SOME REMARKS ON FUNDAMENTAL SOLUTIONS OF PARABOLIC DIFFERENTIAL EQUATIONS OF SECOND ORDER

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1. INTRODUCTION

Let E^n be the real n-dimensional Euclidean space of points $x = (x_1, \cdots, x_n)$, and $D \subset E^n$ an open simply connected domain. Let

$$(1.1) \qquad L(u) = \sum_{i,k=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ik}\frac{\partial u}{\partial x_k}\right) - V(x)u \qquad (a_{ik} = a_{ki},\ V \geq 0)$$

be a uniformly elliptic operator in D with coefficients depending on x. Let I_T be the interval $0 < t < T$, and Δ_T the product $D \times I_T$. A fundamental solution of the parabolic equation

$$(1.2) \qquad \Lambda \equiv L(u) - \frac{\partial u}{\partial t} = 0$$

in $\Delta = \Delta_\infty$ may be defined as a function $\Gamma(x, \xi, t)$ which as function of (x, t) satisfies (1.2) in Δ, and in addition has the following property: for each function $h(x)$ which is continuous in the closure $\overline{D}$ of D and for each (proper or improper) subdomain D_1 of D, the limit relation

$$(1.3) \qquad \lim_{t \to 0} \int_{D_1} h(\xi)\, \Gamma(x, \xi, t)\, d\xi = \begin{cases} h(x) & \text{for } x \text{ interior to } D_1, \\ 0 & \text{for } x \text{ interior to } D - D_1 \end{cases}$$

holds [19], [6], [7].

It is known that if D is bounded and has a smooth enough boundary $\dot{D}$, then such a fundamental solution may be constructed as follows: let $\{u_1(x), u\ (x), \cdots\}$ be a full orthonormal set of eigenfunctions, and $\{-\lambda_1, -\lambda_2, \cdots\}$ the set of corresponding eigenvalues of the elliptic eigenvalue problem

$$(1.4) \qquad L(u) - \lambda u = 0 \quad \text{in } D,$$

$$(1.5) \qquad u = 0 \quad \text{on } \dot{D};$$

then

$$(1.6) \qquad G(x, \xi, t) = \sum_{k=1}^{\infty} u_k(x)\, u_k(\xi)\, e^{-\lambda_k t}$$

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is a fundamental solution. Moreover, if $G_0(x, \xi, t)$ is the Green's function belonging to the problem (1.4), (1.5) multiplied by λ, and if for $\nu = 1, 2, \cdots$ the function G_ν is defined by the iteration

$$(1.7) \qquad G_\nu(x, \xi, \lambda) = \int_D G_0(x, \zeta, \lambda)\, G_{\nu-1}(\zeta, \xi, \lambda)\, d\xi ,$$

then the G defined in (1.6) may also be written as

$$(1.8) \qquad G(x, \xi, t) = \lim_{\nu\to\infty} G_\nu(x, \xi, \nu/t) .$$

(See [19, pp. 489, 493]. There only the case $V(x) = 0$ and $n = 3$ was treated. The extension to the more general case considered in the present paper requires only minor changes which concern the proof of the convergence of the series in (1.6) and of (1.8). These changes are indicated in Appendix II.)

On the other hand, F. G. Dressel constructed in a totally different way a fundamental solution for the case of a bounded D as well as for the case $D = E^n$. (See [6] and [7]; Dressel's advance over [19] consists in the fact that he does not confine himself to the selfadjoint operator (1.1), but deals with a general elliptic operator L of order 2 with coefficients which may depend on t. In the present paper, however, we use his results only for the special case indicated.) The function constructed by Dressel for the case $D = E^n$ we shall call Dressel's function, and we reserve for it the notation $\Gamma(x, \xi, t)$; in the case of the one-dimensional heat equation

$$(1.9) \qquad \frac{\partial^2 u}{\partial u^2} - \frac{\partial u}{\partial t} = 0 ,$$

it reduces to the well-known function $(2\sqrt{\pi t})^{-1} \exp[(x - \xi)^2/4t]$. To avoid confusion, we shall refer to the function G defined by (1.6) or (1.8) as Green's function. This terminology is appropriate, since $G(x, \xi, t)$ takes boundary values 0 on the set of points $(\dot{x}, t)$ with $\dot{x} \in \dot{D}$, $t > 0$, and on the set $x \in D$, $t = 0$ $(x \neq \xi)$. The first statement follows from (1.5), (1.6), and the second from (1.6), (1.25).

In Section 2 we shall have to recall the construction of Dressel's function. At this moment we only note that the "principal part" $Z(x, \xi, t)$ characterizing the singularity at $x = \xi$, $t = 0$ is of the form

$$(1.10) \qquad Z(x, \xi, t) = t^{-n/2} \exp[-\sigma(x, (x - \xi)/4t] / F(\xi) ,$$

where $F(\xi)$ is a normalizing factor (see (2.2)), and $\sigma(x, x - \xi)$ is the uniformly positive definite quadratic form in $x - \xi$ whose matrix is the inverse of the matrix of the coefficients a_{ik} in (1.1).

While thus for Dressel's function the singularity is made explicit, the same is not true for Green's function G, either in the form (1.6) or that of (1.8). It is therefore not obvious that the principal part of G is also given by (1.10), in other words, that in the difference

$$(1.11) \qquad \gamma(x, \xi, t) = \Gamma(x, \xi, t) - G(x, \xi, t)$$

the singularity "cancels out"; for there are many "δ-functions" satisfying (1.3), for example, for $n = 1$, the Fejér kernel

$$t(2\pi)^{-1}\{\sin[(x-\xi)/2t]\}^2\{\sin[(x-\xi)/2]\}^{-2}.$$

The main object of this paper is to prove (1.11) with a "regular" γ approaching zero as $t \to 0$, and to show that this equality can be used to prove a number of facts concerning G and Γ as well as facts concerning the eigenfunctions $u_k(x)$ and eigenvalues λ_k of (1.4) and (1.5); not all of these facts are new.

