that T does not separate E^n, let again $n = 3$, and let C consist of (1) a 2-sphere T and (2) a curve interior to T that spirals toward T in such a manner as to have T as limiting set; here T is a frontier set rel C and does not disconnect C.

It is necessary to recall here the distinctions regarding "separation" in the set-theoretic sense and in the sense of homology. If C is a connected set and $T \subset C$, then T *separates* C if C - T is not connected. On the other hand, if C is any set and $T \subset C$, then T *0-separates* C if there exists a bounding compact 0-cycle of C having compact carrier in C - T but not bounding on any compact subset of C - T.

As to local separation: If C is any point set and $T \subset C$, then T *separates* C *locally* if there exists an open, connected subset D of C such that D - T is not connected. And T is a *local* *0-separating* set of C if for some open subset U of C and some compact cycle Z_0 of U - T, Z_0 bounds on a compact subset of U but does not bound on any compact subset of U - T.

(To define, more generally, *r-separation* and *local* *r-separation*, one simply replaces the 0's above by r's.)

LEMMA 2. *If* X *is a locally connected, locally compact space, then a local separating set* T *of* X *is a local* *0-separating set of* X; *and conversely, if a closed set* T *is a local* *0-separating set of* X, *then* T *is a local separating set of* X. *Similar statements hold regarding the relations between "separating" and "0-separating."*

Indication of proof. In the first case, there exists a domain (that is, an open, connected set) D such that $D - T = D_1 \cup D_2$ (separated); and since X is locally connected and locally compact, there exists a continuum C in D containing points x_1 and x_2 of D_1 and D_2, respectively. A nontrivial 0-cycle on $x_1 \cup x_2$ bounds in D but not in D - T. For the converse case, there exist an open set U and a compact cycle Z_0 of U - T that bounds in U but not in U - T. There exists a carrier of Z_0 in U - T that is the union of m continua $K_1, \cdots, K_m$, where m is the number of components of U meeting K [5, p. 105, Corollary 3.4]. We can express Z_0 as $Z_1 + \cdots + Z_m$, where Z_i is the portion of Z_0 on K_i and Z_i bounds in the component C_i of U that contains K_i ($i = 1, \cdots, m$). If $Z_0 \not\sim 0$ in U - T, then some $Z_i \not\sim 0$ in $C_i - T$, and since C_i is a domain of X, the set $C_i - T$ cannot be connected (otherwise, T being closed, $C_i - T$ would be a domain of X in which Z_i bounds).

The following Lemma contains Theorem 2 of [2] as a special case:

LEMMA 3. *In* E^n, *let* C *be an* lc^k *continuum that cuts* E^n. *Let* T *be a closed subset of* C *that is a frontier set* rel C *and not a local* *r-separating set of either* C *or* E^n *for* $r \leq k$, *and such that if* $\varepsilon > 0$ *there exists an* ε*-transformation* $f: C \to (E^n - C) \cup T$ *that carries* C - T *into* $E^n - C$ *and is* (k - 1)*-monotone on* C - T. *Then* $E^n - C$ *is the union of disjoint* ulc^k *domains* U *and* V *having* C *as common boundary.*

The conclusion still holds if T *depends on* ε, *provided it may be assumed that* f *can be chosen so that* $f(T) \subset T$.

(By lc^k we denote local r-connectedness in all dimensions $r \leq k$. By restricting to homology over a field, we can define a mapping $f: X \to Y$ to be n-monotone if, for each $y \in Y$, $H_r(f^{-1}(y)) = 0$ for all $r \leq n$. It should be noted that when $k = 0$, the hypothesis of Lemma 3 imposes no monotoneity on f.)

Proof. By Lemma 1, $E^n - C$ is the union of disjoint domains U and V of which C is the common boundary. Suppose U is not r-ulc for some $r \leq k$. Then there exist $p \in C$ and $\varepsilon > 0$ such that whenever $0 < \delta < 3\varepsilon$, the set $U \cap S(p, \delta)$ contains an r-cycle nonbounding in $U \cap S(p, 3\varepsilon)$.

