ISOTROPY STRUCTURE OF COMPACT LIE GROUPS ON COMPLEXES

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1. INTRODUCTION

In this paper we prove the following conjecture of Floyd [1, page 95]:

THEOREM. *If* G *is a compact Lie group operating on a finite complex* K, *then there are only finitely many distinct conjugate classes of isotropy subgroups.* In the proof we use a decomposition of K into a finite number of invariant open manifolds, each of which has an orientable covering manifold whose integral cohomology (with compact supports) is finitely generated. Lifting the action of G to the covering manifolds, we apply a result of Mann [4] to establish the theorem.

The theorem is false when K is a locally-finite complex having finitely generated integral cohomology. To see this, consider the 2-complex consisting of a line and a sequence of closed discs, with centers on the line and going off to infinity. Define an action of the circle group S^1 on this complex by defining θ in S^1 $(0 < \theta < 2\pi)$ to act as the rotation $j\theta$ on the j^{th} disc. Then Z_j (the subgroup of S^1 isomorphic to the integers, modulo j) leaves the j-th disc point-wise fixed, and therefore Z_j is an isotropy subgroup for each j. On the other hand, the one-point compactification of this complex is of the same homotopy type as the circle, and therefore the finitely generated integral cohomology condition is satisfied.

2. CONSTRUCTION OF COVERING MANIFOLDS WITH FINITELY GENERATED COHOMOLOGY

Let K be any n-dimensional complex. Denote by F(K) the subset of K consisting of points which have neighborhoods homeomorphic to E^n. Then F(K) is an n-manifold. If F(K) is connected and K = Cl [F(K)], then K will be said to be *F-connected.*

LEMMA 1. *Let* K *be a finite* n*-complex which is* F*-connected and such that* F(K) *is a non-orientable* n*-manifold. Then there exist a finite* n*-complex* K* *and a simplicial map* p: K* → K *such that*

$$p \,|\, p^{-1}(F(K)) \colon p^{-1}(F(K)) \to F(K)$$

is the orientable double covering of F(K) *and* $p^{-1}(F(K))$ *has finitely generated cohomology.*

Proof. Let K^{n-1} be the (n - 1)-skeleton of K, let A be the subcomplex formed from the union of all (n - 1)-simplexes which are faces of exactly two n-simplexes, and let B be the union of all the other (n - 1)-simplexes in K^{n-1}. Note that

$$F(K) \subset K - B \subset (K - K^{n-1}) \cup A.$$

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Let $\sigma_1, \sigma_2, \cdots, \sigma_\ell$ be the n-simplexes in K. For each $i = 1, \cdots, \ell$, take σ_i^+ and σ_i^-, two oriented n-simplexes with opposite orientation. K^* is formed from these 2ℓ n-simplexes as follows: let $p_{i\varepsilon}: \sigma_i^\varepsilon \to \sigma_i$ be a simplicial map identifying σ_i^ε with σ_i ($\varepsilon = +$ or $-$). We determine intersections of simplexes in K^* as follows: Let τ be an (n - 1)-simplex in A which is a face of σ_i and σ_j, say. Then choose ε and ε' such that if τ^ε is the face of σ_i^ε corresponding to τ under $p_{i\varepsilon}$, and $\tau^{\varepsilon'}$ is the face of $\sigma_j^{\varepsilon'}$ corresponding to τ under $p_{j\varepsilon'}$, then $p_{i\varepsilon}(\tau^\varepsilon)$ and $p_{j\varepsilon'}(\tau^{\varepsilon'})$ induce opposite orientations on τ. We identify $p_{i\varepsilon}^{-1}(x)$ with $p_{j\varepsilon'}^{-1}(x)$ for all x in τ. Do this for each τ in A and each pair ε and ε'. The resulting identification space is the complex K^* having simplexes $\{\sigma_i^\varepsilon\}$, and the map $p: K^* \to K$ is obtained on each σ_i^ε by using $p_{i\varepsilon}$.