The proof for (1.11) given in Section 2 is along the following lines: it is seen easily from (1.3) (with Γ replaced by G) and from (1.5) and (1.6) that the function

$$U(x, t) = \int_D h(\xi)\, G(x, \xi, t)\, d\xi \tag{1.12}$$

is in Δ a solution of (1.2) and satisfies the boundary conditions

$$\lim_{t\to 0} U(x, t) = h(x) \qquad (x \in D,\ t > 0), \tag{1.13}$$

$$\lim_{(x,t)\to(\dot{x},t)} U(x, t) = 0 \qquad (x \in D,\ \dot{x} \in \dot{D},\ t > 0). \tag{1.14}$$

The function obtained from (1.12) by replacing G by Γ satisfies (1.2) and (1.13), but not (1.14). However, it is easy to construct a function $\gamma(x, \xi, t)$, without singularities in Δ, having the following property: if G^1 is defined by

$$G^1(x, \xi, t) = \Gamma(x, \xi, t) - \gamma(x, \xi, t), \tag{1.15}$$

then the function

$$U_1(x, t) = \int_D h(\xi)\, G^1(x, \xi, t)\, d\xi \tag{1.16}$$

satisfies (1.2), (1.13), (1.14); that is, U_1 is a solution of the same boundary value problem as the function U defined by (1.12). (For details and literature see Section 2.)

Unfortunately this fact does not allow us to conclude that U_1 is identical with U. For the solution of the boundary value problem in question is not unique, as is well known (see [5, p. 299]) even in the simple case of the equation (1.9).

However it is also known that our boundary value problem has at most one solution which is continuous in the closure $\overline{\Delta}$ of Δ. (See [18, pp. 253, 254], where the "maximum principle" is used for the uniqueness proof. The application of this principle to parabolic equations in certain cases was already used in [12, pp. 372, 373]. See also [16].) For the existence of such a solution, it is obviously necessary, because of (1.13) and (1.14), that

$$h(x) = 0 \qquad \text{for } x \in \dot{D}. \tag{1.17}$$

Under the assumption (1.17), it will be shown in Appendix I that U and U_1 are continuous in $\overline{\Delta}$. Thus the integrals in (1.12) and (1.16) represent the same function for all continuous $h(x)$ satisfying (1.17). This in turn implies

$$G(x, \xi, t) = G^1(x, \xi, t) \qquad (x, \xi \text{ in } D,\ t > 0), \tag{1.18}$$

which because of (1.15) and the regularity of γ is what we wanted to prove.

The first of the applications of (1.11) given in Section 3 is a proof of asymptotic relations for eigenfunctions and eigenvalues of (1.4), (1.5): we shall see that (1.18), together with certain properties of γ and with (1.6) and (1.10) easily yields, for $x = \xi \in D$, the asymptotic relation

$$\sum_{k=1}^{\infty} u_k^2(x)\, e^{-\lambda_k t} \sim \frac{t^{-n/2}}{F(x)} \quad \text{for } t \to 0+ . \tag{1.19}$$

Applying a suitable Tauberian theorem (see, for example, [22, p. 192]) to (1.19), we obtain

$$\sum_{\lambda_k<\lambda} u_k^2(x) \sim \frac{1}{F(x)} \frac{\lambda^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)} \quad \text{for } \lambda \to \infty , \tag{1.20}$$

where $\Gamma(\tau)$ denotes Euler's gamma function. Integrating over D we obtain, for the number $A(\lambda)$ of eigenvalues less than λ of the boundary value problem (1.4), (1.5),

$$A(\lambda) = \sum_{\lambda_k<\lambda} 1 \sim \frac{\lambda^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)} \int_D \frac{1}{F(x)}\, dx . \tag{1.21}$$

(The method of using Tauberian theorems to derive estimates for eigenfunctions and eigenvalues is standard. It was used first by Carleman [4]. In case that L is the Laplace operator Δ, M. Kac [14] combined this method with probabilistic arguments which he used to derive (1.19) for $L = \Delta$. The same approach was used by him for certain integral operators [15], and by D. Ray for $L(u) = \Delta(u) + V(x)u$ in [17]. Corresponding results for equations whose order is greater than 2 with coefficients in C^∞ were obtained by Browder [2], [3], Gårding [11], and Bergendal [1] by means of Hilbert space methods. The present paper uses straight-forward "classical analysis." In all three methods, a central role is played by what M. Kac [14] calls "the principle of not feeling the boundary" namely, the fact that the Green's function satisfying certain boundary conditions shows for $t \to 0+$ the same asymptotic behavior as the fundamental solution constructed. In the present paper, this principle is expressed in the relation (1.11) between G and Γ, and in [1] in equation (0.4), p. 244.)

The next application concerns the so-called bilinear series

$$\sum_{k=1}^{\infty} \frac{u_k(x)\, u_k(\xi)}{\lambda_k}. \tag{1.22}$$

It is well known from the theory of integral equations that if the $u_k(x)$ form a full orthonormal set of eigenfunctions of a homogeneous Fredholm integral equation with a continuous and symmetric kernel $K(x, \xi)$ and if the λ_k are the corresponding eigenvalues, then

$$K(x, \xi) = \sum_{k=1}^{\infty} \frac{u_k(x)\, u_k(\xi)}{\lambda_k}, \tag{1.23}$$

provided that the series converges uniformly. (See for example [21, p. 303, Satz 57].) It is also well known that in general the series does not converge. (See [21, p. 305], where the classical sufficient conditions by Mercer and Hammerstein for the convergence of (1.23) are discussed, and where an example exhibiting divergence is given.) Our application is concerned with the case where the u_k and $-\lambda_k$ are defined as before, that is, are the eigenfunctions and eigenvalues of (1.4), (1.5). Here the above kernel K is the Green's function of the operator L defined in (1.1) belonging to the boundary condition (1.5), and the question is the validity of (1.23). If L is the Laplace operator in E^n with $n = 2$ or 3, then for $x \in D$ and for ξ in a closed subdomain D_1 of D, A. Hammerstein constructed convergence factors (depending on D_1) and a lower bound ρ for $|x - \xi|$ such that, after insertion of these convergence factors, (1.23) is valid if ξ is restricted to D_1 and $|x - \xi| \geq \rho$ (see [13, p. 285, Satz 6, and p. 296]). In Section 3 it will be shown that for the general operator L given by (1.1) with arbitrary n and arbitrary $x \neq \xi$ in D, (1.24) becomes valid after insertion of the factor $\exp(-\lambda_k t)$; in other words, that

$$(1.24) \qquad K(x, \xi) = \lim_{t \to 0+} \sum \frac{u_k(x)\, u(\xi)}{\lambda_k} e^{-\lambda_k t}.$$

It will also be shown that

$$(1.25) \qquad \lim_{t \to 0+} \sum_{k=1}^{\infty} u_k(x)\, u_k(\xi)\, e^{-\lambda_k t} = 0 \qquad \text{for } x \neq \xi,$$

a relation which supplements (1.19). For a special class of elliptic operators, it is known that (1.24) still holds if $\exp(-\lambda_k t)$ is replaced by $(1 + t\lambda_k)^{-1}$, and that (1.25) holds if $\exp(-\lambda_k t)$ is replaced by $(1 + t\lambda_k)^{-2}$ [19, p. 674]. This special case may be described as follows: each selfadjoint operator L of order 2 in n variables $(n > 3)$ may (up to a factor) be written as the second Beltrami operator belonging to a Riemannian line element [8, p. 637]. Our special case is that where this Riemannian line element has constant curvature.