Since C is lc^k, there exists a positive number $\delta < \varepsilon$ such that every r-cycle of $C \cap S(p, \delta)$ bounds on $C \cap S(p, \varepsilon)$. Let γ_r be a cycle of $U \cap S(p, \delta)$, nonbounding in $U \cap S(p, 3\varepsilon)$. Let $\eta > 0$ be chosen so that every η-transformation of C is linked by γ_r in $S(p, 2\varepsilon)$ [2, p. 159, Lemma]. Let δ' be selected so that $0 < \delta' < \delta$ and so that some compact carrier M of γ_r lies in $U \cap S(p, \delta')$. Since $r \leq k$ and T is not a local r-separating set of E^n, the cycle γ_r bounds on a compact set A, containing M, of $S(p, \delta') - T$. Let η' be a positive number, less than η and less than $d(A, T)$. By hypothesis, there exists an η'-transformation $f: C \to (E^n - C) \cup T$ carrying $C - T$ into $E^n - C$, and if possible we select f so that $f(C - T) \subset V$. Then there exists on $A \cap F(S(p, \delta') - U)$ —and hence on $(C - T) \cap S(p, \delta')$ —a cycle Z_r such that $Z_r \sim \gamma_r$ on $A \cap \overline{U}$ (see [5, p. 203, Lemma 1.13]). By the choice of δ, $Z_r \sim 0$ on $C \cap S(p, \varepsilon)$, and since T is not a local r-separating set of C, $Z_r \sim 0$ on $(C - T) \cap S(p, \varepsilon)$. But then $\gamma_r \sim 0$ on the set

$$[U \cup (C - T)] \cap S(p, \varepsilon) \subset S(p, 2\varepsilon) - f(C),$$

and this contradicts the choice of f. We may assume, then, that if $\eta' < \eta$ and $\eta' < d(A, T)$, there does not exist an f such that $f(C - T) \subset V$.

In this case, we select η' as before, and a mapping f as before, but so that now

(1) f is $(r - 1)$-monotone on $C - T$,

(2) $f(C) = C'$ separates M from V,

(3) $f(C \cap S(p, \varepsilon)) \subset S(p, 2\varepsilon)$, and

(4) $f(C \cap (E^n - S(p, \delta)) \cap S(p, \delta') = \emptyset$.

Since C' separates M and V, M lies in an open subset W of $S(p, \delta')$ whose boundary is a subset of $(C' \cap S(p, \delta')) \cup F(p, \delta')$. With A as above, there exists on $A \cap F(W)$ —and hence on $C' \cap S(p, \delta')$ —a cycle Z_r such that $\gamma_r \sim Z_r$ on $A \cap \overline{W}$. By the choice of η' and A, Z_r is on $C' - T - f(T)$. Let F be a carrier of Z_r on $A \cap (C' - T - f(T))$. Since f is $(r - 1)$-monotone on $C - T$, there exists a cycle Z'_r on

$$f^{-1}(F) \subset (C - T) \cap S(p, \delta)$$

(see condition (4) above) such that $f(Z'_r) \sim Z_r$ on F. Since $Z'_r \sim 0$ on $C \cap S(p, \varepsilon)$ and T is not a local r-separating set of C, $Z'_r \sim 0$ on some compact subset F' of $C \cap S(p, \varepsilon) - T$ containing F. By (3), $f(F') \subset S(p, 2\varepsilon)$, and therefore $Z_r \sim 0$ on $f(C - T) \cap S(p, 2\varepsilon)$. But then $\gamma_r \sim 0$ on $S(p, 2\varepsilon) - C$, contrary to the choice of η' and f. We conclude that U is ulc^k, and similarly V is ulc^k.

For the case where T is independent of ε but $f(T) \subset T$, the proof differs only in that η' cannot be subjected to the condition "less than $d(A, T)$" and A is of course selected after the η'-transformation f has been selected. The essential requirement that Z_r be on $C' - T - f(T)$ in the second part of the proof is automatically satisfied under the circumstances.

THEOREM 1. *In* E^n, *let* C *be an* lc^k *continuum that cuts* E^n, *where* $k = m - 1$ *if* $n = 2m$ *or* $n = 2m + 1$. *Let* T *be a closed subset of* C *which is a frontier set* rel C *and is not a local* r-*separating set of either* C *or* E^n *for* $r \leq k$, *and such that if* $\varepsilon > 0$ *there exists an* ε-*transformation* $f: C \to (E^n - C) \cup T$ *that carries* $C - T$ *into* $E^n - C$ *and is* $(k - 1)$-*monotone on* $C - T$. *Then, with the additional assumption that in case* n *is odd,* $H_m(C)$ *is finitely generated, the set* C *is an orientable* $(n - 1)$-gcm.

