CHAPTER 3

LINEAR RANK AND SIGNED RANK STATISTICS

3.1. INTRODUCTION

Let $\{X_{ni}, F_{ni}\}$ be as in (2.2.33) and $\{\mathbf{c}_{ni}\}$ be p×1 real vectors. The rank and the absolute rank of the ith residual are defined, respectively, as

$$(1)\quad R_{i\mathbf{u}} = \sum_{j=1}^{n} I(X_{nj} - \mathbf{u}'\mathbf{c}_{nj} \le X_{ni} - \mathbf{u}'\mathbf{c}_{ni}),$$

$$R^{+}_{i\mathbf{u}} = \sum_{j=1}^{n} I(|X_{nj} - \mathbf{u}'\mathbf{c}_{nj}| \le |X_{ni} - \mathbf{u}'\mathbf{c}_{ni}|), \qquad 1 \le i \le n, \quad \mathbf{u} \in \mathbb{R}^p.$$

Let φ be a nondecreasing real valued function on $[0, 1]$ and define

$$(2)\quad T_d(\varphi, \mathbf{u}) = \sum_{i=1}^{n} d_{ni}\, \varphi(R_{i\mathbf{u}}/(n+1)),$$

$$T^{+}_d(\varphi, \mathbf{u}) = \sum_{i=1}^{n} d_{ni}\, \varphi^{+}(R^{+}_{i\mathbf{u}}/(n+1))\, s(X_{ni} - \mathbf{u}'\mathbf{c}_{ni}), \quad \mathbf{u} \in \mathbb{R}^p,$$

where $\varphi^{+}(s) = \varphi((s+1)/2)$, $0 \le s \le 1$, and $s(x) = I(x > 0) - I(x < 0)$.

The processes $\{T_d(\varphi, \mathbf{u}),\ \mathbf{u} \in \mathbb{R}^p\}$ and $\{T^{+}_d(\varphi, \mathbf{u}),\ \mathbf{u} \in \mathbb{R}^p\}$ are used to define rank (R) estimators of $\boldsymbol{\beta}$ in the linear regression model (1.1.1). See, e.g., Adichie (1967), Koul (1971), Jurečková (1971) and Jaeckel (1972). One key property used in studying these R-estimators is the asymptotic uniform linearity (a.u.l.) of $T_d(\varphi, \mathbf{u})$ and $T^{+}_d(\varphi, \mathbf{u})$ in $\mathbf{u} \in \mathcal{N}(B)$. Such results have been proved by Jurečková (1969) for $T_d(\varphi, \mathbf{u})$ for general but fixed functions φ, by Koul (1969) for $T^{+}_d(I, \mathbf{u})$ (where I is the identity function) and by Van Eeden (1971) for $T^{+}_d(\varphi, \mathbf{u})$ for general but fixed φ functions. In all of these papers $\{X_{ni}\}$ are assumed to be i.i.d..

In Sections 3.2 and 3.3 below we prove the a.u.l. of $T_d(\varphi, .)$, $T^{+}_d(\varphi, .)$, uniformly in those φ which have $\|\varphi\|_{tv} < \infty$, and under fairly general independent setting. These proofs reveal that this a.u.l. property is also a consequence of the asymptotic continuity of certain w.e.p.'s and the smoothness of $\{F_{ni}\}$.

Besides being useful in studying the asymptotic distributions of R-estimators of $\boldsymbol{\beta}$ these results are also useful in studying some rank based minimum distance estimators, some goodness-of-fit tests for the error distributions of (1.1.1) and the robustness of R-estimators against certain heteroscedastic errors.

3.2. ASYMPTOTIC UNIFORM LINEARITY OF LINEAR RANK STATISTICS

At the outset we shall assume

(1) $\varphi \in \mathscr{C} := \{\varphi: [0,1] \longrightarrow \mathbb{R},\ \varphi \in \mathbb{D}\mathcal{I}[0,1], \text{ with } \|\varphi\|_{tv} := \varphi(1) - \varphi(0) = 1\}.$

Define the w.e.p. based on ranks, with weights $\{d_{ni}\}$,

(2) $Z_d(t, \mathbf{u}) := \Sigma_i\, d_{ni}\, I(R_{i\mathbf{u}} \le nt), \qquad 0 \le t \le 1,\ \mathbf{u} \in \mathbb{R}^p.$

Note that

(3) $$T_d(\varphi, \mathbf{u}) = \int \varphi(nt/(n+1))\, Z_d(dt, \mathbf{u})$$
$$= -\int Z_d((n+1)t/n, \mathbf{u})\, d\varphi(t) + n\bar{d}_n\, \varphi(1), \qquad n\bar{d}_n = \sum_{i=1}^{n} d_{ni}.$$

The representation (3) shows that in order to prove the a.u.l. of $T_d(\varphi, .)$, it suffices to prove it for $Z_d(t, .)$, uniformly in $0 \le t \le 1$. Thus, we shall first prove the a.u.l. property for the Z_d-process. Define, for $x \in \mathbb{R}$, $0 \le t \le 1$, $\mathbf{u} \in \mathbb{R}^p$,

(4) $$H_{n\mathbf{u}}(x) = n^{-1}\, \Sigma_i\, I(X_{ni} - \mathbf{c}_{ni}'\mathbf{u} \le x), \qquad H_{\mathbf{u}}(x) := n^{-1}\Sigma_i\, F_{ni}(x + \mathbf{c}_{ni}'\mathbf{u}),$$
$$H_{n\mathbf{u}}^{-1}(t) = \inf\{x;\, H_{n\mathbf{u}}(x) \ge t\}, \qquad H_{\mathbf{u}}^{-1}(t) = \inf\{x;\ H_{\mathbf{u}}(x) \ge t\}.$$

Note that H_0 is the H of (2.2a.33). We shall write H_n for H_{n0}.

Recall that for any d.f. G,

$$G(G^{-1}(t)) \ge t, \quad 0 \le t \le 1 \quad \text{and} \quad G^{-1}(G(x)) \le x, \quad x \in \mathbb{R}.$$

This fact and the relation $nH_{n\mathbf{u}}(X_i - \mathbf{c}_i'\mathbf{u}) \equiv R_{i\mathbf{u}}$ yield that $\forall\ 0 \le t \le 1$,

(5) $$[X_i - \mathbf{c}_i'\mathbf{u} \ge H_{n\mathbf{u}}^{-1}(t)] \Rightarrow [R_{i\mathbf{u}} \ge nt] \Rightarrow [X_i - \mathbf{c}_i'\mathbf{u} \ge H_{n\mathbf{u}}^{-1}(t)], \qquad 1 \le i \le n.$$

For technical convenience, it is desirable to center the weights of linear rank statistics appropriately. Accordingly, let

(6) $$w_{ni} := (d_{ni} - \bar{d}_n), \qquad 1 \le i \le n.$$

Then, with Z_w denoting the Z_d when weights are $\{w_{ni}\}$,

$$Z_d(t, u) = Z_w(t, \mathbf{u}) + \bar{d}_n \cdot [nt], \qquad 0 \le t \le 1, \mathbf{u} \in \mathbb{R}^p.$$