If x is in F(K), then x has a neighborhood contained in K - B, hence the intersection pattern in K^*, locally at each of the two points in $p^{-1}(x)$, is identical with the intersection pattern in K around x, hence $p \mid p^{-1}(F(K)): p^{-1}(F(K)) \to F(K)$ is a local homeomorphism. We can see that $p^{-1}(F(K))$ is orientable, by triangulating it in such a way that every simplex in the triangulation occurs in some rectilinear subdivision of K^*. The orientation of simplexes in K^* induces an orientation of each simplex in the triangulation. Clearly, any two n-simplexes meeting in an (n - 1)-face τ induce opposite orientations in τ, hence $p^{-1}(F(K))$ is orientable. Note that the cohomology of $p^{-1}(F(K))$ is finitely generated, since $K^* - p^{-1}(F(K))$ is a subcomplex.

3. LIFTING THE ACTION OF G

All spaces will be locally compact, metric, connected and locally connected. The term *covering space* is used in the sense of Chevalley [3]. The following lemma was suggested by Section 3 in [5].

LEMMA 2. *Let* M^* *be a covering space for* M *with the projection map* π *finite-to-one. Let* G *be a compact connected Lie group operating on* M, *and let* K *be a compact* G*-invariant subset of* M *on which* G *acts effectively. Then there exist a compact connected Lie group* G^*, *a neighborhood* U^* *of* 1 *in* G^*, *and a homomorphism* $\phi: G^* \to G$ *such that*

(1) $\phi \mid U^*$ *is a homeomorphism of* U^* *onto a neighborhood* U *of* 1 *in* G,

(2) G^* *acts effectively on* $\pi^{-1}(K) = K^*$, *and* $\pi g^*(x^*) = \phi(g^*)\pi(x^*)$ *for* x^* *in* K^*, g^* *in* G^*.

Proof. Cover K by $N_1, \cdots, N_\ell$, evenly covered connected open sets in M. Let U be a neighborhood of 1 in G which is compact, connected, symmetric, and such that $U(x) \equiv \{g(x) \mid g \text{ in } U\}$ is contained in some N_i, for each x in K. For each g in U, we define $\theta(g)$, a mapping of K^* into itself, as follows: if x^* is in K^* and $x = \pi(x^*)$, then $U(x) \subset N_i$ for some i. Let N_i^* be the component of $\pi^{-1}(N_i)$ containing x^*, and let $y^* = \pi^{-1}(g(x)) \cap N_i^*$. Then y^* is independent of i, hence, if we define $\theta(g)(x^*) = y^*$, then $\theta(g)$ is well-defined and continuous. It is one-to-one, and its inverse is continuous, since $\theta(g^{-1})$ is the inverse map. Let $\mathscr{H}(K^*)$ denote the set of all homeomorphisms of K^* onto itself with the compact-open topology. Then $\theta: U \to \mathscr{H}(K^*)$ is continuous and, since G is effective on K, θ is one-to-one. Let $U^* = \theta(U)$, and define $\phi: U^* \to G$ to be θ^{-1}. Since U is compact, ϕ is a homeomorphism, and condition (1) is satisfied. Note that $\pi g^* = \phi(g^*)\pi$ for all g^* in U^*.

Now let G^* be the subgroup of $\mathscr{H}(K^*)$ generated by U^*. Then G^* is connected, since U^* is connected. Let $g^* = g_1^* g_2^* \cdots g_k^*$ be in G^*, where g_i^* is in U^*. Then

$$\pi g^* = \pi g_1^*(g_2^* \cdots g_k^*) = \phi(g_1^*)\,\pi(g_2^* \cdots g_k^*)\,,$$

and by iteration

$$\pi g^* = \phi(g_1^*)\,\phi(g_2^*) \cdots \phi(g_k^*)\pi\,.$$

Therefore, if we define $\phi(g^*) = \phi(g_1^*)\,\phi(g_2^*) \cdots \phi(g_k^*)$, then $\phi\colon G^* \to G$ is a well-defined homomorphism, and condition (2) is satisfied. The kernel of ϕ contains only covering transformations, hence is finite, and G^* is a compact Lie group. If g^* is near 1 in G^*, then $\phi(g^*)$ is near 1 in G, hence in U, and this implies that g^* lies in U^*. Therefore U^* is a neighborhood of 1 in G^*, and the proof is complete.

The next lemma is a strong form of the uniqueness in Lemma 2.

LEMMA 3. *Suppose that* M^*, M, G *and* K_i $(i = 1, 2)$ *satisfy the hypotheses in Lemma* 2, *with* $K_1 \subset K_2$, *and that* G_i^*, U_i^* *and* ϕ_i *satisfy conditions* (1) *and* (2), *for* $i = 1, 2$. *Then* K_1^* *is invariant under* G_2^*, *and the restriction map* ρ *of the action of* G_2^* *to* K_1^* *is a homomorphism of* G_2^* *onto* G_1^* *such that* $\phi_1 \rho = \phi_2$.