Proof. If n is even, the theorem follows from Lemma 3 and [5, p. 308, Theorem 7.1]. If n is odd, then $H_{k+1}(U)$ is finitely generated by the Alexander Duality Theorem, and the theorem follows from [5, p. 308, Theorem 7.3].

THEOREM 2. *Under the same hypothesis as in Theorem 1, except that* T *need not be assumed closed and may depend on* ε, *and that* f *may be so chosen that* $f(T) \subset T$, *the same conclusion follows.*

Corollaries of these theorems for E^3 have a special interest; below, we state one for the case of the 2-sphere. We recall that if C is a common boundary of two domains such that $H_1(C) = 0$, and T is a closed, totally disconnected subset of C, then T cannot separate C locally [7, p. 120, Corollary].

COROLLARY. *In* E^3, *let* C *be a 1-acyclic, locally connected continuum which cuts* E^3 *such that for each* $\varepsilon > 0$ *there exist a totally disconnected subset* T *of* C *and an* ε*-transformation* $f: C \rightarrow (E^3 - C) \cup T$ *that carries* $C - T$ *into* $E^3 - C$ *and* T *into a subset of* T. *Then* C *is a 2-sphere.*

Before proceeding, we point out that the definition of local r-separation actually refers to a medial, rather than to a local, property, since it is stated relative to all open subsets of a space (for a discussion of medial properties, see my paper [6]). In particular, a set that is not a local r-separating set cannot r-separate. For some purposes it is useful to know when the property is equivalent to a local property.

LEMMA 4. *Let* T *be a closed, totally disconnected subset of a locally compact space, and for each point* x *in* T, *let there exist arbitrarily small open sets* U, *containing* x, *such that each compact* r*-cycle of* $U - T$ *that bounds in* U *bounds also in* $U - T$. *Then* T *is not a local* r*-separating set of the space.*

Proof. Let U be any open set, Z_r a cycle with compact carrier K in $U - T$, and A a compact set carrying the homology $Z_r \sim 0$ in U. Let $A \cap T = T'$. Then T' is a closed, totally disconnected subset of T not meeting K. By hypothesis, if $x \in T'$, there exists an open set $U_x \subset U - K$ containing x such that if γ_r is a compact cycle of $U_x - T$ that bounds in U_x, then it bounds in $U_x - T$. Since T is closed and totally disconnected, there exists in U_x an open set V_x such that $\overline{V}_x \subset U_x$ and $T \cap F(V_x) = \emptyset$. And since A is compact, T' is compact, and a finite number of such sets V_x, say $V_1, V_2, \cdots, V_k$, cover T'. Replacing V_1 by $W_1 = V_1 - \bigcup_{i=2}^{k} V_i$, and so on, we obtain a covering of T' by disjoint open sets whose boundaries do not meet T and lie in $U - K$.

From the elementary exact sequence of the pair A, $A - W_1$, it follows that Z_r is homologous on $A - W_1$ to a cycle Z_1 on $F(W_1)$; and since $Z_1 \sim 0$ in $U_1 - T$, where U_1 is the U_x to which V_1 and W_1 correspond, we can replace A by a compact set A_1 that agrees with A on the complement of W_1 and meets T' only in the sets W_i $(i > 1)$. Proceeding in this manner, we replace A by a set A_k that lies in $U - T$.

Remark. If T is a closed point set forming an annular ring in a plane X, and $r = 0$, then T is a set satisfying the hypothesis of the Lemma with the exception of the total disconnectedness of T; yet T is a local 0-separating set. Analogously, if T is a solid torus in a 3-space X and $r = 1$, then a similar remark holds.

We recall that we call a space X r-*declinable* at $x \in X$ if there exists an open set U containing x such that every compact r-cycle of X is homologous to a compact r-cycle of $X - U$. If X is r-declinable at every point, we call X itself r-declinable. We can now prove the following lemma:

LEMMA 5. *Let* X *be an* (r + 1)-*declinable, locally compact space, and* T *a closed, totally disconnected subset of* X *that is not an* r-*separating set of* X. *Then* T *is not a local* r-*separating set of* X.

Proof. Let $x \in X$, and let U be an open set containing x. Since x is (r + 1)-declinable, there exists an open set P containing x such that every compact r-cycle of X is homologous to a compact r-cycle of X - P. Since T is closed and totally disconnected, there exists an open set Q, containing x, such that $\overline{Q} \subset U \cap P$ and $T \cap F = \emptyset$, where F is the boundary of Q. Clearly every compact r-cycle of X is homologous to a compact r-cycle in $X - \overline{Q}$.