Hence

$$(7) \qquad Z_d(t, u) - Z_d(t, 0) = Z_w(t, \mathbf{u}) - Z_w(t, 0), \qquad 0 \le t \le 1, \mathbf{u} \in \mathbb{R}^p.$$

Next define, for arbitrary real weights $\{d_{ni}\}$,

$$(8) \qquad \mathcal{Y}_d(t, \mathbf{u}) := \Sigma\, d_{ni}\, I(X_{ni} - \mathbf{c}_{ni}'\mathbf{u} \le H_{n\mathbf{u}}^{-1}(t)), \qquad 0 \le t \le 1, \mathbf{u} \in \mathbb{R}^p.$$

By (5) and direct algebra, for any weights $\{d_{ni}\}$,

$$(9) \qquad \sup_{t,\mathbf{u}} |Z_d(t, \mathbf{u}) - \mathcal{Y}_d(t, \mathbf{u})| \le 2 \max_i |d_i|.$$

Consider the condition

$$(\mathbf{N3}) \qquad \tau_w^2 = 1, \qquad \max_i w_{ni}^2 \longrightarrow 0.$$

In view of (7) and (9), (N3) implies that the problem of proving the a.u.l. for the Z_d-process is reduced to proving it for the $\mathcal{Y}_w$-process.
Recall the definitions in (2.3.1) and define

$$(10) \qquad \tilde{T}_d(t, \mathbf{u}) := \mathcal{Y}_d(t, \mathbf{u}) - \mu_d(t, \mathbf{u}), \qquad 0 \le t \le 1, \mathbf{u} \in \mathbb{R}^p.$$

Note the basic decomposition: for any real numbers $\{d_{ni}\}$ and for all $0 \le t \le 1$, $\mathbf{u} \in \mathbb{R}^p$,

$$(11) \qquad \tilde{T}_d(t, \mathbf{u}) = Y_d(HH_{n\mathbf{u}}^{-1}(t), \mathbf{u}) + \mu_d(HH_{n\mathbf{u}}^{-1}(t), \mathbf{u}) - \mu_d(t, \mathbf{u}),$$

provided H is strictly increasing for all $n \ge 1$. Decomposition (11) is basic to the following proof of the a.u.l. property of Z_d.

Theorem 3.2.1. *Suppose that* $\{X_{ni}, F_{ni}\}$ *satisfy* (2.2a.34), **(N3)** *holds, and* $\{c_{ni}\}$ *satisfy* (2.3.4) *and* (2.3.5) *with* $d_{ni} \equiv w_{ni}$. *In addition, assume that* (C*) *holds with* $d_{ni} \equiv w_{ni}$, H *is strictly increasing, the densities* $\{f_{ni}\}$ *of* $\{F_{ni}\}$ *satisfy* (2.3.3b), *and that*

$$(12) \qquad \lim_{\delta\to 0} \limsup_n \max_i \sup_{|H(x)-H(y)|<\delta} |f_{ni}(x) - f_{ni}(y)| = 0.$$

Then, for every $0 < B < \infty$,

$$(13) \qquad \sup |\tilde{T}_w(t, \mathbf{u}) - Y_w(t, 0) - \mu_w(HH_{n\mathbf{u}}^{-1}(t), 0) + \mu_w(t, 0)| = o_p(1)$$

where the supremum is being taken over $0 \le t \le 1$, $\mathbf{u} \in \mathbb{R}^p$.

Before proceeding to prove the theorem, we prove the following lemma which is of independent interest. In this result, no assumptions other than independence of $\{X_{ni}\}$ are being used.

Lemma 3.2.1. *Let* H, H_n, $H_{\mathbf{u}}$ *and* $H_{n\mathbf{u}}$ *be as in* (4) *above. Assuming only* (2.2a.34), *we have*

$$\|H_n - H\|_\infty \longrightarrow 0 \quad \text{a.s..} \tag{14}$$

If, in addition, (2.3.4) *holds and if, for any* $0 < B < \infty$,

$$\sup_{|x-y| \le 2m_n B} |H(x) - H(y)| \longrightarrow 0, \qquad (m_n = \max_i \|\mathbf{c}_i\|), \tag{15}$$

then,

$$\sup_{|x| < \infty, \|\mathbf{u}\| \le B} |H_{n\mathbf{u}}(x) - H_{\mathbf{u}}(x)| \longrightarrow 0 \quad \text{a.s..} \tag{16}$$

Proof. Note that $H_n(x) - H(x)$ is a sum of centered independent Bernoulli r.v.'s. Thus $E[H_n(x) - H(x)]^4 = O(n^{-2})$. Apply the Markov inequality with the 4th moment and the Borel-Cantelli lemma to obtain

$$|H_n(x) - H(x)| \longrightarrow 0, \text{ a.s.}, \quad \text{for every } x \in \mathbb{R}.$$

Now proceed as in the proof of the Glivenko-Cantelli Lemma (Loève (1963), p.21) to conclude (14).

To prove (16), note that $\mathbf{u} \in \mathcal{N}(B)$ implies that $-m_n B \le \mathbf{c}_i' \mathbf{u} \le m_n B$, $1 \le i \le n$. The monotonicity of $H_{n\mathbf{u}}$ and $H_{\mathbf{u}}$ yields that for $\mathbf{u} \in \mathcal{N}(B)$, $x \in \mathbb{R}$,

$$\begin{aligned} &H_n(x-Bm_n) - H(x-Bm_n) + H(x-Bm_n) - H(x+Bm_n) \\ &\quad \le H_{n\mathbf{u}}(x) - H_{\mathbf{u}}(x) \\ &\qquad \le H_n(x+Bm_n) - H(x+Bm_n) + H(x+Bm_n) - H(x-Bm_n). \end{aligned}$$

Hence (16) follows from (15) and the following inequality:

$$\text{l.h.s. } (16) \le 2 \sup_{|x| < \infty} |H_n(x) - H(x)| + \sup_{|x-y| \le 2m_n B} |H(x) - H(y)|. \qquad \square$$

Proof of Theorem 3.2.1. From (11), for all $0 \le t \le 1$, $\mathbf{u} \in \mathbb{R}^p$,

$$\begin{aligned}\tilde{T}_w(t,\mathbf{u}) = {} & [Y_w(HH_{n\mathbf{u}}^{-1}(t), \mathbf{u}) - Y_w(HH_{n\mathbf{u}}^{-1}(t), 0]\\ & + [Y_w(HH_{n\mathbf{u}}^{-1}(t), 0) - Y_w(t,0)]\\ & + Y_w(t, 0) - [\mu_w(t, \mathbf{u}) - \mu_w(t, 0) - \mathbf{u}'\nu_w(t)]\\ & + [\mu_w(HH_{n\mathbf{u}}^{-1}(t), \mathbf{u}) - \mu_w(HH_{n\mathbf{u}}^{-1}(t), 0) - \mathbf{u}'\nu_w(HH_{n\mathbf{u}}^{-1}(t))]\\ & + \mu_w(HH_{n\mathbf{u}}^{-1}(t), 0) - \mu_w(t, 0) + \mathbf{u}'[\nu_w(HH_{n\mathbf{u}}^{-1}(t)) - \nu_w(t)].\end{aligned}$$

Therefore,

$$\begin{aligned}\text{l.h.s. (13)} \le {} & \sup|Y_w(t, \mathbf{u}) - Y_w(t, 0)| + \sup|Y_w(HH_{n\mathbf{u}}^{-1}(t), 0) - Y(t,0)|\\ & + 2\sup|\mu_w(t, \mathbf{u}) - \mu_w(t, 0) - \mathbf{u}'\nu_w(t)|\\ & + \sup|\mathbf{u}'[\nu_w(HH_{n\mathbf{u}}^{-1}(t)) - \nu_w(t)]|\end{aligned}$$

$$= A_1 + A_2 + A_3 + A_4, \quad \text{say,} \tag{17}$$

where, as usual, the supremum is being taken over $0 \le t \le 1$, $\mathbf{u} \in \mathcal{N}(B)$. In what follows, the range of x and y over which the supremum is being taken is $\mathbb{R}$, unless specified otherwise.

