COMBINATORIAL IDENTITIES DERIVED FROM THE KOU JUMP-DIFFUSION MODEL

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Abstract. We derive interesting combinatorial identities by employing the instantaneous centralized moments obtained from the Lévy measure of the Kou jump-diffusion market.

1. Introduction

The Kou [4] Jump-Diffusion model is an example of a financial market that is driven by a Lévy process [1, 2]. Kou employs it in the pricing of options. The attendant Lévy measure and density are built from a double exponential distribution, which allows the process to jump up and down. Attached to this measure are objects called instantaneous centralized moments, which Buckley [3] employs to approximate optimal portfolios of stocks in Lévy markets where asymmetric information prevails. The moments are then used to derive a plethora of combinatorial identities.

The rest of the paper is organized as follows: Section 2 gives a brief review of the Kou [4] jump-diffusion model. Section 3 presents kernels and instantaneous centralized moments of the Kou Jump-Diffusion model and combinatorial identities. Section 4 concludes the paper.

2. The Kou Jump–Diffusion Model

Let S_t be the price of a stock at time t. The log-stock price dynamic for this model with continuous return μ_t and volatility σ_t, is given by:

$$d(\log S_t) = \left(\mu_t - \frac{1}{2}\,\sigma_t^2\right)\,dt + \sigma_t\,dB_t + d\left(\sum_{i=1}^{N(t)}(V_i - 1)\right), \tag{2.1}$$

where B_t is standard Brownian motion and the log-jump amplitude $X = \log(V)$ has *double exponential* distribution with density f_{kou}, dependent on 3 parameters p, η_1, and η_2. It is defined by

$$f_{kou}(x) = p\,\eta_1\,\exp\left(-\eta_1\,x\right)I_{\{x>0\}} + q\,\eta_2\,\exp\left(-\eta_2\,|x|\right)I_{\{x<0\}}, \tag{2.2}$$
$$\eta_1 > 1,\ \eta_2 > 0,\ p + q = 1,\ p \geq 0,\ q \geq 0.$$

The log-jump amplitude X, can be expressed as:

$$X = \begin{cases} X^u, & \text{with probability } p. \\ X^d, & \text{with probability } q. \end{cases}$$

Equivalently, $X = X^u I_{\{x>0\}} - X^d I_{\{x<0\}}$ where $X^u \sim \exp(\eta_1)$ and $X^d \sim \exp(\eta_2)$ are exponential random variables with means $\frac{1}{\eta_1}$ and $\frac{1}{\eta_2}$, respectively. The upward jump log amplitude, X^u, occurs with probability p, and is not expected to exceed 100%. This leads to the constraint $\mathbf{E}(X^u) = \frac{1}{\eta_1} < 1$. The log amplitude of the downward movements in returns is X^d, which occurs with probability $q = 1 - p$. For this model (cf Kou [4]), the expectation and variance of X are respectively

$$\mathbf{E}(X) \;\; = \;\; \frac{p}{\eta_1} - \frac{q}{\eta_2} \text{ and } \mathbf{Var}(X) = p\,q\left(\frac{1}{\eta_1} + \frac{1}{\eta_2}\right)^2 + \left(\frac{p}{\eta_1^2} + \frac{q}{\eta_2^2}\right).$$

The expected jump amplitude is $\mathbf{E}(V) = \frac{q\,\eta_2}{(\eta_2+1)} - \frac{p\,\eta_1}{(\eta_1-1)}$. The Kou Lévy density is

$$\begin{aligned} v_{kou}(x) &= \lambda f_{kou}(x) \\ &= \lambda\, p\, \eta_1 \, \exp\left(-\eta_1\, x\right) I_{\{x>0\}} + \lambda\, q\, \eta_2 \, \exp\left(-\eta_2\, |x|\right) I_{\{x<0\}}, \end{aligned}$$

where λ is the intensity of the Poisson process $N(t)$ as presented in Buckley [3]. The log return stock dynamic (2.1) can be written as percentage return:

$$\frac{dS_t}{S_t} = \mu_t\, dt + \sigma_t\, dB_t + \int_R (e^x - 1)\, N(dt, dx),$$

where the total return

$$b_t = \mu_t + \lambda \int_R (e^x - 1)\, f_{kou}(x) dx = \mu_t + \lambda \left(\frac{p}{\eta_1 - 1} - \frac{q}{\eta_2 + 1}\right),$$

is due to both diffusion and jump processes.