Let Z_r be a cycle on a compact subset K of Q - T which bounds on some compact subset A_1 of Q. Since T is not an r-separating set of X, Z_r also bounds on a compact subset A_2 of X - T. We may assume that $K \subset A_1 \cap A_2$ (augmenting A_1 and A_2 by K, if necessary, to justify this relation). Consider the sequence of homomorphisms

$$H_{r+1}(A_1 \cup A_2) \xrightarrow{\triangle} H_r(A_1 \cap A_2) \xrightarrow{i} H_r(A_1) + H_r(A_2)$$

forming a portion of the Mayer-Vietoris sequence of the triad $A_1 \cup A_2$, A_1, A_2. If we indicate homology classes by brackets, then $i[Z_r] = 0$, and consequently $[Z_r]$ has antecedent $[Z_{r+1}]$ in $H_{r+1}(A_1 \cup A_2)$. By the choice of Q, in the homomorphism

$$H_{r+1}(A_1 \cup A_2) \xrightarrow{j} h_{r+1}(X)$$

induced by inclusion, $j[Z_{r+1}]$ is represented by a compact cycle γ_{r+1} of $X - \overline{Q}$. (We use h to denote homology groups based on compact supports.)

Let us extend X to a space X' as follows: Let M be a compact subset of $X - \overline{Q}$ carrying γ_{r-1}, and let C be the join of M to an ideal point p; $C \cap X = M$. Then X' is X augmented by C; it is topologized in an obvious manner so as to be locally compact. Note, however, that $\gamma_{r+1} \sim 0$ on the compact subset C of X', and hence $Z_{r+1} \sim 0$ in X'.

Let $T \cap Q = T_1$ and $T - T_1 = T_2$; and let $U_1 = X' - T_1$, $U_2 = X' - F - T_2$. Note that $U_1 \cup U_2 = X'$ and $U_1 \cap U_2 = X' - T - F$.

Consider the diagram

$$\begin{tikzcd} H_{r+1}(A_1 \cup A_2) \arrow[r, "\triangle"] \arrow[d, "f"] & H_r(A_1 \cap A_2) \arrow[r, "i"] \arrow[d, "g"] & H_r(A_1) + H_r(A_2) \\ h_{r+1}(X') \arrow[r, "\triangle'"] & h_r(X' - T - F) \arrow[r, "i'"] & h_r(U_1) + h_r(U_2). \end{tikzcd}$$

By commutativity, $g\triangle[Z_{r+1}] = \triangle' f[Z_{r+1}]$, and since $f[Z_{r+1}] = 0$, it follows that $g\triangle[Z_{r+1}] = 0$ and that $Z_{r+1} \sim 0$ in X' - T - F. It then follows easily that $Z_{r+1} \sim 0$ in Q - T and, since $Q \subset U$ and U and x were arbitrary, that T is not a local r-separating set by virtue of Lemma 4.

Remark. Since X is (r + 1)-declinable whenever $h_{r+1}(X)$ is trivial, Lemma 5 is really a generalization of the Lemma of [7] (compare Lemma 2).

Definition. A subset M of a space X is called *semi*-r-*connected at* $x \in X$ if there exists a neighborhood U of x such that $h_r(M \cap U \mid M) = 0$. If M is semi-r-connected at all $x \in X$, we say that M is *semi*-r-*connected* rel X. [By $h_r(A \mid B)$ we mean the image of $h_r(A)$ in $h_r(B)$ induced by inclusion. Our concept is a relativization of the notion of semi-r-connectedness as given in [5, p. 167, Definition 19.4]. If M is an open set with compact closure, it provides a "uniform semi-r-connectedness" such as was used in [5] in Theorems 3.9, 3.15, and 3.17 of Chapter XII (where the word "uniformly" was inadvertently omitted).]

We recall that by the symbol $M^n_{r,r+1}$ we denote an orientable n-gcm that is spherelike in homology in dimensions r and r + 1.

LEMMA 6. *In order that a closed subset* M *of an* $M^n_{r,r+1}$, S, *should be* r-*declinable, it is necessary and sufficient that* S - M *be semi-*(n - r - 1)-*connected* rel S.