As a further application we consider the behaviour of Γ and G for $t \to \infty$. As will be seen, the relation

$$(1.26) \qquad \Gamma(x, \xi, t) = O(t^{-(n+1)/2}) \qquad \text{for } t \to \infty,$$

where O denotes Landau's symbol, follows nearly immediately from inequalities contained in Dressel's work. But it will also be shown that, for arbitrary positive r,

$$(1.27) \qquad G(x, \xi, t) = O(t^{-r}) \qquad \text{for } t \to \infty.$$

Finally, let $D^{(1)}, D^{(2)}, D^{(3)}, \cdots$ be a sequence of bounded domains which exhaust E^n, and denote by $G^{(p)}$ the Green's function (1.6) belonging to $D = D^{(p)}$. Then it will be shown that

$$(1.28) \qquad \lim_{p \to \infty} G^{(p)}(x, \xi, t) = \Gamma(x, \xi, t) \qquad \text{for } t > 0.$$

The assumptions of this paper concerning the smoothness of $\dot{D}$ and the coefficients of (1.1) are those made in [6], [7] and [19].

2. PROOF OF (1.18)

We start by recalling the construction of Dressel's function. First we complete the definition (1.10) of the function Z by giving the definition of $F(\xi)$: if the coordinates $\xi_1, \xi_2, \cdots, \xi_n$ are expressed in spherical coordinates r, θ with the point x as pole, where

$$r = |x - \xi| = \{ \sum (x_i - \xi_i)^2 \}^{1/2} \quad \text{and} \quad \theta = \{\theta_1, \theta_2, \cdots, \theta_{n-1}\},$$

and if $\phi(x, \theta)$ is defined by setting

$$\sigma(x, x - \xi) = r^2 \phi(x, \theta), \tag{2.1}$$

then

$$F(\xi) = \frac{(2\sqrt{\pi})^n}{m(S^{n-1})} \int_{S^{n-1}} \phi^{-n/2} ds, \tag{2.2}$$

where S^{n-1} is the surface of the unit sphere in E^n, $m(S^{n-1})$ its measure, and ds its surface element. (If L is the Laplace operator, then $F = (2\sqrt{\pi})^n$ and $Z = (4\pi t)^{-n/2} \exp[-r^2/4t]$.) This completes the definition of Z. Dressels function $\Gamma(x, \xi, t)$ is then defined by

$$\Gamma(x, \xi, t) = Z(x, \xi, t) + \int_0^t \int_{E^n} Z(x, \zeta, t - \theta) f(\zeta, \xi, \theta) \, d\xi \, d\theta, \tag{2.3}$$

where $f(\zeta, \xi, \theta)$ is the solution of a certain integral equation (see [7, (9)]) and satisfies, for suitable positive constants C and h, the following inequality ([7, Theorem 2]):

$$|f(x, \xi, t)| \leq C t^{-(n+1)/2} \exp\left[\frac{-h|x - \xi|^2}{4t}\right]. \tag{2.4}$$

We now define the function γ appearing in the definition (1.15) of G'. For fixed $\xi \in D$, let $\gamma = \gamma(x, \xi, t)$ as function of x and t be in Δ a solution of the boundary value problem

$$L(\gamma) - \frac{\partial \gamma}{\partial t} = 0, \tag{2.5}$$

$$\lim_{x \to \dot{x}} \gamma(x, \xi, t) = \Gamma(x, \xi, t) \qquad (\dot{x} \in \dot{D}, \ t > 0), \tag{2.6}$$

$$\lim_{t \to 0+} \gamma(x, \xi, t) = 0 \qquad (x \in D). \tag{2.7}$$

(The classical way is to define $\gamma(x, \xi, t, \tau)$ as function of ξ and τ as that solution of the equation adjoint to (1.2) which satisfies certain boundary conditions, and then to prove that γ as function of x and t satisfies the original equation (1.2). See for example [12, p. 323] or [10]. In the present paper we proceed in a slightly different way which avoids the use of the adjoint to (1.2).)

According to well-known existence and uniqueness theorems [18], problem (2.5) to (2.7) has one and only one solution which is continuous in $\overline{\Delta}$, provided that the given boundary values are continuous. To show that this condition is satisfied in our case we must obviously, because of (2.6) and (2.7), prove that

$$\lim_{(\dot{x},t)\to(\dot{x}_0,0+)} \Gamma(\dot{x}, \xi, t) = 0 \quad \text{for } \dot{x} \in \dot{D},\ \dot{x}_0 \in \dot{D}. \tag{2.8}$$

For this purpose we recall first that, because of the uniform ellipticity of L and the definition of σ, there exist two positive constants a and A such that

$$a|x - \xi|^2 \leq \sigma(x, x - \xi) \leq A|x - \xi|^2 \tag{2.9}$$

(see [7] and [6]). Consequently, by (1.10), (2.2), and (2.1), there exists a positive constant C_1 such that

$$0 \leq Z(x, \xi, t) \leq C_1 t^{-n/2} \exp\left[-a\frac{|x - \xi|^2}{4t}\right]. \tag{2.10}$$

If we denote the integral in (2.3) by $I(x, \xi, t)$ then, by Lemma 2 of [7], conditions (2.4) and (2.10) imply that for suitable constants C_2 and g_2,

$$|I(x, \xi, t)| \leq C_2 t^{(-n-1)/2} \exp\left[-g_2\frac{|x - \xi|^2}{4t}\right]. \tag{2.11}$$

Since for fixed ξ in D and variable $\dot{x}$ on $\dot{D}$, $|\dot{x} - \xi|$ is bounded away from zero, (2.11), (2.10) and (2.3) imply (2.8).

We can now define γ as the uniquely determined solution of the boundary value problem (2.5), (2.6), (2.7) which is continuous in $\overline{\Delta}$. The following two lemmas deal with properties of γ which will be needed later on.

LEMMA 2.1. *For fixed* x *in* D *and* $t > 0$, γ *is continuous in* ξ *as this point varies over* D.