Now, (2.3.3b) implies that $|H(x) - H(y)| \le |x-y|\, k$. This and (2.3.4) together imply (15). It also implies that

$$\sup_{|x-y|<\delta} |f_{ni}(y) - f_{ni}(x)| \le \sup_{|H(x)-H(y)|<k\delta} |f_{ni}(y) - f_{ni}(x)|.$$

for all $1 \le i \le n$ and all $\delta > 0$. Hence, by (12), it follows that $\{f_{ni}\}$ satisfy (2.3.3a). Now apply Lemma 2.3.1 and (2.3.25), with $d_{ni} = w_{ni}$, $1 \le i \le n$, to conclude that

$$A_j = o_p(1), \quad j = 1, 3. \tag{18}$$

Next, observe that

$$\sup|HH_{n\mathbf{u}}^{-1}(t) - t| \le \sup_{x,\mathbf{u}} |H_{n\mathbf{u}}(x) - H_{\mathbf{u}}(x)| + \sup_{x,\mathbf{u}} |H_{\mathbf{u}}(x) - H(x)| + n^{-1}, \tag{19}$$

$$\sup_{x,\mathbf{u}}|H_{\mathbf{u}}(x) - H(x)| \le \sup_x|H(x + m_n B) - H(x - m_n B)|.$$

Hence, in view of (19) and Lemma 3.2.1, we obtain

$$(20)\qquad \sup_{t,\mathbf{u}} |HH_{n\mathbf{u}}^{-1}(t) - t| \longrightarrow 0, \text{ a.s..}$$

(We need to use the convergence in probability only).

Now, fix a $\delta > 0$ and let $B_n^\delta = [\sup_{t,\mathbf{u}} |\, HH_{n\mathbf{u}}^{-1}(t) - t| < \delta]$. By (20),

$$(21)\qquad \limsup_n P((B_n^\delta)^c) = 0.$$

Now observe that $Y_d(., 0) = W_d^*(.)$ of (2.2a.33). Hence, with A_2 as in (17), for every $\eta > 0$,

$$(22)\qquad \limsup_n P(|A_2| \geq \eta) \leq \limsup_n P(\sup_{|t-s|<\delta} |W_w^*(t) - W_w^*(s)| \geq \eta, B_n^\delta).$$

Upon letting $\delta \longrightarrow 0$ in (22), (2.2a.35) implies

$$(23)\qquad A_2 = o_p(1).$$

Next, we have

$$(24)\qquad \lim_{\delta\to 0} \limsup_n \sup_{|t-s|<\delta} \|\nu_w(t) - \nu_w(s)\|$$

$$\leq \lim_{\delta\to 0} \limsup_n \max_i \sup_{|H(x)-H(y)|<\delta} |f_{ni}(y) - f_{ni}(x)|(\Sigma_i \|w_i c_i\|)$$

$$= 0, \qquad \text{by (12) and (2.3.5).}$$

From (24) and (21) one obtains, in a fashion similar to (23), that

$$(25)\qquad A_4 = o_p(1).$$

This completes the proof of the theorem. □

From a practical point of view, it is worthwhile to state the a.u.l. result in the i.i.d. case separately. Accordingly, we have

Theorem 3.2.2. *Suppose that* $X_{n1}, \dots, X_{nn}$ *are i.i.d.* F. *In addition, assume that* **(F1)**, **(F2)**, **(N3)**, (2.3.4) *and* (2.3.5) *with* $d_{ni} \equiv w_{ni}$ *hold. Then,* $\forall\ 0 < B < \infty$,

$$(26)\qquad \sup_{0\leq t\leq 1,\, \|\mathbf{u}\|\leq B} |Z_d(t, \mathbf{u}) - Z_d(t, 0) - \mathbf{u}'\Sigma_i w_{ni}\mathbf{c}_{ni}\, q(t)| = o_p(1),$$

$$(27)\qquad \sup_{\varphi\in\mathscr{C}, \|\mathbf{u}\|\le B} |T_d(\varphi, \mathbf{u}) - T_d(\varphi, 0) + \mathbf{u}'\Sigma_i w_{ni}\,\mathbf{c}_{ni} \int q\, d\varphi| = o_p(1).$$

where $q = f(F^{-1})$.

Proof. Let $\rho = \Sigma\, w_{ni}\,\mathbf{c}_{ni}$. From (7),

$$(28)\qquad \text{l.h.s. } (26) = \sup_{t,\mathbf{u}} |Z_w(t, \mathbf{u}) - Z_w(t, 0) - \mathbf{u}'\rho\, q(t)|.$$

Take $F_{ni} \equiv F$ in Theorem 3.2.1. Then **(F1)** and **(F2)** imply that q is uniformly continuous on $[0, 1]$ and ensure the satisfaction of all assumptions pertaining to F in Theorem 3.2.1. In addition, $\mu_w(t, 0) = 0$, $0 \le t \le 1$. Thus, Theorem 3.2.1 is applicable and one obtains

$$\sup_{t,\mathbf{u}} |\tilde{T}_w(t, \mathbf{u}) - Y_w(t, 0)| = o_p(1)$$

which in turn yields

$$(29)\qquad \sup_{t,\mathbf{u}} |\tilde{T}_w(t, \mathbf{u}) - \tilde{T}_w(t, 0)| = o_p(1).$$

From (10) and (28),

$$\begin{aligned}\text{l.h.s. } (26) \le \sup_{t,\mathbf{u}}\{&|Z_w(t, \mathbf{u}) - \mathscr{Y}_w(t, \mathbf{u})| + |Z_w(t, 0) - \mathscr{Y}_w(t, 0)| + \\ &+ |\tilde{T}_w(t, \mathbf{u}) - \tilde{T}_w(t, 0)| + |\mu_w(t, \mathbf{u}) - \mathbf{u}'\rho\, q(t)|\} \\ = o_p(1),&\end{aligned}$$

by (9), (10), **(N3)**, (29) and Lemma 2.3.1 applied to $F_{ni} \equiv F$, $d_{ni} \equiv w_{ni}$.