Proof of sufficiency. Suppose S - M is semi-(n - r - 1)-connected rel S, but that M is not r-declinable at $x \in M$. Then there exists an open subset P of S containing x such that

(1) some cycle Z_r on a compact subset F of M is not homologous on M to any cycle of M - P, and

(2) all (n - r - 1)-cycles of P - M are bounding in S - M.

Let $T = (S - P) \cup (M \cap P)$. Then $Z_r \not\sim 0$ on T, so that by the Alexander Duality Theorem there exists a cycle Z_{n-r-1} in P - M linked with Z_r. But $Z_{n-r-1} \sim 0$ in $S - M \subset S - F$, so that Z_{n-r-1} cannot be linked with Z_r.

Proof of necessity. Let M be r-declinable, and suppose that S - M is not semi-(n - r - 1)-connected at $x \in S$. Since S is r-lc, $x \notin S - M$. Hence x is a point of M such that every open set P which contains x also contains an (n - r - 1)-cycle of S - M that fails to bound in S - M. Let P be an open subset of S such that every r-cycle of M is homologous to a cycle of M - P. Since S is also (n - r - 1)-lc, there exists an open set Q such that $x \in Q \subset P$ and $h_{n-r-1}(Q \mid P) = 0$. Let Z_{n-r-1} be a cycle of Q - M that is nonbounding in S - M. Then [5, p. 266, Theorem 8.3] Z_{n-r-1} is linked with a cycle Z_r of M. But there exists a cycle γ_r on M - P that is in the same homology class of M as Z_r and is therefore also linked with Z_{n-r-1}. But $Z_{n-r-1} \sim 0$ in P, so that it cannot be linked with γ_r.

LEMMA 7. *Let* M *be a closed subset of an* $M^n_{r,r+1}$, S, *and* T *a closed, totally disconnected subset of* M *which is not an* r-*separating set of* M. *If* S - M *is semi-*(n - r - 2)-*connected* rel S, *then* T *is not a local* r-*separating set of* M.

Proof. By Lemma 6, M is (r + 1)-declinable and hence, by Lemma 5, T is not a local r-separating set of M.

THEOREM 3. *In* E^n $(n > 2)$, *let* C *be an* lc^k *continuum* (*with* $k = m - 1$ *if* $n = 2m$ *or* $n = 2m + 1$), *that cuts* E^n, *and let* T *be a closed, totally disconnected subset of* C *that is not an* r-*separating set of* C *for* $r \le k$. *Suppose that for each* $\varepsilon > 0$ *there exists an* ε-*transformation* $f: C \to (E^n - C) \cup T$ *that carries* C - T *into* $E^n - C$ *and is* (k - 1)-*monotone on* C - T, *and that* $E^n - C$ *is semi-*(n - r - 2)-*connected* rel E^n *for all* $r \le k$. *Then* C *is an orientable* (n - 1)-gcm.

Proof. By Lemma 7, T is not a local r-separating set of C for $r \le k$. By Lemma 3, $E^n - C$ is the union of two disjoint ulc^k domains having C as common boundary. If $n = 2m$, the theorem now follows from [5, p. 308, Theorem 7.1]; and if $n = 2m + 1$, the theorem follows from [5, p. 308, Theorem 7.2].

Remark. If T depends on ε, then we may of course add the assumption that $f(T) \subset T$, again apply Lemma 3, and so forth, and arrive at the same conclusion as in Theorem 3.

It is interesting to note that by increasing the requirement of semi-connectivity by one dimension when n is odd, the other conditions of the hypothesis of Theorem 3 can be modified. Thus we can state the following theorem:

THEOREM 4. *In* E^n $(n > 2)$, *let* C *be an* lc^k *continuum* $(k = m - 1$ *if* $n = 2m$ *or* $n = 2m + 1)$ *that cuts* E^n, *and let* T *be a closed, totally disconnected subset of* C *that is not an* r*-separating set of* C *for* $r \leq k - 1$. *Suppose that for arbitrary* $\varepsilon > 0$ *there exists an* ε*-transformation* $f: C \to (E^n - C) \cup T$ *carrying* $C - T$ *into* $E^n - C$ *which is* $(k - 1)$*-monotone on* $C - T$, *and that* $E^n - C$ *is semi-*$(n - r - 2)$*-connected* rel E^n *for* $r \leq k$ *when* n *is even, and for* $r \leq k + 1$ *when* n *is odd. Then* C *is an orientable* $(n - 1)$-gcm.