Definition 1. *Define the objects $\widehat{M}_j(\eta)$ and $\widehat{K}_s(\eta)$ by the prescriptions:*

$$\begin{aligned} \widehat{M}_j(\eta) &= \int_0^\infty (e^x - 1)^j e^{-\eta x} dx, \; \eta > 0, \; j < \eta. \\ \widehat{K}_s(\eta) &= \int_0^\infty (e^{sx} - 1) e^{-\eta x} dx, \; s < \eta. \\ \mathbf{B}(\alpha, \beta) &= \int_0^\infty x^{\alpha-1} (x+1)^{-\alpha-\beta} dx, \; \alpha > 0, \; \beta > 0. \end{aligned}$$

Lemma 1. *Let $\mathbf{B}(\alpha, \beta)$ be the Beta function above.*

(1) *If $j < \eta$, then $\widehat{M}_j(\eta) = \mathbf{B}(j+1, \eta - j) = \frac{\Gamma(j+1)\Gamma(\eta-j)}{\Gamma(\eta+1)}$.*
(2) *If $s < \eta$, then $\widehat{K}_s(\eta) = \frac{s}{\eta(\eta-s)}$.*

Proof. (1) Let $y = e^x - 1$. Then $x = \log(1+y)$ and $dy = e^x\,dx$. Thus,

$$\begin{aligned}
\widehat{M}_j(\eta) &= \int_0^\infty y^j e^{-\eta \log(1+y)} dx = \int_0^\infty y^j (1+y)^{-\eta-1} dy \\
&= \int_0^\infty y^{(j+1)-1} (1+y)^{-(j+1)-(\eta-j)} dy \\
&= \mathbf{B}(j+1, \eta - j) = \frac{\Gamma(j+1)\Gamma(\eta-j)}{\Gamma(\eta+1)}.
\end{aligned}$$

(2) If $\eta > s$, then $\widehat{K}_s(\eta) = \int_0^\infty (e^{sx}-1)\, e^{-\eta x} dx = \frac{1}{\eta - s} - \frac{1}{\eta} \;=\; \frac{s}{\eta(\eta-s)}$. □

With Lemma 1 in hand, we are now able to compute the kernels $K_s(\eta)$ and instantaneous centralized moments of returns $M_j(\eta)$, defined in the following definition.

Definition 2 (**Instantaneous Centralized Moments of Return**).

$$\begin{aligned}
M_j \equiv M_j(\eta_1, \eta_2, p, \lambda) &\stackrel{\triangle}{=} \int_R (e^x - 1)^j v_{kou}(x) dx \\
&= \lambda \int_R (e^x - 1)^j f_{kou}(x) dx, \qquad (2.3) \\
K_s \equiv K_s(\eta_1, \eta_2, p, \lambda) &\stackrel{\triangle}{=} \int_R (e^{sx} - 1) v_{kou}(x) dx \\
&= \lambda \int_R (e^{sx} - 1) f_{kou}(x) dx. \qquad (2.4)
\end{aligned}$$

Lemma 2. *For the Kou jump–diffusion market given by (2.1), we have the following. Let $\eta_1 > 1$, $\eta_2 > 0$, $p + q = 1$, $p \geq 0$, $q \geq 0$, with Poisson intensity rate $\lambda > 0$. If $\max(s, j) < \eta_1$, then*

$$M_j = (-1)^j\,(j!)\,\frac{\lambda\, q\, \eta_2\, \Gamma(\eta_2)}{\Gamma(\eta_2 + j + 1)} + (j!)\,\frac{\lambda\, p\, \eta_1\, \Gamma(\eta_1 - j)}{\Gamma(\eta_1 + 1)},$$

and

$$K_s = \frac{\lambda\, p\, s}{(\eta_1 - s)} + \frac{\lambda\, q\, s}{(\eta_2 + s)} = \lambda \frac{s\,(p\,\eta_2 - q\,\eta_1 + s)}{(\eta_1 - s)(\eta_2 + s)}.$$