To conclude (27), observe that

$$\begin{aligned}\text{l.h.s.}(27) \le \sup_{t,\mathbf{u}} \{&|Z_d(t, \mathbf{u}) - Z_d(t, 0) - \mathbf{u}'\rho\, q(t)| \\ &+ |\mathbf{u}'\rho|\ |q((n+1)t/n) - q(t)|\} \\ = o_p(1),&\end{aligned}$$

by (26), the uniform continuity of q and (2.3.5) with $d_{ni} \equiv w_{ni}$. □

Remark 3.2.1. Theorem 3.2.2 continues to hold if F depends on n, provided now that the $\{q\}$ are uniformly equicontinuous on $[0, 1]$. □

Remark 3.2.2. An analogue of Theorem 3.2.2 was first proved in Koul (1970) under somewhat stronger conditions on various underlying entities. In Jurečková (1969) one finds yet another variant of (27) for a fixed but a fairly general function φ and with p in c_{ni} equal to 1. Because of the importance of the a.u.l. property of $T_d(\varphi, .)$, it is worthwhile to compare Theorem 3.2.2 above with that of Jurečková's Theorem 3.1 (1969). For the sake of completeness we state it as

Theorem 3.2.3. (Theorem 3.1, Jurečková (1969)). *Let* $X_{n1}, \ldots, X_{nn}$ *be i.i.d.* F. *In addition, assume the following:*

(a) *F has an absolutely continuous density* f *whose a.e. derivative* $\dot{f}$ *satisfies*

$$0 < I(f) < \infty, \qquad I(f) := \int (\dot{f}/f)^2 dF.$$

(b) $\{w_{ni}\}$ *satisfy* (N3).

(c) 1. $\Sigma(c_{ni} - \bar{c}_n)^2 \le M < \infty$ (*recall here* c_{ni} *is* 1×1)

2. $\max(c_{ni} - \bar{c}_n)^2 = o(1), \qquad \bar{c}_n = n^{-1}\sum_{i=1}^{n} c_{ni}.$

(d) φ *is a nondecreasing function on* (0, 1) *with*

$$\int_0^1 (\varphi(t) - \bar{\varphi})^2 \, dt > 0, \quad \bar{\varphi} := \int_0^1 \varphi(u)du.$$

(e) *Either* $(d_{ni} - d_{nj})(c_{ni} - c_{nj}) \ge 0, \ \forall \ 1 \le i, j \le n,$

or $(d_{ni} - d_{nj})(c_{ni} - c_{nj}) \le 0, \ \forall \ 1 \le i, j \le n.$

Then, $\forall \ 0 < B < \infty$,

(f) $$\sup_{\|u\| \le B} |T_d(\varphi, u) - T_d(\varphi, 0) + u\, \Sigma_i\, w_{ni}\, c_{ni}\, b(\varphi, f)| = o_p(1)$$

where $b(\varphi, f) := -\int_{-\infty}^{\infty} \varphi(F(x))\, \dot{f}(x)\, dx.$ □

The strongest point of Theorem 3.2.3 is that it allows for unbounded score functions, such as the "Normal scores" that corresponds to $\varphi = \Phi^{-1}$, Φ being the d.f. of N(0, 1) r.v.. However, this is balanced by requiring (a), (c1) and (e). Note that (b) and (c1) together imply (2.3.5) with $d_{ni} = w_{ni}$, $1 \le i \le n$. Moreover, Theorem 3.2.2 does not require anything like (e).

Claim 3.2.1. (a) *implies that* f *is* $Lip(1/2)$.

First, from Hájek – Šidák (1967), pp 19–20, we recall that (a) implies that $f(x) \longrightarrow 0$ as $x \longrightarrow \pm\infty$. Now, absolute continuity and nonnegativity of f implies that

$$|f(x) - f(y)| \leq \int_x^y |(\dot{f}/f)| \, dF, \qquad x < y.$$

Therefore, by the Cauchy–Schwarz inequality, for $x < y$,

$$\text{(i)} \qquad |f(x) - f(y)| \leq \{\int_x^y (\dot{f}/f)^2 dF \cdot [F(y) - F(x)]\}^{1/2}$$

$$\text{(ii)} \qquad \leq I^{1/2}(f).$$

Letting $y \longrightarrow \infty$ in (ii) yields

$$\text{(iii)} \qquad \|f\|_\infty \leq I^{1/2}(f).$$

Now (i) and (iii) together imply

$$|f(x) - f(y)| \leq I^{1/2}(f) \, \{\int_x^y f(t) \, dt\}^{1/2} \leq I^{3/4}(f) \, (y-x)^{1/2}.$$

A similar inequality holds for $x > y$, thereby giving

$$|f(x) - f(y)| \leq I^{3/4}(f) \, |y-x|^{1/2}, \qquad \forall \; x, y \in \mathbb{R},$$

and proving the claim. Consequently, (a) *implies* (**F1**).

Note that f can be uniformly continuous, bounded, positive a.e., yet need not satisfy $I(f) < \infty$. For example, consider

$$\begin{aligned} f(x) &:= (1-x)/2, \; 0 \leq x \leq 1 \\ &:= (x-2j+1)/2^{j+2}, \;\; 2j-1 \leq x \leq 2j \\ &:= (2j+1-x)/2^{j+2}, \;\; 2j \leq x \leq 2j+1, \;\; j \geq 1; \\ f(x) &:= f(-x), \qquad x \leq 0. \end{aligned}$$

The above discussion shows that both Theorems 3.2.2 and 3.2.3 are needed. Neither displaces the other. If one is interested in the a.u.l. property of, say, Normal scores type rank statistics, then Theorem 3.2.3 gives an answer. On the other hand if one is interested in the a.u.l. property of, say, the Wilcoxon type rank statistics, then Theorem 3.2.2 provides a better result.

The proof of Theorem 3.2.3 uses contiguity and projection technique *a la* Hájek (1962) to approximate $T_d(\varphi, u)$ for each fixed u. Then condition (e) implies the monotonicity of $T_d(\varphi, .)$ which yields the uniformity with respect to u. Such a proof is harder to extend to the case where u and c_{ni} are p×l vectors; this has been done by Jurečková (1971).

The proof of Theorem 3.2.2 exploits the monotonicity inherent in the w.e.p.'s Y_d and certain smoothness properties of F. It would be desirable to *extend this proof to include unbounded* φ. □

We now return to Theorem 3.2.1 with general $\{F_{ni}\}$. We wish to state an a.u.l. theorem for $\{Z_d\}$ and $\{T_d(\varphi, .)\}$ under general $\{F_{ni}\}$. Theorem 3.2.1 still does not quite do it because there is $\mathbf{u}$ in μ_w-expressions. We need to carry out an expansion of these terms in order to recover a term that is linear in $\mathbf{u}$. To that effect we have

Lemma 3.2.2. *In addition to the assumptions of Theorem* 3.2.1, *suppose that*

$$(30) \qquad n^{-1/2} \Sigma_i \|c_{ni}\| = 0(1).$$

Then, $\forall\ 0 < B < \infty$,

$$(31) \qquad \sup_{0\le t\le 1, \|u\|\le B} |n^{1/2}(HH_{nu}^{-1}(t) - t) + Y_1(t, 0) + \mathbf{u}'\nu_1(t)| = o_p(1)$$

where Y_1, ν_1 etc. *are* Y_d, ν_d *of* (2.3.1), (2.3.8) *with* $d_{ni} \equiv n^{-1/2}$. *Consequently,*

$$(32) \qquad \sup_{0\le t\le 1, \|u\|\le B} |n^{1/2}(HH_{nu}^{-1}(t) - t)| = O_p(1).$$