Proof. By Lemma 3, $E^n - C$ is the union of disjoint ulc^{k-1} domains U and V having C as common boundary. We shall show that T is not a k-separating set of C, from which it will follow that C is an orientable $(n - 1)$-gcm.

Let Z_k be a cycle with compact carrier $K \subset C - T$ that bounds on some compact subset M of C. Let $\varepsilon > 0$ be such that all $(n - k - 1)$-cycles of $E^n - C$ of diameter less than ε bound in $E^n - C$. Suppose $Z_k \not\sim 0$ in $C - T$. Then [5, p. 184, Theorem 4.4] $Z_k \not\approx 0$ in $C - T$, and consequently, by [5, p. 269, Theorem 9.1], Z_r is linked with an $(n - k - 1)$-cycle γ_{n-k-1} of $E^n - (C - T)$.

Since $E^n - C$ consists of the domains U and V, and $n - k - 1 > 0$, the cycle γ_{n-k-1} can be expressed as a sum of cycles γ_U and γ_V that have carriers in $U \cup T$ and $V \cup T$, respectively.

Since T is closed and totally disconnected, there exist a finite number of disjoint open subsets $W_1, \cdots, W_j$ of C, of diameters less than $\varepsilon/3$, covering T, whose boundaries (rel C) are disjoint and do not meet T. Consequently Z_k is homologous on $M - \bigcup_{h=1}^{j} W_h$ to a sum of cycles $Z_k^1, \cdots, Z_k^j$, where Z_k^h is a cycle on $F(W_h)$. By well-known methods, the cycles Z_k^h can be approximated in U by cycles U_k^h, of diameter less than ε, such that $Z_k^h \sim U_k^h$ in $E^n - \|\gamma_V\|$. By the choice of ε, the cycles U_k^h all bound in U, and it follows that Z_k bounds in $E^n - \|\gamma_V\|$. We conclude that Z_k is not linked with γ_V, and similarly that it is not linked with γ_U. But then Z_k is not linked with γ_{n-k-1}, and a contradiction results. We conclude that T cannot be a k-separating set of C.

By Lemma 3, the domains U and V are ulc^k, and when $n = 2m$, the theorem follows from [5, p. 308, Theorem 7.1]. If $n = 2m + 1$, then $E^n - C$ is semi-m-connected, and the theorem follows from [5, p. 308, Theorem 7.2].

In the next theorem, we exploit the semi-connectedness of $E^n - C$ in arriving at conditions where T is dependent upon ε.

THEOREM 5. *In* E^n $(n > 2)$, *let* C *be an* lc^k *continuum* $(k = m - 1$ *if* $n = 2m$ *or* $n = 2m + 1)$ *cutting* E^n, *and let* $E^n - C$ *be semi-*r*-connected* rel E^n *for* $r = n - k - 2, \cdots, n - 2$. *For arbitrary* $\varepsilon > 0$, *let there exist a closed and totally disconnected subset* T *of* C *and an* ε*-transformation* $f: C \to (E^n - C) \cup T$ *that carries* $C - T$ *into* $E^n - C$ *and is* $(k - 1)$*-monotone on* $C - T$, *and such that if* $n > 3$, T *is not an* r*-separating set of* C *for* $r = 1, \cdots, k$ *and* f *is* 0*-monotone on* T. *Then* C *is an orientable* $(n - 1)$-gcm.

Proof. By Lemma 1, $E^n - C$ is the union of two disjoint domains U and V of which C is common boundary. Since $n > 2$, a closed and totally disconnected subset of C is not a separating set of C; therefore, by Lemma 2, it is not a 0-separating set of C; and since $E^n - C$ is semi-$(n - 2)$-connected rel E^n, such a set would by Lemma 7 not be a local 0-separating set of C (and hence not a local separating set of C). For $0 < r \leq k$ and $n > 3$, the set T (dependent upon ε) is not a local r-separating set of $\bar{C}$ (by application of Lemma 7).

We first show that U is ulc^k. For the case where for every $\varepsilon > 0$ we may assume that $f(C - T) \subset V$, the proof is obtained by methods similar to those used in proving Lemma 3. Where we must assume that $f(C - T) \subset U$ and $n > 3$, the chief difference from the proof used in the corresponding case of the proof of Lemma 3 is that, since f is assumed 0-monotone on T, the set f(T) and hence the set $T \cup f(T)$ is closed and totally disconnected, whence the set A of the proof of Lemma 3 may be selected in $S(p, \delta') - T - f(T)$.