Proof. (1) Let $j < \eta_1$ and $s < \eta_2$. From (2.2), we obtain

$$\frac{M_j}{\lambda} = \int_R (e^x - 1)^j f_{kou}(x)dx$$
$$= \int_R (e^x - 1)^j \left(p\eta_1 \exp(-\eta_1 x) I_{\{x>0\}} + q\eta_2 \exp(-\eta_2 |x|) I_{\{x<0\}}\right) dx$$
$$= q\eta_2 \int_{-\infty}^0 (e^x - 1)^j \exp(-\eta_2|x|)dx + p\eta_1 \int_0^\infty (e^x - 1)^j \exp(-\eta_1 x)dx$$
$$= q\eta_2 \int_0^\infty (1 - e^x)^j \exp(-jx) \exp(-\eta_2 x)dx + p\eta_1 \widehat{M_j}(\eta_1),$$

and from Lemma 1,

$$\frac{M_j}{\lambda} = (-1)^j\, q\, \eta_2 \int_0^\infty (e^x - 1))^j \exp(-(j + \eta_2)\, x)dx + p\, \eta_1\, \widehat{M_j}(\eta_1)$$
$$= (-1)^j\, q\, \eta_2\, \widehat{M_j}\,(j + \eta_2) + p\, \eta_1\, \widehat{M_j}\,(\eta_1)$$
$$= (-1)^j q\, \eta_2\, \frac{\Gamma(j+1)\, \Gamma(j + \eta_2 - j)}{\Gamma(\eta_2 + j + 1)} + p\, \eta_1\, \frac{\Gamma(j+1)\, \Gamma(\eta_1 - j)}{\Gamma(\eta_1 + 1)}$$
$$= (-1)^j\, j!\, \frac{\lambda\, q\, \eta_2\, \Gamma(\eta_2)}{\Gamma(\eta_2 + j + 1)} + j!\, \frac{\lambda\, p\, \eta_1\, \Gamma(\eta_1 - j)}{\Gamma(\eta_1 + 1)},$$

and the result follows.

For part (2):

$$\frac{K_s}{\lambda} = \int_R (e^{sx} - 1) f_{kou}(x)dx$$
$$= \int_R (e^{sx} - 1) \left(p\eta_1 \exp(-\eta_1 x) I_{\{x>0\}} + q\eta_2 \exp(-\eta_2 |x|) I_{\{x<0\}}\right) dx$$
$$= q\eta_2 \int_{-\infty}^0 (e^{sx} - 1) \exp(-\eta_2|x|)dx + p\eta_1 \int_0^\infty (e^{sx} - 1) \exp(-\eta_1 x)dx$$
$$= q\eta_2 \int_0^\infty (e^{-sx} - 1) \exp(-\eta_2 x)dx + p\eta_1 \int_0^\infty (e^{sx} - 1) \exp(-\eta_1 x)dx$$
$$= q\eta_2 \int_0^\infty (\exp(-(s + \eta_2)x) - \exp(-\eta_2 x))dx$$
$$\quad + p\eta_1 \int_0^\infty (\exp((s - \eta_1)x) - \exp(-\eta_1 x))dx$$
$$= q\eta_2 \widehat{K}_{-s}(\eta_2) + p\eta_1 \widehat{K}_s(\eta_1) = q\eta_2 \left[\frac{-s}{\eta_2(\eta_2 + s)}\right] + p\eta_1 \left[\frac{s}{\eta_1(\eta_1 - s)}\right]$$
$$= \frac{s\,(p\,\eta_2 - q\,\eta_1 + s)}{(\eta_1 - s)(\eta_2 + s)}.$$

□

For the Kou model $\eta_1 = \frac{1}{\mathbb{E}(X^u)} > 1$, and M_j exists provided $j < \eta_1$. Thus M_1 always exists for this model, where

$$M_1 = \frac{\lambda p}{\eta_1 - 1} - \frac{\lambda q}{\eta_2 + 1} = \lambda(\mathbf{E}(e^X - 1)) = \lambda(\mathbf{E}(V) - 1)),$$

and $\mathbf{E}(V)$ is the mean jump amplitude. In this case M_1 is the jump component of the total stock appreciation rate $b = \mu + M_1$, where μ is its continuous component.