Proof. For ξ_1, ξ_2 in D, the function $\delta(x, t) = \gamma(x, \xi_1, t) - \gamma(x, \xi_2, t)$ satisfies the boundary value problem obtained from the one which defines γ by replacing the right member of (2.6) by $\Gamma(\dot{x}, \xi_1, t) - \Gamma(\dot{x}, \xi_2, t)$. Noting that δ satisfies (2.7), we see therefore from the maximum principle that the difference $\delta(x, t)$ is in absolute value not greater than the maximum of $|\Gamma(\dot{x}, \xi_1, \tau) - \Gamma(\dot{x}, \xi_2, \tau)|$ as $\dot{x}$ varies over $\dot{D}$ and τ varies over the interval $0 \leq \tau \leq t$. This proves the lemma.

LEMMA 2.2. *For* x, ξ *in* D *and* $t > 0$,

$$0 \leq \gamma(x, \xi, t) \leq \Gamma(x, \xi, t). \tag{2.12}$$

Proof. The right part of this inequality can be proved by application of the maximum principle (such a proof is given in [10]).

If the first part of (2.12) were false, then γ would take on a negative minimum on $\overline{\Delta}_{t_0} = \overline{D} \times \{0 \leq t \leq t_0\}$, which because of the maximum principle and (2.7) would be taken in some point of the product $\dot{D} \times \{0 \leq t \leq t_0\}$, and we see from (2.6) that in such a point Γ would take on a negative value. Consequently it will be sufficient to prove

LEMMA 2.3. *For all* x, ξ *and* $t > 0$, (2.13) $$\Gamma(x, \xi, t) \geq 0.$$

Proof. We claim first: corresponding to any positive numbers t_0 and ε, there exists a number R such that

(2.14) $$|\Gamma(x, \xi, t)| \leq \varepsilon \quad \text{for } |x - \xi| \geq R,\ 0 < t \leq t_0.$$

This follows easily from the definitions (2.3), (2.2), (1.10) together with the inequality (2.11), if we note that, for any positive constants β and k and for $p > \beta$,

$$0 < \frac{1}{t^\beta e^{kr^2/t}} < \frac{1}{t^\beta (kr^2/t)^p/p!} = \frac{p!\, t^{p-\beta}}{k^p\, r^{2p}} \qquad (t > 0,\ r = |x - \xi|).$$

It follows from (2.14) that, for any continuous function $h(x)$ with bounded support S,

(2.15) $$V(x, t) = \int_{E^n} h(\xi)\Gamma(x, \xi, t)\, d\xi \to 0$$

uniformly for $0 < t \leq t_0$ as $|x| \to \infty$. For the case where in addition $h(x)$ is non-negative, we prove next that

(2.16) $$V(x, t) \geq 0 \quad \text{for all } x \text{ in } E^n \text{ and } t > 0.$$

For the proof, let us suppose that, for some x_1, t_1 with $t_1 > 0$,

(2.17) $$V(x_1, t_1) = -p_1 < 0.$$

We first choose the arbitrary number t_0 of the beginning of our proof to be not smaller than t_1. Since the support S of $h(x)$ is bounded, it follows from (2.14) that there exists an R_1 such that

(2.18) $$|V(x, t)| < p_1/2 \quad \text{for } |x| = R_1,\ 0 < t \leq t_0,$$

and that in addition S is contained in the open ball with center 0 and radius R_1. Now, in the domain $B(t_0)$ defined as the product of this ball with the interval $0 < t \leq t_0$, the function $V(x, t)$ is a solution of (1.2), as can be seen from the following considerations. Since h has compact support, the integral in (2.15) is for $|x| \leq R_1$, $0 < \delta \leq t \leq t_0$ an ordinary Riemann integral, the integrand being uniformly continuous in all its variables. The approximating Riemann suns of the form

$$S_p(x, t) = \sum_\nu \Gamma(x, \xi_\nu, t)\, h(\xi_\nu)\triangle\xi$$

are solutions of (1.2) which satisfy the boundary conditions

$$S_p(\dot{x}, t) = \sum_\nu \Gamma(\dot{x}, \xi_\nu, t)\, h(\xi_\nu)\triangle\xi \qquad (|\dot{x}| = R_1,\ \delta \leq t \leq t_0),$$
$$S_p(x, \delta) = \sum_\nu \Gamma(x, \xi_\nu, \delta)\, h(\xi_\nu)\triangle\xi \qquad (|x| \leq R_1).$$

For $p \to \infty$, the sums in the last two lines converge to the corresponding integrals, uniformly on $|x| = R_1$, $\delta \leq t \leq t_0$, and on $|x| \leq R$, $t = \delta$. By Harnack's convergence theorem, we conclude that the solutions $S_p(x, t)$ satisfying these boundary conditions converge to a solution of (1.2). This completes the proof that $V(x, t)$ is a solution of

(1.2), since the limit of the $S_p(x, t)$ is the integral (2.15). (For a proof of Harnack's theorem for linear parabolic equations of second order, see [12, pp. 386-387] or [10]).

Moreover, $V(x, t)$ is continuous in the closure of $B(t_0)$. Because of (2.17), its minimum in this closure is at most $-p_1$. By the maximum principle, the minimum must be taken either on $|x| = R_1$ or on $t = 0$. By (2.18), the first possibility must be excluded. The same is true of the second possibility, for it is clear from the definition (2.15) of V and the main property of Dressel's function Γ that the limit of $V(x, t)$ as $t \to 0$ is $h(x)$, and this function is nonnegative by assumption.

Thus the assumption (2.17) leads to a contradiction, and (2.16) is established for every nonnegative continuous $h(x)$ with compact support. We omit the routine proof that this fact implies the assertion (2.13) of Lemma 2.3.

Our aim is to prove that the U_1 defined in (1.16) is a solution of (1.2), (1.13), (1.14). Now, as function of ξ, $\gamma(x, \xi, t)$ is continuous as ξ varies over D, and it follows from Lemma 2.2 that it is also bounded if a positive t is fixed. Consequently the integral

$$w(x, t) = \int_D h(\xi)\gamma(x, \xi, t)\, d\xi \qquad (t > 0)$$

exists for continuous $h(\xi)$. We then have, from (1.15),

$$U_1(x, t) = W(x, t) - w(x, t), \tag{2.19}$$

where we have set

$$W(x, t) = \int_D h(\xi)\Gamma(x, \xi, t)\, d\xi .$$

Since Γ satisfies (1.2) and (1.3), and γ satisfies (2.5), (2.6) and (2.7), it will be seen from (2.19) that U_1 is indeed a solution of the boundary value problem (1.2), (1.13), (1.14) of which U, defined by (1.12), is also a solution. (That $W(x, t)$ satisfies (1.2) may be seen by a method quite similar to the one outlined above for the case of the integral (2.15).)