If $n = 3$ and only the 0-ulc is needed, but f is not assumed 0-monotone on T, we may proceed as follows: Suppose U is not ulc; then there exist $\varepsilon > 0$ and $x \in C$ such that in every $S(x, \delta)$ $(0 < \delta < 3\varepsilon)$ there exist points of U in different components of $U \cap S(x, 3\varepsilon)$. Since C is locally connected, there exists a δ $(0 < \delta < \varepsilon)$ such that $C \cap S(x, \delta)$ lies in a single component M of $C \cap S(x, \varepsilon)$. Select δ' so that $0 < \delta' < \delta$, and let $p, q \in S(x, \delta')$ in different components of $U \cap S(x, 3\varepsilon)$, and $r \in V \cap S(x, \delta')$. Let $\eta > 0$ be such that every η-transformation of C separates each of the point pairs (p, q), (p, r), (q, r) in $S(p, 2\varepsilon)$. Let f and T be as in the hypothesis (with η replacing ε) and with η small enough so that $f(M) \subset S(x, 2\varepsilon)$ and $f(C) \cap S(x, \delta') \subset f(M)$. Let $C' = f(C)$. As in the proof of Lemma 1, the set $C'' = C' - C$ is connected and lies in U or V; the latter case is handled by obvious methods, and we consider only the case where $C'' \subset U$.

Let A_1 be an arc from p to r in $S(p, \delta') - T$, and in the order from p to r let a_1 be the first point of M on A_1, and b the last point of M. Let A_2 be an arc from q to r in $S(p, \delta') - T$, and in the order from q to r let a_2 be the first point of M on A_2. If the subarc pa_1 of A_1 fails to meet C', then $pa_1 \cup (M - T) \cup rb$ (where rb is also a subarc of A_1) is a connected subset of $S(x, 2\varepsilon)$ not meeting C'; and if the subarc qa_2 of A_2 fails to meet C', then $qa_2 \cup (M - T) \cup rb$ is a connected subset of $S(x, 2\varepsilon)$ not meeting C'; in either case a contradiction results, and therefore we may suppose that there exist points $c_1 \in C' \cap pa_1$ and $c_2 \in C' \cap qa_2$.

Consider the set $M_1 = (M - T) \cup f^{-1}(f(M \cap T) \cap S(x, \delta') - T)$. By the choice of η, $f^{-1}(f(M \cap T) \cap S(x, \delta') - T) \subset M$, and since T is a frontier set rel C, M_1 is a connected subset of M such that $f(M_1) = M''$ contains all points of $C' - T$ in $S(p, \delta')$. Hence $c_1, c_2 \in M''$, and $pc_1 \cup M'' \cup qc_2$ (where pc_1 and qc_2 are subarcs of A_1 and A_2, respectively) is a connected subset of $S(x, 2\varepsilon)$ not meeting C. This contradicts the fact that p and q lie in different components of $U \cap S(x, 3\varepsilon)$.

In every case, then, U is ulc^k, and likewise V. That C is an orientable (n - 1)-gcm now follows from [5, p. 308, Theorem 7.1] when n is even, and from [5, p. 308, Theorem 7.2] when n is odd.

The following corollary of Theorem 5 is of interest in connection with Theorem 1 of [7].

COROLLARY. *In* E^3, *let* C *be a* 1-*acyclic, locally connected continuum cutting* E^3 *such that for each* $\varepsilon > 0$ *there exists an* ε-*transformation* $f: C \to (E^3 - C) \cup T$, *where* T *is a closed and totally disconnected subset of* C *and* f *carries* C - T *into* $E^3 - C$. *Then* C *is a* 2-*sphere.*

The corresponding case, the spherelike gcm, is determined by the following corollary of Theorem 5:

COROLLARY. *In* E^n $(n > 3)$, *let* C *be an* lc^k *continuum* $(k = m - 1$ *if* $n = 2m$ *or* $n = 2m + 1)$ *cutting* E^n *such that* $H_r(C) = 0$ *when* $1 \leq r \leq k + 1$. *For each* $\varepsilon > 0$, *let there exist a closed and totally disconnected subset* $\bar{T}$ *of* C *that is not an* *r-separating set of* C *if* $1 \leq r \leq k$, *and an* ε*-transformation* $f: C \to (E^n - C) \cup T$ *that carries* C - T *into* $E^n - C$, *is* (k - 1)*-monotone on* C - T, *and is* 0*-monotone on* T. *Then* C *is a spherelike* (n - 1)-gcm.