Corollary 1. *For the double exponential Kou jump-diffusion model with $p = q = \frac{1}{2}$ and $\eta_1 = \eta_2 = \eta$, we have the following results.*

(1) *If $j < \eta$, then*

$$M_j = \frac{\lambda}{2}\, j! \left[\frac{(-1)^j}{\Pi_{r=1}^{j}(\eta + r)} + \frac{1}{\Pi_{r=1}^{j}(\eta - r)} \right].$$

(2) *If $|s| < \eta$, then*

$$K_s = \frac{\lambda s^2}{\eta^2 - s^2}.$$

Proof. This result follows from Lemma 2 with $p = q = \frac{1}{2}$ and $\eta_1 = \eta_2 = \eta$. □

There are many combinatorial identities resulting from Lemma 2 and its corollary. We now present them.

3. Combinatorial Identities

Recall that the objects M_j and their respective kernels K_s, are defined by the prescriptions:

$$M_j = \int_R (e^x - 1)^j v(dx), \quad K_s = \int_R (e^{sx} - 1) v(dx), \quad s \geq 0.$$

M_j is called the jth instantaneous centralized moment of return of the Lévy process X, with measure $v_{kou}(\cdot)$. The kernel K_s is used to calculate M_j. We have the following result which will be quite useful in the sequel.

Lemma 3. *If there exists $k \in N$, such that $\int_R (e^{jx} - 1)v(dx) < \infty$ for each $0 \leq j \leq k$, then K_j and M_j exist, and*

$$M_j = \sum_{i=1}^{j} (-1)^{j-i} \binom{j}{i} K_i. \tag{3.1}$$

Proof. If there exists $k \in N$, such that $\int_R (e^{jx}-1)v(dx) < \infty$ for each $0 \le j \le k$, then $K_j = \int_R (e^{jx}-1)v(dx) < \infty$. Now $M_j = \int_R (e^x-1)^j\, v(dx)$. From the Binomial Theorem

$$\begin{aligned}
(e^x-1)^j &= \sum_{i=0}^{j} (-1)^{j-i} \binom{j}{i} e^{ix} \\
&= \sum_{i=0}^{j} (-1)^{j-i} \binom{j}{i} (e^{ix}-1) + \sum_{i=0}^{j} (-1)^{j-i} \binom{j}{i} \\
&= \sum_{i=1}^{j} (-1)^{j-i} \binom{j}{i} (e^{ix}-1) + (1-1)^j \\
&= \sum_{i=1}^{j} (-1)^{j-i} \binom{j}{i} (e^{ix}-1).
\end{aligned}$$

Therefore,

$$\begin{aligned}
M_j &= \int_R (e^x-1)^j\, v(dx) \\
&= \sum_{i=1}^{j} (-1)^{j-i} \binom{j}{i} \int_R (e^{ix}-1)\, v(dx) \\
&= \sum_{i=1}^{j} (-1)^{j-i} \binom{j}{i} K_i,
\end{aligned}$$

which is clearly finite for each integer $0 \le j \le k$. $\square$

We obtain new combinatorial identities by applying Lemma 3 to the Kou jump-diffusion model. The central result follows.

Theorem 1. *Let $k \ge 1$ be an integer, $\eta_1 > k$, $\eta_2 > 0$, $p+q=1$, $p \ge 0$, $q \ge 0$. Then*

$$\begin{aligned}
&\sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} \frac{j\,(j + p\,\eta_2 - q\,\eta_1)}{(\eta_1 - j)(\eta_2 + j)} \\
&\qquad = k! \left[(-1)^k\, q\, \eta_2 \frac{\Gamma(\eta_2)}{\Gamma(\eta_2 + k + 1)} + p\, \eta_1 \frac{\Gamma(\eta_1 - k)}{\Gamma(\eta_1 + 1)} \right].
\end{aligned}$$

Proof. From Lemma 3, if $\eta_1 > k$ then K_j and M_j exist for each $0 \le j \le k$, and

$$M_k = \sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} K_j.$$

For the Kou model, K_j and M_j are given explicitly by Lemma 2, with

$$M_j = (-1)^j\,(j!)\,\frac{\lambda\, q\, \eta_2\, \Gamma(\eta_2)}{\Gamma(\eta_2 + j + 1)} + (j!)\,\frac{\lambda\, p\, \eta_1\, \Gamma(\eta_1 - j)}{\Gamma(\eta_1 + 1)},$$

and

$$K_s = \frac{\lambda\, p\, s}{(\eta_1 - s)} + \frac{\lambda\, q\, s}{(\eta_2 + s)} = \lambda \frac{s\,(p\,\eta_2 - q\,\eta_1 + s)}{(\eta_1 - s)(\eta_2 + s)}.$$

The result then follows by dividing both sides by λ, the intensity rate of the driving Poisson process. □

In the sequel, we assume without loss of generality that $\lambda = 1$.