Proof. Let V^* be a small neighborhood of 1 in $\mathscr{H}(K_1^*)$, so that g_1^*, g_2^* in V^* with $\pi g_1^* = \pi g_2^*$ implies $g_1^* = g_2^*$. Clearly, K_1^* is G_2^*-invariant, by (2), and ϕ is a homomorphism of G_2^* into $\mathscr{H}(K_1^*)$. Without loss of generality (using (1)), assume that U_1^* and $\rho(U_2^*)$ are contained in V^* and that U is the common image of U_i^* under ϕ_i $(i = 1, 2)$. It suffices to show that

$$\rho(g^*) = (\phi_1|\, U)^{-1}\phi_2(g^*)$$

for g^* in U_2^*. Now

$$\pi[\rho(g^*)\,(x^*)] = \pi[g^*(x^*)] = \phi_2(g^*)[\pi(x^*)] = \phi_1((\phi_1\,|\,U)^{-1}\phi_2(g^*))[\pi(x^*)]$$
$$= \pi(\phi_1\,|\,U)^{-1}\phi_2(g^*)(x^*)\,,$$

for all x^* in K_1^*; hence $\pi\rho(g^*) = \pi(\phi_1\,|\,U)^{-1}\phi_2(g^*)$, and, by the argument above, $\rho(g^*) = (\phi_1|\, U)^{-1}\phi_2(g^*)$.

LEMMA 4. *Let* M^*, M, G *and* π *satisfy the hypotheses in Lemma* 2. *Then there exist a compact connected Lie group* G^*, *a homomorphism* $\phi\colon G^* \to G$, *and a neighborhood* U^* *of* 1 *in* G^* *such that*

(1) $\phi\,|\,U^*$ *is a homeomorphism of* U^* *onto a neighborhood* U *of* 1 *in* G,

(2) G^* *acts on* M^*, *and* $\pi g^*(x^*) = \phi(g^*)\,\pi(x^*)$ *for* x^* *in* M^*, g^* *in* G^*.

Proof. Express M as the union of an ascending sequence of compact invariant subsets $\{K_i\}$. Employ Lemma 2 to get G_i^* and ϕ_i, for each i. Let ρ_{ij} be the "restriction" map of G_i^* into G_j^* for $i \geq j$. Then $\{G_i^*, \rho_{ij}\}$ forms an inverse system. Let G^* be the inverse limit. By Lemma 3, $\phi_j\,\rho_{ij} = \phi_i$, hence G^* operates in an obvious manner on M^*, and the ϕ_i define a homomorphism of G^* into G satisfying (2). For large j, ρ_{ij} is an isomorphism and G_j^* is isomorphic to G^*. From this, (1) readily follows.

4. PROOF OF THE THEOREM

By [1; p. 94], it suffices to prove the theorem for the case where G is a toral group. If K is n-dimensional, then K - F(K) is a G-invariant subcomplex having dimension less than n. Thus if we can show that the isotropy groups $\{G_x \mid x$ in $F(K)\}$ fall into finitely many conjugate classes, we can work inductively to obtain the theorem. Finally, since F(K) has finitely many components and each is G-invariant, we may assume in the proof that F(K) is connected and that K is Cl[F(K)], in other words, that K is F-connected.

If F(K) is an orientable manifold, we use the main result in [4] directly, and we are done. If F(K) is non-orientable, we use Lemma 1 to show that its orientable double covering M* has finitely generated cohomology. We use Lemma 4 to lift the action of G to an action of G*, an abelian group, on M*. By [4], G* has finitely many distinct isotropy groups. The map ϕ: G* → G is at most two-to-one, since the kernel of ϕ consists of covering transformations of M*. If x* is in M*, then $\phi^{-1}(G_{\pi(x^*)})$ contains $G^*_{x^*}$ as a subgroup of index at most two. By [5, Lemma 2] there are at most a finite number of distinct $\phi^{-1}(G_{\pi(x^*)})$, hence only a finite number of distinct $G_{\pi(x^*)}$, and the theorem is proved.

The referee points out that Bredon [2] has proved the special case of Lemma 4 where M* is the orientable covering.

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