It was pointed out in the Introduction that this fact alone does not ensure the equality

$$U(x, t) = U_1(x, t), \tag{2.20}$$

but that for its validity it will be sufficient to show that both members of (2.20) are continuous in the closure $\overline{\Delta}$ of Δ. The proof that this is true if the necessary condition (1.17) is satisfied consists essentially of some modifications in the relevant arguments in [6], [7], [19]. These modifications will be indicated in Appendix I (Section 4). A complete independent proof would involve copying great parts of [19] and [7].

The proof that the validity of (2.20) for all continuous $h(x)$ satisfying (1.17) implies (1.18) is routine, and it will therefore be omitted.

3. APPLICATIONS

As a first application of the main result (1.18) of the preceding section we give a proof of the asymptotic relations (1.20), (1.21). It is well known, as pointed out in the Introduction, that it suffices to prove (1.19). If $I(x, \xi, t)$ again denotes the integral in (2.3), we see from (1.18) and the definitions (1.6), (1.15) and (1.10) that

$$\sum_{k=1}^{\infty} u^2(x)\, e^{-\lambda_k t} = t^{-n/2}(F(x))^{-1} + I(x, x, t) - \gamma(x, x, t). \tag{3.1}$$

Because of (2.7) and the fact that, by (2.11), $I(x, x, t) = O(t^{-(n-1)/2})$ for $t \to 0$, (1.19) follows immediately.

We now turn to the proof of (1.24) and (1.25), where $K(x, \xi)$ is the Green's function with boundary conditions zero belonging to the operator L defined in (1.2). We then have

$$u_k(x) = \lambda_k \int_D K(x, \alpha)\, u_k(\alpha)\, d\alpha, \tag{3.2}$$

and since (see [19, Section 1]), for fixed $t > 0$ the series in (1.6) converges absolutely and uniformly for x, ξ in D, we obtain from (3.2)

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{u_k(x)\, u_k(\xi)}{\lambda_k} e^{-\lambda_k t} &= \int_D K(\xi, \alpha) \sum_{k=1}^{\infty} u_k(x)\, u_k(\alpha)\, e^{-\lambda_k t}\, d\alpha \\ &= \int_D K(\xi, \alpha)\, G(x, \alpha, t)\, d\alpha. \end{aligned} \tag{3.3}$$

Let now x and ξ be a pair of distinct fixed points in D, and let ε and ρ be positive numbers such that (i) $|x - \xi| \geq 2\rho$, (ii) the ball $B_\rho = B_\rho(\xi)$ with center ξ and radius ρ is contained in D, and (iii)

$$0 < \int_{B_\rho} K(\xi, \alpha)\, d\alpha < \varepsilon \tag{3.4}$$

(note that $K(\xi, \alpha)$ is nonnegative). We then write

$$\int_D K(\xi, \alpha)\, G(x, \alpha, t)\, d\alpha = \int_{B_\rho} \cdots d\alpha + \int_{D-B_\rho} \cdots d\alpha, \tag{3.5}$$

where the dots indicate the integrand appearing in the left member of this equation. Now from (1.15), (1.18) and (2.13) we have

$$0 \leq G(x, \alpha, t) \leq \Gamma(x, \alpha, t). \tag{3.6}$$

On the other hand, we see from (2.3), (2.10), (2.11) that, for suitable positive constants C_3 and g_3,

$$(3.7)\qquad 0 \leq \Gamma(x, \alpha, t) \leq C_3[t^{-n/2} + t^{-(n-1)/2}]\exp\left[-g_3\frac{\rho^2}{4t}\right] \quad \text{for } |x - \alpha| \geq \rho.$$

It follows from (3.6) and (3.7) that there exists a positive t_1 such that

$$0 \leq G(x, \alpha, t) < 1 \quad \text{for } 0 < t \leq t_1,\ |x - \alpha| \geq \rho,$$

and we see from (3.4) that

$$(3.8)\qquad 0 < \int_{B_\rho} K(\xi, \alpha)\, G(x, \alpha, t)\, d\alpha < \varepsilon \quad \text{for } 0 < t \leq t_1,$$

since $|x - \alpha| \geq \rho$ for $\alpha \in B_\rho$.

Now for $\alpha \in D - B_\rho$ and $|\xi - \alpha| \geq \rho$, the function $K(\xi, \alpha)$ is continuous in α, and we can conclude from the basic property of G (see (1.3) with Γ replaced by G) that

$$(3.9)\qquad \lim_{t\to 0}\int_{D-B_\rho} K(\xi, \alpha)\, G(x, \alpha, t)\, d\alpha = K(\xi, x).$$

Obviously (3.3), (3.5), (3.8), (3.9) together imply (1.24).

For the proof of (1.25) we have only to note that (3.7) remains true if $\Gamma(x, \alpha, t)$ is replaced by $G(x, \alpha, t)$, as is seen from (3.6). The inequality thus obtained together with (1.6) implies (1.25).

Obviously (3.7) implies (1.26) directly, and we turn to the proof of (1.27).

We recall first that the singularity of the Green's function $K(x, \xi)$ is $O(|x - \xi|^{-(n-2)})$. Consequently, as is well known (see [9, p. 546]), the p-th iteration $K^{(p)}$ of K will be bounded and continuous if

$$(3.10)\qquad p > \frac{1}{1 - \frac{n-2}{n}} = \frac{n}{2}.$$

Let then p be such an integer. The $u_k(x)$ form a full orthonormal system of eigenfunctions of the continuous kernel $K^{(p)}(x, \xi)$, and the corresponding eigenvalues are the p-th powers of the eigenvalues λ_k of $K(x, \xi)$. From the theory of integral equations it is known (see, for example, [21, p. 301]) that

$$(3.11)\qquad \sum_{k=1}^{\infty} \frac{u_k^2(x)}{(\lambda_k^p)^2} = \int_D [K^{(p)}(x, \xi)]^2\, d\alpha.$$