Corollary 2. *Let $k, \alpha \in N$ and $\eta > k$. Then*

$$\sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} \frac{j^2}{(\eta^2 - j^2)} = \frac{\eta}{2}\, k! \left[(-1)^k \frac{\Gamma(\eta)}{\Gamma(\eta + k + 1)} + \frac{\Gamma(\eta - k)}{\Gamma(\eta + 1)} \right].$$

$$\sum_{j=1}^{k} (-1)^{k-j} \frac{1}{j!(k-j)!} \frac{j^2}{(\eta^2 - j^2)} = \frac{\eta}{2} \left[(-1)^k \frac{\Gamma(\eta)}{\Gamma(\eta + k + 1)} + \frac{\Gamma(\eta - k)}{\Gamma(\eta + 1)} \right].$$

$$\sum_{j=1}^{k} (-1)^{k-j} \frac{1}{j!(k-j)!} \frac{j^2}{((k+\alpha)^2 - j^2)} = \frac{1}{2} \left[(-1)^{k+\alpha} \frac{(k+\alpha)!}{(2k+\alpha)!} + \frac{(\alpha - 1)!}{(k + \alpha - 1)!} \right].$$

Proof. Let$p = q = \frac{1}{2}$, $\eta_1 = \eta_2 = \eta > k$ in Theorem 1 to get the first two identities. Set $\eta = k + \alpha$, $\alpha \in N$. Then by said theorem, we get the last identity as follows:

$$\begin{aligned}
&\sum_{j=1}^{k} (-1)^{k-j} \frac{1}{j!(k-j)!} \frac{j^2}{((k+\alpha)^2 - j^2)} \\
&= \frac{1}{2}\eta \left[(-1)^k \frac{\Gamma(\eta)}{\Gamma(\eta + k + 1)} + p\eta \frac{\Gamma(\eta - k)}{\Gamma(\eta + 1)} \right] \\
&= \frac{1}{2}\eta \left[(-1)^k \frac{(\eta - 1)!}{(\eta + k)!} + \frac{(\eta - k - 1)!}{(\eta)!} \right] \\
&= \frac{1}{2} \left[(-1)^k \frac{(\eta)!}{(\eta + k)!} + \frac{(\eta - k - 1)!}{(\eta - 1)!} \right] \\
&= \frac{1}{2} \left[(-1)^{k+\alpha} \frac{(k+\alpha)!}{(2k+\alpha)!} + \frac{(\alpha - 1)!}{(k + \alpha - 1)!} \right].
\end{aligned}$$

□

Theorem 1 yields other results based on the choice of the parameters p, η_1, and η_2.

Theorem 2. *Let $k \in N$ and $\eta > k$. Then*

$$\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\frac{j}{\eta-j} = k!\frac{\Gamma(\eta-k)}{\Gamma(\eta)} = \frac{k!}{\Pi_{j=1}^{k}(\eta-j)}. \tag{3.2}$$

$$\sum_{j=0}^{k}(-1)^{k-j}\frac{1}{j!(k-j)!}\frac{j}{\eta-j} = \frac{\Gamma(\eta-k)}{\Gamma(\eta)} = \frac{1}{\Pi_{j=1}^{k}(\eta-j)}. \tag{3.3}$$

$$\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\frac{j}{k+1-j} = 1. \tag{3.4}$$

Proof. Let $p = 1$, $q = 0$, and $\eta_1 = \eta_2 = \eta$ in Theorem 1. Then

$$\sum_{j=1}^{k}(-1)^{k-j}\binom{k}{j}\frac{j(j+\eta)}{(\eta-j)(\eta+j)} = k!\,p\,\eta\frac{\Gamma(\eta-k)}{\Gamma(\eta+1)}.$$

Thus starting with $j = 0$, we get

$$\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\frac{j}{\eta-j} = k!\frac{\Gamma(\eta-k)}{\Gamma(\eta)} = \frac{k!}{\Pi_{j=1}^{k}(\eta-j)}.$$