Proof. Since C is an orientable (n - 1)-gcm by Theorem 5, the spherelike character of C follows from the Poincaré Duality.

In all of the above theorems, local connectedness of the continuum imbedded in E^n has been assumed only to the dimension [n/2] - 1. The next two theorems exploit the local connectedness to higher dimensions.

THEOREM 6. *In* E^n $(n > 2)$, *let* C *be an* lc^{n-3} *continuum that cuts* E^n, *such that* $E^n - C$ *is semi-*(n - 2)*-connected* rel E^n. *For each* $\varepsilon > 0$, *let there exist a closed and totally disconnected subset* T *of* C *that is not a local* r*-separating set of* C *for* $1 \leq r \leq n - 3$, *and an* ε*-transformation* $f: C \to (E^n - C) \cup T$ *that carries* C - T *into* $E^n - C$. *Then* C *is an orientable* (n - 1)-gcm.

Proof. By Lemma 7, no closed and totally disconnected subset T of C is a local 0-separating set of C. By methods similar to those used above we can now show that one of the domains complementary to C, say U, is ulc^{n-3}. And, by the argument that was used in the proof of Theorem 5 to show, when n = 3 and $C'' \subset U$, that U is 0-ulc, it may be shown in the present case that V is 0-ulc. The theorem now follows from [5, p. 308, Theorem 7.2].

Remark. It is interesting to note that when n = 3, Theorems 5 and 6 are equivalent, but that when n = 4, Theorem 6 is stronger than Theorem 5.

For the case where the given continuum C is lc^{n-2}, we can modify the condition that f carry C - T into $E^n - C$. For this, we need the following analogue of Lemma 1:

LEMMA 8. *In* E^n $(n > 2)$, *let* C *be a continuum such that for each* $\varepsilon > 0$ *there exist a closed and totally disconnected subset* T *of* C *not separating* C, *and an* ε*-transformation* $f(C) = C'$ *into* $(E^n - C) \cup T$ *such that* $C' - C$ *lies in one component of* $E^n - C$. *Then* C *is a frontier set* rel E^n; *and if* C *cuts* E^n, *then* $E^n - C$ *consists of exactly two domains of which* C *is the common boundary.*

Proof. That $E^n - C$ has at most two components and that C is a frontier set rel E^n may be shown by methods similar to those used in proving Lemma 1. To show that when C cuts E^n, its complement consists of exactly two domains having C as common boundary, we begin the argument as in the proof of Lemma 1; but in order to show that $C' - C \subset V$, we select the ε-transformation f so that not all points of C - T in $S(p', \varepsilon)$ map into T. To do this, we note that the closure of $C \cap S(p', \varepsilon)$ contains a continuum K which meets $F(p', \varepsilon)$ and contains p', so that if F is an η-transformation (where $\eta < \varepsilon/4$), not all points of K can map into T, inasmuch as this would imply that f(K) is a single point. With this further restriction on f, we can again select $p \in (C' - T) \cap S(p', \varepsilon)$ in such a way that $f(p) \in C' - C$ and moreover $f(p) \in V$. The proof is now concluded as that of Lemma 1.

THEOREM 7. *In* E^n $(n > 2)$, *let* C *be an* lc^{n-2} *continuum that cuts* E^n. *For arbitrary* $\varepsilon > 0$, *let there exist a closed and totally disconnected subset* T *of* C *that is not a local* r*-separating set of* C *for* $r \leq n - 2$, *and an* ε*-transformation* $f: C \to (E^n - C) \cup T$ *such that* $f(C) - C$ *lies in one component of* $E^n - C$. *Then* C *is an orientable* (n - 1)-gcm.

Proof. By Lemma 8, $E^n - C$ is the union of disjoint domains U and V having C as common boundary. And since for every $\varepsilon > 0$, there exists an ε-transformation $f(C) = C'$ such that $C' - C$ lies in a single component of $E^n - C$, we may assume that for arbitrarily small ε, f may be so chosen that $C' - C$ lies in V. We may now use the argument that was used in proving Theorem 3 of [7] to show that U is ulc^{n-2}.

That C is an orientable $(n - 1)$-gcm now follows from [5, p. 311, Theorem 8.3].

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