Let now r be an arbitrary positive integer greater than $2p$. Since all eigenvalues λ_k are positive and λ_1 may be assumed to be the smallest one, we have

$$(3.12)\qquad \frac{1}{e^{\lambda_k t}} < \frac{r!}{\lambda_k^r t^r} = \frac{r!}{\lambda_k^{r-2p}}\frac{1}{\lambda_k^{2p}}\frac{1}{t^r} \leq \frac{r!}{\lambda_1^{r-2p}}\frac{1}{\lambda_k^{2p}}\frac{1}{t^r}.$$

Since G is nonnegative, we see from (1.6), (3.12), Schwarz' inequality, and (3.11) that

$$
(3.13) \quad
\begin{aligned}
0 \leq G(x, \xi, t) &= \sum_{k=1}^{\infty} u_k(x) u_k(\xi) e^{-\lambda_k t} \\
&\leq \frac{r!}{t^r \lambda_1^{r-2p}} \sum_{k=1}^{\infty} \frac{|u_k(x)|}{\lambda_k^p} \frac{|u_k(\xi)|}{\lambda_k^p} \\
&\leq \frac{r!}{t^r \lambda_1^{r-2p}} \left\{ \sum_{k=1}^{\infty} \frac{u_k^2(x)}{\lambda_k^{2p}} \sum_{k=1}^{\infty} \frac{u_k^2(\xi)}{\lambda_k^{2p}} \right\}^{1/2} \\
&= \frac{r!}{t^r \lambda_1^{r-2p}} \left\{ \int_D [K^{(p)}(x, \alpha)]^2 \, d\alpha \int_D [K^{(p)}(\xi, \alpha)]^2 \, d\alpha \right\}^{1/2}.
\end{aligned}
$$

This proves (1.27)

It remains to prove (1.28). We use the notations of the last paragraph of the Introduction. In addition, we denote by $\gamma^{(p)}(x, \xi, t)$ the function $\gamma(x, \xi, t)$ appearing in (1.15) if $D = D^{(p)}$. Let then x, ξ be an arbitrary pair of points in E^n, and let $t > 0$. Then there exists an integer p_0 such that for $p \geq p_0$ the points x and ξ are in $D^{(p)}$, and from (2.12) (1.15) and (1.18) we have

$$
(3.14) \qquad 0 \leq \Gamma(x, \xi, t) - G^{(p)}(x, \xi, t) = \gamma^{(p)}(x, \xi, t) \qquad \text{for } p \geq p_0 .
$$

Now $\gamma^{(p)}$ is that solution of the boundary value problem (2.5) to (2.7) which is continuous in $\overline{D^{(p)}} \{0 \leq t\}$. By an argument used repeatedly, we conclude from this that

$$
(3.15) \qquad |\gamma^{(p)}(x, \xi, t)| \leq \max_{\substack{\dot{x} \in \dot{D}^{(p)} \\ 0 \leq \tau \leq t}} \Gamma(\dot{x}, \xi, \tau) .
$$

Now it follows from (2.14) that corresponding to each $\varepsilon > 0$ we can choose a $p_1 > p_0$ such that

$$
0 < \Gamma(\dot{x}, \xi, \tau) \leq \varepsilon \qquad \text{for } \dot{x} \in \dot{D}^{(p)},\ 0 < \tau \leq t,\ p \geq p_1 .
$$

It then follows from (3.15) and (3.14) that $0 \leq \Gamma(x, \xi, t) - G^{(p)}(x, \xi, t) \leq \varepsilon$ for $p \geq p_1$. This proves (1.28).

4. APPENDIX I.

For $t_0 > 0$ we have to prove that the functions $U(x, t)$ and $U_1(x, t)$ defined in (1.12) and (1.16) are continuous in the closure $\overline{\Delta t_0}$ of the domain

$$
\Delta t_0 = D \times \{0 < t < t_0\}
$$

provided that $h(x)$ is continuous in D and satisfies (1.17). Since both functions are certainly continuous in the interior of this domain, only the continuity at the boundary points is in question. Now both functions satisfy the boundary conditions (1.13), (1.14). It is easily seen that it will be sufficient to show (i) that the limit (1.13) (for U and U_1) is uniform for $x \in \overline{D}$, and (ii) that each point (x_1, t_1) on

$B_0 = \dot{D} \times \{0 < t \leq t_0\}$ has a neighborhood N_1 such that for all points (x, t) of the intersection $N_1 \cap \overline{B}_0$ the limit (1.14) (for U and U_1) is uniform. Of these two statements, (ii) is the easier one to prove, since it deals only with positive t-values. We shall therefore confine ourselves to the proof of (i). For this proof it will obviously be sufficient to establish the following two statements (A) and (B):

(A) If for any small enough positive β, S_β denotes the "boundary strip" consisting of all points of $\overline{D}$ whose distance from $\dot{D}$ is not greater than β, then corresponding to each pair of positive numbers ε and t_1 there exists an α such that

$$|U(x_0, t) - h(x_0)| < \varepsilon \quad \text{for } x_0 \in S_\alpha,\ 0 < t \leq t_1, \tag{4.1}$$

$$|U_1(x_0, t) - h(x_0)| < \varepsilon \quad \text{for } x_0 \in S_\alpha,\ 0 < t \leq t_1. \tag{4.2}$$

(B) The limit (1.13) (for U and U_1) is uniform in each closed subset C of D. Now a careful reading of the proof given in [18] for (1.13) shows that this proof actually implies the uniformity in each set having a positive distance from $\dot{D}$ (note that the number $\sigma = \sigma(x)$ on which the estimates in [19, p. 496] are based is subject only to the restriction that it be less than the distance from x to the boundary, and that it can therefore be chosen to be the same for all x in C.) As to U_1, it is clear from (1.15) and the fact that $\gamma(x)$ is continuous in $\overline{D}$ that it will be sufficient to prove the uniformity in C of the limit relation

$$\lim_{t \to 0} \int_D \Gamma(x, \xi, t)\, h(\xi)\, d\xi = h(x). \tag{4.3}$$

Again, a perusal of Dressel's papers [6], [7] will show that the limit (4.3) is indeed uniform in a set having a positive distance from $\dot{D}$.

It remains to prove statement (A). Since by assumption h(x) is continuous in $\overline{D}$ and vanishes on $\dot{D}$, there exists corresponding to each $\varepsilon > 0$, an α such that

$$|h(x)| < \varepsilon/3 \quad \text{for } x \in S_{2\alpha}. \tag{4.4}$$

We have to prove the existence of a t_1 such that (4.1) and (4.2) hold.