Proof. Write $Y_1(\cdot)$, $\mu_1(\cdot)$ for $Y_1(\cdot, 0)$, $\mu_1(\cdot, 0)$, respectively. Let I denote the identity function and set $\Delta_{nu} := n^{1/2}(H_{nu}H_{nu}^{-1} - I)$. Then,

$$(33) \quad n^{1/2}(HH_{nu}^{-1} - I) = n^{1/2}(HH_{nu}^{-1} - H_uH_{nu}^{-1} + H_uH_{nu}^{-1} - H_{nu}H_{nu}^{-1}) + \Delta_{nu}$$
$$= -[\mu_1(HH_{nu}^{-1}, \mathbf{u}) - \mu_1(HH_{nu}^{-1}) - \mathbf{u}'\nu_1(HH_{nu}^{-1})] + \Delta_{nu}$$
$$- \mathbf{u}'[\nu_1(HH_{nu}^{-1}) - \nu_1] - \mathbf{u}'\nu_1 - Y_1$$
$$- [Y_1(HH_{nu}^{-1}, \mathbf{u}) - Y_1(HH_{nu}^{-1})] - [Y_1(HH_{nu}^{-1}) - Y_1].$$

Now, note that $\sup_{t,\mathbf{u}} |\Delta_{n\mathbf{u}}(t)| \le n^{-1/2}$. Hence

$$(34)\quad \sup_{t,\mathbf{u}} |n^{1/2}(HH_{n\mathbf{u}}^{-1}(t) - t) + Y_1(t) + \mathbf{u}'\nu_1(t)|$$

$$\le \sup_{t,\mathbf{u}} |\mu_1(t, \mathbf{u}) - \mu_1(t) - \mathbf{u}'\nu_1(t)| + B \sup_t \|\nu_1(HH_{n\mathbf{u}}^{-1}(t)) - \nu_1(t)\|$$

$$+ \sup_{t,\mathbf{u}} | Y_1(t, \mathbf{u}) - Y_1(t)| + \sup_{t,\mathbf{u}} |W_1^*(HH_{n\mathbf{u}}^{-1}(t)) - W_1^*(t)|,$$

where we have used the fact that $Y_1(t) = W_1^*(t)$ of (2.2a.33). The first term on the r.h.s. of (34) tends to zero by Lemma 2.3.1 when applied with $d_{ni} \equiv n^{-1/2}$. The third term tends to zero in probability by (2.3.25) applied with $d_{ni} \equiv n^{-1/2}$. To show that the other two terms go to zero in probability, use Lemma 3.2.1, (2.2a.35) and an analogue of (24) for ν_1 and an argument similar to the one that yielded (23) and (26) above. Thus we have (31). Since $\sup_{t,\mathbf{u}} | Y_1(t, 0) + \mathbf{u}'\nu_1(t)| = O_p(1)$, (32) follows. □

Lemma 3.2.3. *In addition to the assumptions of Theorem* 3.2.1 *and* (30), *suppose that for every* $0 < k < \infty$,

$$(35)\qquad \max_i \sup_{|t-s| \le kn^{-1/2}} n^{1/2} | L_{ni}(t) - L_{ni}(s) - (t-s)\ell_{ni}(s)| = o_p(1)$$

where $L_{ni} := F_{ni}\, H^{-1}$, $\ell_{ni} := f_{ni}(H^{-1})/h(H^{-1})$, $1 \le i \le n$ *with* $h := n^{-1} \sum_{i=1}^{n} f_{ni}$.

Moreover, suppose that, with $\tilde{w}(t) := n^{-1} \sum_{i=1}^{n} w_{ni}\, \ell_{ni}(t)$, $0 \le t \le 1$,

$$(36)\qquad \sup_{0 \le t \le 1} n^{1/2} |\tilde{w}(t)| = O(1).$$

Then, $\forall\ 0 < B < \infty$,

$$(37)\qquad \sup | \mu_w(HH_{n\mathbf{u}}^{-1}(t)) - \mu_w(t) + \{ Y_1(t) + \mathbf{u}'\nu_1(t)\} n^{1/2}\, \tilde{w}(t)| = o_p(1)$$

where $\mu_w(t)$, $Y_1(t)$ *stand for* $\mu_w(t, 0)$, $Y_1(t, 0)$, *respectively, and where the supremum is being taken over* $0 \le t \le 1$, $\mathbf{u} \in \mathcal{N}(B)$.

Proof. Let $M_{\mathbf{u}} := \mu_w(HH_{n\mathbf{u}}^{-1}) - \mu_w$. From (32) it follows that $\forall\ \epsilon > 0$ $\exists\ K_\epsilon$ and $N_{1\epsilon}$ such that

$$(38)\qquad P(A_n^\epsilon) \ge 1 - \epsilon, \qquad n \ge N_{1\epsilon},$$

where

$$A_n^\epsilon = [\sup_{t,\mathbf{u}} |HH_{n\mathbf{u}}^{-1}(t) - t| \leq K\epsilon n^{-1/2}].$$

By assumption (35), there exists $N_{2\epsilon}$ such that $n \geq N_{2\epsilon}$ implies

$$\max_i \sup_{|t-s| \leq K\epsilon n^{-1/2}} n^{1/2} |L_i(t) - L_i(s) - (t-s)\ell_i(s)| < \epsilon. \tag{39}$$

Define

$$Z_{\mathbf{u}i}^\epsilon := \{L_i(HH_{n\mathbf{u}}^{-1}) - L_i - [HH_{n\mathbf{u}}^{-1} - I]\,\ell_i\}\, I(A_n^\epsilon), \qquad 1 \leq i \leq n.$$

In view of (39) and (38),

$$P(\max_i \sup_{t,\mathbf{u}} n^{1/2} |Z_{\mathbf{u}i}^\epsilon(t)| > \epsilon) < \epsilon, \qquad n \geq N_{1\epsilon} \vee N_{2\epsilon} =: N_\epsilon. \tag{40}$$

Moreover,

$$\begin{aligned} M_{\mathbf{u}} &= M_{\mathbf{u}}\, I(A_n^\epsilon) + M_{\mathbf{u}}\, I((A_n^\epsilon)^c) \\ &= \Sigma_i\, w_i\, Z_{\mathbf{u}i}^\epsilon + Z_{\mathbf{u}0}^\epsilon + n^{1/2}[HH_{n\mathbf{u}}^{-1} - I]\, n^{1/2}\, \tilde{\mathbf{w}}, \end{aligned} \tag{41}$$

where

$$Z_{\mathbf{u}0}^\epsilon := \{M_{\mathbf{u}} - n^{1/2}[HH_{n\mathbf{u}}^{-1} - I]\cdot n^{1/2}\, \tilde{\mathbf{w}}\}\, I((A_n^\epsilon)^c).$$

Note that

$$P(\sup_{t,\mathbf{u}} |Z_{\mathbf{u}0}^\epsilon| \neq 0) \leq P((A_n^\epsilon)^c) < \epsilon, \qquad n > N_\epsilon. \tag{42}$$