Dividing by $k!$ yields the second identity. Letting $\eta = k+1$ in equation (3.2), yields

$$\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\frac{j}{k+1-j} = \frac{k!}{\Pi_{j=1}^{k}(k+1-j)} = \frac{k!}{k!} = 1.$$

$\square$

Theorem 3. *Let $\eta > 0$ and k, m be positive integers. Then*

$$\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\frac{j}{\eta+j} = (-1)^k k!\frac{\Gamma(\eta+1)}{\Gamma(\eta+k+1)} = (-1)^k\frac{k!}{\Pi_{j=1}^{k}(\eta+j)}. \tag{3.5}$$

$$\sum_{j=0}^{k}(-1)^{k-j}\frac{1}{j!(k-j)!}\frac{j}{\eta+j} = (-1)^k\frac{\Gamma(\eta+1)}{\Gamma(\eta+k+1)} = (-1)^k\frac{1}{\Pi_{j=1}^{k}(\eta+j)}. \tag{3.6}$$

$$\sum_{j=0}^{2m}(-1)^{j-1}\binom{2m}{j}\frac{j}{\eta+j} = \frac{\Gamma(2m+1)\Gamma(\eta+1)}{\Gamma(\eta+2m+1)} = \frac{1}{\Pi_{j=1}^{2m}(\eta+j)}. \tag{3.7}$$

$$\sum_{j=0}^{2m-1}(-1)^{j}\binom{2m-1}{j}\frac{j}{\eta+j} = \frac{\Gamma(2m)\Gamma(\eta+1)}{\Gamma(\eta+2m)} = \frac{(2m-1)!}{\Pi_{j=1}^{2m-1}(\eta+j)}. \tag{3.8}$$

Proof. Let $p = 0$, $q = 1$, and $\eta_1 = \eta_2 = \eta$ in Theorem 1. Then

$$\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\frac{j(j-\eta)}{(\eta-j)(\eta+j)} = \sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\frac{j}{\eta+j}$$

$$= (-1)^k k!\eta\frac{\Gamma(\eta)}{\Gamma(\eta+k+1)} = (-1)^k\frac{k!}{\Pi_{j=1}^{k}(\eta+j)}.$$

Equation (3.6) follows by dividing both sides by $k!$. Let $k = 2m$, where m is an integer. Then

$$\sum_{j=0}^{k}(-1)^{2m-j}\binom{2m}{j}\frac{j}{\eta+j} = (-1)^{2m}(2m)!\,\frac{\Gamma(\eta+1)}{\Gamma(\eta+2m+1)}$$

$$= (2m)!\,\frac{\Gamma(\eta+1)}{\Gamma(\eta+2m+1)} = \frac{2m!}{\Pi_{j=1}^{2m}(\eta+j)}.$$

Similarly, setting $k = 2m - 1$ and multiplying by (-1), we have

$$\sum_{j=0}^{2m-1} (-1)^{2m-1-j} \binom{2m-1}{j} \frac{j}{\eta + j} = \sum_{j=0}^{2m-1} (-1)^j \binom{2m-1}{j} \frac{j}{\eta + j}$$
$$= (-1)^{2m-1}(2m-1)! \frac{\Gamma(\eta+1)}{\Gamma(\eta+2m)}$$
$$= (2m-1)! \frac{\Gamma(\eta+1)}{\Gamma(\eta+2m)} = \frac{(2m-1)!}{\Pi_{j=1}^{2m-1}(\eta+j)}.$$

□

We have more interesting identities.

Theorem 4. *For any positive integers k and m, we have*

$$\sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \frac{j}{j+1} = (-1)^k \frac{1}{k+1}. \tag{3.9}$$

$$\sum_{j=0}^{k} (-1)^{j+1} \binom{k}{j} \frac{j}{j+1} = \frac{1}{k+1}. \tag{3.10}$$

$$\sum_{j=0}^{2m} (-1)^{j+1} \binom{2m}{j} \frac{j}{j+1} = \frac{1}{2m+1}. \tag{3.11}$$

$$\sum_{j=0}^{2m-1} (-1)^{j} \binom{2m-1}{j} \frac{j}{j+1} = -\frac{1}{2m}. \tag{3.12}$$