We start with (4.1). With x_0 in S_α, we write

$$U(x_0, t) = I(S_{2\alpha}) + I(D - S_{2\alpha}), \tag{4.5}$$

where I(B) denotes the integral with the integrand $G(x, \xi, t)\, h(\xi)$ and with B as integration domain. We recall [19, last line on p. 494, and p. 495 (34)] that the G_ν defined in (17) satisfy the inequalities

$$G_\nu(x, \xi, \lambda) \geq 0 \quad (\nu = 0, 1, 2, \cdots;\ \lambda > 0) \tag{4.6}$$

and

$$0 \leq \int_D G_\nu(x, \xi, t)\, d\xi \leq 1, \tag{4.7}$$

which by (1.8) imply

$$G(x, \xi, t) \geq 0, \tag{4.8}$$

$$0 \leq \int_D G(x, \xi, t)\, d\xi \leq 1. \tag{4.9}$$

We see from (4.8), (4.4), (4.9) that

$$0 \leq I(S_{2\alpha}) \leq \frac{\varepsilon}{3} \int_{S_{2\alpha}} G(x, \xi, t)\, d\xi \leq \frac{\varepsilon}{3} \int_D G(x, \xi, t)\, d\xi \leq \frac{\varepsilon}{3}. \tag{4.10}$$

To estimate $I(D - S_{2\alpha})$, we need the following result: if $\psi(x)$ is any function whose derivatives up to and including the third exist and are continuous, and which is zero on $\dot{D}$, then, for $x \in \overline{D}$,

$$\left| \psi(x) - \int_D \psi(\xi)\, G_\nu(x, \xi, \lambda)\, d\xi \right| \leq \frac{\nu + 1}{\lambda} \operatorname{Max} |L(\psi)| \tag{4.11}$$

[19, p. 496 (37) and p. 495 (34)]. We now choose for $\psi(x)$ a nonnegative function which satisfies the above requirements and which is 1 in $D - S_{2\alpha}$ and vanishes in S_α. Since then $\psi(x^0) = 0$, we see from (4.6), (4.11) that

$$\begin{aligned} 0 \leq \int_{D-S_{2\alpha}} G_\nu(x^0, \xi, \lambda)\, d\xi &= \int_{D-S_{2\alpha}} G_\nu(x^0, \xi, \lambda)\psi(\xi)\, d\xi - \psi(x^0) \\ &\leq \int_D G_\nu(x^0, \xi, \lambda)\psi(\xi)\, d\xi - \psi(x^0) \leq \frac{\nu + 1}{\lambda} \operatorname{Max} |L(\psi)|. \end{aligned} \tag{4.12}$$

Setting $\lambda = \nu/t$ and letting $\nu \to \infty$, we obtain from (1.8) and (4.12)

$$0 \leq \int_{D-S_{2\alpha}} G(x^0, \xi, t)\, d\xi \leq t \operatorname{Max} |L(\psi)|. \tag{4.13}$$

If we set

$$H = \operatorname*{Max}_{x \in \overline{D}} |h(x)|, \tag{4.14}$$

we see from (4.6) and (4.13) that

$$0 \leq I(D - S_{2\alpha}) \leq tH \operatorname{Max} |L(\psi)|. \tag{4.15}$$

Thus, with $t_1 = \varepsilon(3H \operatorname{Max} |L(\psi)|)^{-1}$, we see that

$$0 \leq I(D - S_{2\alpha}) < \varepsilon/3 \text{ for } 0 < t \leq t_1. \tag{4.16}$$

Finally it follows from (4.4), (4.5), (4.10) and (4.16) that

$$|U(x^0, t) - h(x^0)| \leq |U(x^0, t)| + |h(x^0)| \leq \varepsilon \quad \text{for } 0 < t < t_1,\ x^0 \in S_\alpha.$$

We have thus proved (4.1), and we turn now to the proof of (4.2). Since γ is continuous in $\overline{\Delta}_t$, we see from (1.15) and (2.7) that it will be sufficient to prove the following: corresponding to each $\varepsilon > 0$, there exist α and t_1 such that

$$(4.17)\qquad \left|\int_D h(\xi)\Gamma(x^0, \xi, t)\,d\xi - h(x^0)\right| < \varepsilon \quad \text{for } 0 < t \le t_1,\ x^0 \in S_\alpha .$$

We denote the integral in (4.17) by $U_2(x^0, t)$, and write

$$(4.18)\qquad U_2(x, t) = U_3(x, t) + U_4(x, t),$$

where U_3 and U_4 are obtained from U_2 by replacing the function Γ by the function Z and the integral in (2.3), respectively. Moreover, we set

$$(4.19)\qquad h_1(x) = h(x)/F(x), \qquad F_0 = \operatorname*{Max}_{x\in\overline{D}} F(x),$$

where $F(x)$ is the function defined in (2.2). We now determine an α such that in addition to (4.4)

$$(4.20)\qquad |h_1(x^0)| < \varepsilon/6F_0 \quad \text{for } x^0 \in S_{2\alpha}.$$

We shall now prove the existence of a t_1 such that

$$\left.\begin{array}{ll}(4.21) & |U_3(x^0, t) - h(x^0)| < \varepsilon/2, \\ (4.22) & |U_4(x^0, t)| < \varepsilon/2\end{array}\right\} \text{for } x^0 \in S_{2\alpha},\ 0 < t \le t_1;$$

these inequalities obviously imply (4.17) (as the proof will show, (4.22) is actually valid for all x in $\overline{D}$). To prove (4.21), we write

$$(4.23)\qquad U_3(x^0, t) = I(S_{2\alpha}) + I(D - S_{2\alpha}),$$

where $I(B)$ denotes the integral over B with the same integrand as in U_3, that is, $h(\xi)Z(x^0, \xi, t)$. To estimate $I(S_{2\alpha})$, we remark first that

$$(4.24)\qquad F(x) = \int_{E^n} t^{-n/2} \exp[-\sigma(x, x - \xi)/4t]\,d\xi .$$

This is seen if (with Dressel) we introduce in the integral the spherical coordinates r, θ of the beginning of Section 2, and then in the integral over r the variable $w = r\sqrt{\phi}/2\sqrt{t}$. The resulting integral will be seen to coincide with the definition (2.2), if one notes that

$$m(S^{n-1}) = 2(\sqrt{\pi})^n/\Gamma(n/2),$$

$$\Gamma(n/2) = \int_0^\infty e^{-v} v^{n/2-1}\,dv = 2\int_0^\infty e^{-w^2} w^{n-1}\,dw .$$