By the C-S inequality, (**N3**) and (40),

$$P(\sup_{t,\mathbf{u}} |\Sigma_i\, w_i\, Z_{\mathbf{u}i}^\epsilon(t)| > \epsilon) < \epsilon, \qquad n > N_\epsilon. \tag{43}$$

Hence, (37) follows from (43), (42), (41), Lemma 3.2.2 and (36). □

We combine Theorem 3.2.1, Lemmas 3.2.2 and 3.2.3 to obtain the following

Theorem 3.2.4. *Under the notation and assumptions of Theorem* 3.2.1, *Lemmas* 3.2.2 *and* 3.2.3, $\forall\ 0 < B < \infty$,

$$\sup |Z_d(t, \mathbf{u}) - Z_d(t, 0) - \mathbf{u}' \Sigma_i \left(d_{ni} - \tilde{d}_n(t)\right) c_{ni}\, q_{ni}(t)| = o_p(1), \tag{44}$$

(45) $\sup|T_d(\varphi, \mathbf{u})-T_d(\varphi, 0) + \mathbf{u}'\int \Sigma_i\,(d_{ni}-\tilde{d}_n(t))c_{ni}\,q_{ni}(t)\,d\varphi(t)| = o_p(1),$

where the supremum in (44) *is over* $0\le t\le 1$, $\|\mathbf{u}\|\le B$, *in* (45) *over* $\varphi\in\mathscr{C}$, $\|\mathbf{u}\|\le B$, *and where* $\tilde{d}_n(t) := n^{-1}\Sigma_i\, d_{ni}\,\ell_{ni}(t)$, $q_{ni} := f_{ni}(H^{-1}(t))$, $0\le t\le 1$, $1\le i\le n$.

Proof. Let $\rho(t) := \Sigma_i\,(d_i - \tilde{d}(t))c_iq_i(t)$. Note that the fact that $n^{-1}\Sigma_i\,\ell_i(t)\equiv 1$ implies that $\rho(t) = \Sigma_i\,(w_i - \tilde{w}(t))c_iq_i(t)$, where $\{w_i\}$ are as in (6). From (7), (8) and (9),

(46) $$\text{l.h.s. } (44) = \sup_{0\le t\le 1,\,\|\mathbf{u}\|\le B} |Z_w(t, \mathbf{u}) - Z_w(t, 0) - \mathbf{u}'\rho(t)|$$

$$\le 4\max_i|w_i| + \sup_{0\le t\le 1,\,\|\mathbf{u}\|\le B} |\mathscr{T}_w(t, \mathbf{u}) - \mathscr{T}_w(t, 0) - \mathbf{u}'\rho(t)|.$$

Now, from Theorem 3.2.1 and Lemma 3.2.3, uniformly in $0\le t\le 1$, $\|\mathbf{u}\|\le B$,

(47) $$\sup|T^*_w(t, \mathbf{u}) - Y_w(t) + \{Y_1(t) + \nu_1(t)\,\mathbf{u}\}\,n^{1/2}\,\tilde{w}(t)| = o_p(1),$$

where $Y_d(t)$ stands for $Y_d(t, 0)$ for arbitrary weights $\{d_{ni}\}$. Therefore,

$$\sup|\mathscr{T}_w(t, \mathbf{u}) - \mathscr{T}_w(t, 0) - \mathbf{u}'\rho(t)|$$

$$= \sup|\tilde{T}_w(t, \mathbf{u}) - \tilde{T}_w(t, 0) + \mu_w(t, \mathbf{u}) - \mu_w(t, 0) - \mathbf{u}'\rho(t)|$$

$$\le \sup|\tilde{T}_w(t, \mathbf{u}) - \tilde{T}_w(t, 0) + \mathbf{u}'\nu_1(t)\,n^{1/2}\,\tilde{w}(t)|$$

$$+ \sup|\mu_w(t, \mathbf{u}) - \mu_w(t, 0) - \mathbf{u}'\nu_w(t)| = o_p(1),$$

by (47) and Lemma 2.3.1 and the fact that $\rho(t) = \nu_w(t) - \nu_1(t)\,n^{1/2}\,\tilde{w}(t)$. This completes the proof (44). The proof of (45) follows from (44) in the same fashion as does that of (27) from (26). □

Remark 3.2.3. As in Remark 2.2a.3, suppose we strengthen **(N3)** to require

(B1) $$n\max_i w_{ni}^2 = O(1), \qquad \tau_w^2 = 1.$$

Then **(C*)** and (36) are *a priori* satisfied by L_w. □

Remark 3.2.4. If one is interested in the i.i.d. case only, then Theorem 3.2.2 gives a better result than Theorem 3.2.4. □

3.3. A.U.L. OF LINEAR SIGNED RANK STATISTICS

In this section our aim is to prove analogs of Theorems 3.2.2 and 3.2.4 for the signed rank processes $\{\overset{+}{T}_d(\varphi, \mathbf{u}), \mathbf{u}\in\mathbb{R}^p\}$, using as many results from the previous sections as possible. Many details are quite similar. Define, for $\mathbf{u}\in\mathbb{R}^p$, $0 \le t \le 1$, $x \ge 0$,

$$\text{(1)} \qquad Z_d^+(t, \mathbf{u}) := \Sigma_i\, d_{ni}\, I(R_{i\mathbf{u}}^+ \le nt)\, s(X_{ni} - \mathbf{c}_{ni}'\mathbf{u}),$$

$$J_{n\mathbf{u}}(x) := n^{-1}\Sigma_i\, I(|X_{ni} - \mathbf{c}_{ni}'\mathbf{u}| \le x) = H_{n\mathbf{u}}(x) - H_{n\mathbf{u}}(-x),$$

$$J_{\mathbf{u}}(x) := n^{-1}\, \Sigma_i\, [F_{ni}(x+\mathbf{c}_{ni}'\mathbf{u}) - F_{ni}(-x+\mathbf{c}_{ni}'\mathbf{u})] = H_{\mathbf{u}}(x) - H_{\mathbf{u}}(-x),$$

$$\mathscr{T}_d^+(t, \mathbf{u}) := \Sigma_i\, d_{ni} I(|X_{ni} - \mathbf{c}_{ni}'\mathbf{u}| \le J_{n\mathbf{u}}^{-1}(t))\, s(X_{ni} - \mathbf{c}_{ni}'\mathbf{u}),$$

$$S_d^+(t, \mathbf{u}) := \Sigma\, d_{ni}\, I(|X_{ni} - \mathbf{c}_{ni}'\mathbf{u}| \le J^{-1}(t))\, s(X_{ni} - \mathbf{c}_{ni}'\mathbf{u}),$$

$$\mu_d^+(t, \mathbf{u}) := \Sigma_i\, d_{ni}\, \mu_{ni}^+(t, \mathbf{u}) = E\, S_d^+(t, \mathbf{u}),$$

$$\mu_{ni}^+(t, \mathbf{u}) := F_{ni}(J^{-1}(t)+\mathbf{c}_{ni}'\mathbf{u}) + F_{ni}(-J^{-1}(t)+\mathbf{c}_{ni}'\mathbf{u}) - 2F_{ni}(\mathbf{c}_{ni}'\mathbf{u}), \quad 1\le i\le n.$$