Proof. Letting $\eta = 1$ in Theorem 3 yields,

$$\sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \frac{j}{j+1} = (-1)^k k! \frac{\Gamma(2)}{\Gamma(k+2)}$$
$$= (-1)^k \frac{k!}{\Pi_{j=1}^{k}(j+1)} = (-1)^k \frac{1}{k+1}.$$

Multiplying (3.9) by $(-1)^k$ yields (3.10). Let m be an integer. Let $k = 2m$ in (3.9). Then

$$\sum_{j=0}^{2m} (-1)^{2m-j} \binom{2m}{j} \frac{j}{j+1} = \sum_{j=0}^{2m} (-1)^j \binom{2m}{j} \frac{j}{j+1}$$
$$= (-1)^{2m} \frac{1}{2m+1} = \frac{1}{2m+1}.$$

Similarly, with $k = 2m-1$, we get

$$-\sum_{j=0}^{2m-1}(-1)^{2m-1-j}\binom{2m-1}{j}\frac{j}{j+1} = \sum_{j=0}^{2m-1}(-1)^{j}\binom{2m-1}{j}\frac{j}{j+1}$$
$$= (-1)^{2m}\frac{1}{2m} = \frac{1}{2m}.$$

□

We now present some results involving double sums.

Corollary 3.

$$\sum_{n=0}^{\infty}\sum_{j=0}^{2^n-1}(-1)^{j}\binom{2^n-1}{j}\frac{j}{j+1} = 1. \tag{3.13}$$

$$\sum_{k=1}^{\infty}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\frac{j}{j+1} = \log(2) - 1. \tag{3.14}$$

Proof. Let $m = 2^{n-1}$ in the third identity of Theorem 4. Then $2m-1 = 2^n - 1$, and so

$$\sum_{n=0}^{\infty}\sum_{j=0}^{2^n-1}(-1)^{j}\binom{2^n-1}{j}\frac{j}{j+1} = \sum_{n=0}^{\infty}\frac{1}{2^n} = 1.$$

Recall that

$$\log(1+x) = \sum_{j=1}^{\infty}(-1)^{j-1}\frac{x^j}{j},$$

hence

$$\log 2 = \sum_{j=1}^{\infty}(-1)^{j-1}\frac{1}{j} = \sum_{k=1}^{\infty}\frac{(-1)^k}{k+1}.$$

By Theorem 3,

$$\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\frac{j}{j+1} = (-1)^k\frac{1}{k+1}.$$

Therefore,

$$\sum_{k=1}^{\infty}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\frac{j}{j+1} = \sum_{k=1}^{\infty}(-1)^k\frac{1}{k+1}$$
$$= \sum_{k=0}^{\infty}(-1)^k\frac{1}{k+1} - 1 = \log 2 - 1.$$

□

We state without proof some identities that follow directly from the third identity of Theorem 2, which states that for all positive integer k, $\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\frac{j}{k-j+1}=1$.

Example 1. *Let n be a positive integer. Then*

$$\sum_{k=1}^{\infty}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\frac{j}{(k-j+1)k^2}=\frac{\pi^2}{3}. \tag{3.15}$$

$$\sum_{k=1}^{\infty}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\frac{j}{(k-j+1)(k(k+1))}=1. \tag{3.16}$$

$$\sum_{k=1}^{n}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\frac{jk}{k-j+1}=\frac{1}{2}n(n+1). \tag{3.17}$$

$$\sum_{k=1}^{n}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\frac{jk^2}{k-j+1} = \frac{1}{6}n(n+1)(2n+1). \tag{3.18}$$

4. Conclusion

We applied the instantaneous centralized moments and their kernels to produce combinatorial identities involving the sum and double sums of sequences. These summations and identities are otherwise very difficult to obtain.

Acknowledgements

The author sincerely thanks the anonymous referee for his kind comments and suggestions. He also thanks Hongwei Long, Sandun Perera and Nicollette Deerr for their helpful comments and suggestions. Special thanks go to Dorothy Graham, IMACS and UTech for partial financial support of this project.

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MSC2010: 97KXX, 97K60, 97K20, 60-XX, 62-XX, 40B99, 91G10

Key words and phrases: Kou jump diffusion, Lévy measure, instantaneous centralised moments, combinatorial identities, series summation

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