From the definitions (1.10) and (4.19) of Z and h_1 and from (4.20), we see that, for x^0 in $S_{2\alpha}$,

$$|I(S_{2\alpha})| = \left| \int_{S_{2\alpha}} h_1(\xi)\, t^{-n/2} \exp[-\sigma(x, x - \xi)/4t]\, d\xi \right.$$
$$\leq \frac{\varepsilon}{6F_0} \int_{S_{2\alpha}} t^{-n/2} \exp[-\sigma(x, x - \xi)/4t]\, d\xi .$$

This inequality together with (4.24) shows that

$$(4.25) \qquad |I(S_{2\alpha})| \leq \frac{\varepsilon}{6F_0} F(x) \leq \varepsilon/6 \qquad \text{for } x^0 \in S_\alpha,\ t > 0 .$$

To estimate $I(D - S_{2\alpha})$, let us set $H_1 = \operatorname{Max} |h_1(x)|$ for x in $\overline{D}$, and denote by $m(D)$ the volume of D. Noting that for $x^0 \in S_\alpha$ and $\xi \in D - S_{2\alpha}$ we have $|x^0 - \xi| \geq \alpha$, and using (2.9), we obtain

$$|I(D - S_{2\alpha})| = \left| \int_{D-S_{2\alpha}} h_1(\xi)\, t^{-n/2} \exp[-\sigma(x^0, x^0 - \xi)/4t] d\xi \right.$$
$$\leq H_1\, m(D)\, t^{-n/2} e^{-a\alpha^2/4t} .$$

This inequality obviously implies the existence of a t_1 such that

$$(4.26) \qquad |I(D - S_{2\alpha})| \leq \varepsilon/6 \qquad \text{for } x^0 \in S\ ,\ 0 < t \leq t_1 .$$

This completes the proof of (4.21), for we see from (4.23), (4.25), (4.26) and (4.4) that, for x^0 in S_α and $0 < t \leq t_1$,

$$|U(x^0, t) - h(x^0)| \leq |U(x^0, t)| + |h(x^0)| \leq 3\varepsilon/6 .$$

Turning to the proof of (4.22), we obtain from (2.11) and (4.14) the inequality

$$(4.27) \qquad |U_4(x^0, t)| \leq C_2 H \sqrt{t} \int_D t^{-n/2} \exp[-g_2 |x^0 - \xi|^2/4t] d\xi .$$

The right member becomes larger if the integration is extended over E^n. But the integral thus obtained has a finite value independent of x^0 and t as long as $t > 0$; this is seen by applying the substitutions indicated in the remarks following (4.24) to the case where $\sigma(x, x - \xi) = g_2 |x - \xi|^2$. The actual value of the integral is $(2\sqrt{\pi/g_2})^n$. Thus $U_4(x^0, t)$ is majorized by $\sqrt{t}$ multiplied by a constant. This certainly proves (4.22).

5. APPENDIX II

The proof of the uniform convergence (for $t \geq \delta > 0$) of the series (1.6) given for $n = 3$ in [19, Section 1] is based on the convergence of the series

$$(5.1) \qquad \sum_{k=1}^{\infty} u_k^2(x)/\lambda_k^2 ,$$

and the convergence of this series in turn is based on the fact, well known in the theory of integral equations, that the partial sums are majorized by $\int_D K^2(x, \xi)\, d\xi$, where K has the same meaning as in (3.2). Now for $n > 3$ this integral does not exist, and the proof has to be modified by introducing the p-th iteration $K^{(p)}$ of K for a p satisfying (3.10), and by using the series in (3.11) instead of (5.1). The way this has to be done will be perfectly clear from the proof for (1.27), if one notes particularly the inequality (3.13) appearing in that proof.

A similar modification has to be made in the proof given in [19, Section 2] for the identity of the definitions (1.6) and (1.8): equations (26) and (27) of [19] will be valid for $\nu \geq 2p$ if p is again an integer satisfying (3.10). We then replace [19, (28)] by

$$\left(1 + \frac{\lambda_k t}{\nu}\right)^{\nu+1} > \binom{\nu+1}{2p}\left(\frac{\lambda_k t}{\nu}\right)^{2p} > \frac{\lambda_k^{2p}}{(2p)!}\, t^p \left(\frac{\nu + 2 - 2p}{\nu}\right)^{2p} \qquad (\nu \geq 2p).$$

We thus see that, for $\nu \geq 4p$,

$$\left(1 + \frac{\lambda_k t}{\nu}\right)^{\nu+1} > \frac{1}{2^{2p}(2p)!}\, \lambda_k^{2p}\, t^p.$$

Therefore, for $t \geq \delta > 0$ and $\nu \geq 4p$, we can replace [19] (29) by

$$\left| \sum_{k=N+1}^{\infty} \frac{u_k(x)\, u_k(\xi)}{\left(1 + \frac{\lambda_k t}{\nu}\right)^{\nu+1}} \right| \leq \frac{2^{2p}(2p)!}{\delta^p} \sum_{k=N+1}^{\infty} \frac{|u_k(x)|\, |u_k(\xi)|}{(\lambda_k)^p\, (\lambda_k)^p}. \tag{5.2}$$

Since an application of Schwarz' inequality together with (3.11) shows that the series in the right member of (5.2) converges uniformly for (x, ξ) in $\overline{D}$, we see that for each $\varepsilon > 0$ we can choose N such that

$$\left| \sum_{k=N+1}^{\infty} \frac{u_k(x)\, u_k(\xi)}{\left(1 + \frac{\lambda_k t}{\nu}\right)^{\nu+1}} \right| < \varepsilon \qquad (\nu \geq 4p,\ t \geq \delta). \tag{5.3}$$

On the other hand, we see from (3.13) that we can choose N such that, in addition to (5.3),

$$\left| \sum_{k=N+1}^{\infty} u_k(x)\, u_k(\xi)\, e^{-\lambda_k t} \right| < \varepsilon. \tag{5.4}$$

It follows from (5.3), (5.4), (1.6) and [19, (27)] that, for $\nu \geq 4p$ and $t \geq \delta$,

$$|G(x, \xi, t) - G_\nu(x, \xi, \nu/t)| \leq \sum_{k=1}^{N} \left| u_k(x)\, u_k(\xi) \left(e^{-\lambda_k t} - \frac{1}{\left(1 + \frac{\lambda_k t}{\nu}\right)^{\nu+1}} \right) \right| + 2\varepsilon.$$

The relation (1.8) follows now immediately, if we let $\nu \to \infty$, for fixed N.

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