In the above and sequel, J and J_n stand for J_0 and J_{n0}, respectively. We also need,

$$\text{(2)} \qquad \mathcal{Y}_d^+(t, \mathbf{u}) := S_d^+(t, \mathbf{u}) - \mu_d^+(t, \mathbf{u}),$$

and

$$\text{(3)} \qquad \tilde{T}_d^+(t, \mathbf{u}) := \mathscr{T}_d^+(t, \mathbf{u}) - \mu_d^+(t, \mathbf{u}), \qquad 0 \le t \le 1, \ \mathbf{u} \in \mathbb{R}^p.$$

Analogous to (3.2.11), we have the basic decomposition: For $0\le t\le 1$, $\mathbf{u}\in\mathbb{R}^p$,

$$\text{(4)} \qquad \tilde{T}_d^+(t, \mathbf{u}) = \mathcal{Y}_d^+(JJ_{n\mathbf{u}}^{-1}(t), \mathbf{u}) + \mu_d^+(JJ_{n\mathbf{u}}^{-1}(t), \mathbf{u}) - \mu_d^+(t, \mathbf{u}),$$

Now, note that, w.p. 1, for all $0 \le t \le 1$, $\mathbf{u} \in \mathbb{R}^p$,

$$\text{(5)} \qquad \mathcal{Y}_d^+(t, \mathbf{u}) = \mathcal{Y}_d(HJ^{-1}(t), \mathbf{u}) + \mathcal{Y}_d(H(-J^{-1}(t)), \mathbf{u}) - 2\, \mathcal{Y}_d(H(0), \mathbf{u}),$$

where $\mathcal{Y}_d$ is as in (2.3.1). Therefore, by Theorem 2.3.1 (see (2.3.25)), under the assumptions of that theorem and strictly increasing nature of J and H,

$$(6) \qquad \sup_{t,\mathbf{u}} | Y_d^+(t, \mathbf{u}) - Y_d^+(t, 0)| = o_p(1).$$

One also has, in view of the continuity of $\{F_{ni}\}$, a relation like (5) between μ_d^+ and μ_d. Thus by Lemma 2.3.1, under the assumptions there,

$$(7) \qquad \sup_{t,\mathbf{u}} |\mu_d^+(t, \mathbf{u}) - \mu_d^+(t, 0) - \mathbf{u}' \nu_d^+(t)| = o(1)$$

where

$$(8) \qquad \nu_d^+(t) := \Sigma\, d_{ni}\, \mathbf{c}_{ni} [f_{ni}(J^{-1}(t)) + f_{ni}(-J^{-1}(t)) - 2f_{ni}(0)], \quad 0 \le t \le 1.$$

We also have an anlogue of Lemma 3.2.1:

Lemma 3.3.1. *Without any assumption except* (2.2a.34),

$$(9) \qquad \sup_{0 \le x \le \infty} |J_n(x) - J(x)| \longrightarrow 0 \ \text{ a.s.}$$

If, in addition, (2.3.4) *and* (3.2.15) *hold, then*

$$(10) \qquad \sup_{0 \le x \le \infty,\, \|\mathbf{u}\| \le B} |J_{n\mathbf{u}}(x) - J_{\mathbf{u}}(x)| \longrightarrow 0 \ \text{ a.s..}$$

Using this lemma, arguments like those in Theorem 3.2.1 and the above discussion, one obtains

Theorem 3.3.1. *Suppose that* $\{X_{ni}, F_{ni}\}$ *satisfy* (2.2a.34), (2.3.3b) *and that* $\{d_{ni}, \mathbf{c}_{ni}\}$ *satisfy* (N1), (N2), (2.3.4) *and* (2.3.5). *In addition, assume that*

$$(11) \qquad \lim_{\delta \to 0} \limsup_n \max_i \sup_{|J(x)-J(y)| < \delta} |f_{ni}(x) - f_{ni}(y)| = 0$$

and that H *is strictly increasing for every* n. *Then, for every* $0 < B < \infty$,

$$(12) \qquad \sup_{0 \le t \le 1,\, \|\mathbf{u}\| \le B} |\tilde{T}_d^+(t, \mathbf{u}) - Y_d^+(t, 0) - \mu_d^+(JJ_{n\mathbf{u}}^{-1}(t), 0) + \mu_d^+(t, 0)| = o_p(1). \quad \square$$

We remark here that (11) implies (3.2.12).

Next, note that if $\{F_i\}$ are symmetric about 0, then

$$(13) \qquad \mu_d^+(t, 0) = 0, \quad 0 \le t \le 1, \quad n \ge 1.$$

Upon combining (13), (12) with (7) one obtains

Theorem 3.3.2. *In addition to the assumptions of Theorem* 3.2.1, *suppose that* $\{F_{ni}, 1 \le i \le n\}$ *are symmetric about* 0.
Then, for every $0 < B < \infty$,

(14)
$$\sup_{0 \le t \le 1, \|\mathbf{u}\| \le B} |Z_d^+(t, \mathbf{u}) - Z_d^+(t, 0) - \mathbf{u}' \Sigma_i d_{ni} c_{ni} \nu_{ni}^+(t)| = o_p(1),$$

(15)
$$\sup_{\varphi \in \mathscr{C}, \|\mathbf{u}\| \le B} |T_d^+(\varphi, \mathbf{u}) - T_d^+(\varphi, 0) + \mathbf{u}' \Sigma_i d_{ni} c_{ni} \int_0^1 \nu_{ni}^+(t)\, d\varphi^+(t)| = o_p(1),$$

where
$$\nu_{ni}^+(t) := 2[f_{ni}(J^{-1}(t) - f_{ni}(0)], \quad 1 \le i \le n, \quad 0 \le t \le 1.$$

Proof. Using a relation like (3.2.5) between R_{iu}^+ and J_{nu}, one obtains, as in (3.2.9),

(16)
$$\sup_{t, \mathbf{u}} |Z_d^+(t, \mathbf{u}) - \mathscr{Z}_d^+(t, \mathbf{u})| \le 2 \max_i |d_i| = o(1), \qquad \text{by } \mathbf{(N2)}.$$

Thus (13) follows from (16), (12), (11) and (7). Conclusion (15) follows from (13) in the same way as (3.2.27) follows from (3.2.26). □

Because of the importance of the i.i.d. symmetric case, we specialize the above theorem to yield

Corollory 3.3.1. *Let* F *be a d.f., symmetric around zero, satisfying* **(F1)**, **(F2)** *and let* $X_{n1}, \ldots, X_{nn}$ *be i.i.d.* F. *In addition, assume that* $\{d_{ni}, c_{ni}\}$ *satisfy* (N1), (N2), (2.3.4) *and* (2.3.5). *Then, for every* $0 < B < \infty$,

(17)
$$\sup_{0 \le t \le 1, \|\mathbf{u}\| \le B} |Z_d^+(t, \mathbf{u}) - Z_d^+(t, 0) - \mathbf{u}' \Sigma_i d_{ni}\, c_{ni}\, q^+(t)| = o_p(1),$$

(18)
$$\sup_{\varphi \in \mathscr{C}, \mathbf{u} \in \mathscr{N}(B)} |T_d^+(\varphi, \mathbf{u}) - T_d^+(\varphi, 0) + \Sigma_i d_{ni} c_{ni}' \mathbf{u} \int_0^1 q^+(t) d\varphi^+(t)| = o_p(1),$$

where $q^+(t) := 2[f(F^{-1}((t+1)/2)) - f(0)], \ 0 \le t \le 1.$ □

Remark 3.3.1. Van Eeden (1972) proved an analogue of (18) without the supremum over φ, but for square integrable φ's. She also needs conditions like those in Theorem 3.2.3 above. Thus Remark 3.2.1 is equally applicable here when comparing Corollory 3.2.1 with Van Eeden's results. □

Now, we return to Theorem 3.3.1 and expand the μ_d^+-terms further so as to recover an extra linearity term. Define, for $0 \le t \le 1$, $\mathbf{u} \in \mathbb{R}^p$,

$$(19)\qquad \overset{*}{Y_d}(t, \mathbf{u}) := \Sigma_i\, d_{ni}[I(|X_{ni} - \mathbf{c}_{ni}\,\mathbf{u}| \le J^{-1}(t)) - F^+_{i\mathbf{u}}(J^{-1}(t))]$$

$$\overset{*}{\nu_d}(t) := \Sigma_i\, d_{ni}\, \mathbf{c}_{ni}\, [f_{ni}(J^{-1}(t)) - f_{ni}(-J^{-1}(t))]$$

where

$$F^+_{i\mathbf{u}}(x) := F_{ni}(x + \mathbf{c}_i'\mathbf{u}) - F_{ni}(-x + \mathbf{c}_i'\mathbf{u}), \qquad x \ge 0.$$

Note the relation: For arbitrary $\{d_{ni}\}$,

$$(20)\qquad \overset{*}{Y_d}(t, \mathbf{u}) \equiv Y_d(HJ^{-1}(t), \mathbf{u}) - Y_d(H(-J^{-1}(t)), \mathbf{u}).$$

From (20) and (2.3.25) applied with $d_{ni} = n^{-1/2}$, we obtain

$$(21)\qquad \sup_{t,\mathbf{u}} |\overset{*}{Y_1}(t, \mathbf{u}) - \overset{*}{Y_1}(t, 0)| = o_p(1).$$

Note that in the case $d_{ni} \equiv n^{-1/2}$, (2.3.5) reduces to (3.2.30).
Next, under (11) and (2.3.5), just as (3.2.24),

$$(22)\qquad \lim_{\delta\to 0} \limsup_n \sup_{|t-s|<\delta} \|\overset{*}{\nu_d}(t) - \overset{*}{\nu_d}(s)\| = 0,$$

for the given $\{d_{ni}\}$ and for $d_{ni} \equiv n^{-1/2}$.
Using (21), (22) and calculations similar to those done in the proof of Lemma 3.2.2, we obtain

Lemma 3.3.2. *Under the conditions of Theorem* 3.2.1 *and* (3.2.30)

$$(23)\qquad \sup_{t,\mathbf{u}} |n^{1/2}(JJ_{n\mathbf{u}}^{-1}(t) - t) + \overset{*}{Y_1}(t, 0) + \mathbf{u}'\,\overset{*}{\nu_1}(t)\,| = o_p(1).$$

Consequently,

$$(24)\qquad \sup_{t,\mathbf{u}} |n^{1/2}(JJ_{n\mathbf{u}}^{-1}(t) - t)| = O_p(1). \qquad \square$$

Similarly arguing as in Lemma 3.2.3, we obtain the following Lemma 3.3.3. In it $\mu^+_d(t)$, $\mu^+_1(t)$ etc. stand for $\mu^+_d(t, 0)$, $\mu^+_1(t, 0)$ etc. of (1).

Lemma **3.3.3.** *In addition to the assumptions of Theorem* 3.2.1, (3.2.30) *assume that for every* $0 < k < \infty$,

$$(25)\qquad \max_i \sup_{|t-s|\le kn^{-1/2}} n^{1/2}\, |\mu^+_{ni}(t) - \mu^+_{ni}(s) - (t-s)\ell^+_{ni}(s)| = o(1)$$

where $\{\mu_{ni}^{+}\}$ *are as in* (1),

$$(26) \qquad \ell_{ni}^{+}(s) := [f_{ni}(J^{-1}(s)) - f_{ni}(-J^{-1}(s))] \,/\, h^{+}(J^{-1}(s)), \qquad 0 \le s \le 1,$$

$$h^{+}(x) := n^{-1}\, \Sigma_i\, [f_{ni}(x) - f_{ni}(-x)], \qquad x \ge 0.$$

Moreover, with $\tilde{d}_n^{+}(t) := n^{-1}\, \Sigma_i\, d_{ni}\ell_{ni}^{+}(t)$, $0 \le t \le 1$, *assume that*

$$(27) \qquad \sup_{0 \le t \le 1} |n^{1/2}\, \tilde{d}_n^{+}(t)| = O(1).$$

Then,

$$(28) \qquad \sup |\mu_d^{+}(JJ_{n\mathbf{u}}^{-1}(t)) - \mu_d^{+}(t) + \{\overset{*}{Y}_1^{+}(t) + \mathbf{u}'\, \overset{*}{\nu}_1(t)\}n^{1/2}\, \tilde{d}_n^{+}(t)| = o_p(1),$$

where the supremum is taken over the set $0 \le t \le 1$, $\|\mathbf{u}\| \le B$. □

Finally, an analogue of Theorem 3.2.3 is

Theorem 3.3.3. *Under the assumptions of Theorem* 3.3.1, (3.2.30), (25) *and* (27), *for every* $0 < B < \infty$,

$$(29) \qquad \sup_{0 \le t \le 1,\, \|\mathbf{u}\| \le B} |Z_d^{+}(t, \mathbf{u}) - Z_d^{+}(t, 0) - \mathbf{u}'[\nu_d^{+}(t) - \overset{*}{\nu}_1(t)n^{1/2}\, \tilde{d}_n^{+}(t)]| = o_p(1),$$

$$(30) \quad \sup |T_d^{+}(\varphi,\mathbf{u}) - T_d^{+}(\varphi,0) + \mathbf{u}' \int_0^1 [\nu_d^{+}(t) - \overset{*}{\nu}_1(t)n^{1/2}\tilde{d}_n^{+}(t)]\, d\varphi^{+}(t)| = o_p(1),$$

where the supremum in (30) *is over* $\varphi \in \mathscr{C}$, $\|\mathbf{u}\| \le B$. □

Remark 3.3.2. Unlike the case in Theorem 3.2.3, there does not appear to be a nice simplification of the term $\nu_d^{+} - \overset{*}{\nu}_1\, n^{1/2}\, \tilde{d}_n^{+}$. However, it can be rewritten as follows:

$$\nu_d^{+}(t) - \overset{*}{\nu}_1(t)n^{1/2}\, \tilde{d}_n^{+}(t) = \Sigma_i\, d_i c_i\, [f_i(J^{-1}(t)) + f_i(-J^{-1}(t)) - 2f_i(0)]$$
$$+ \Sigma_i\, (d_i - \tilde{d}_n^{+}(t))\, c_i\, [f_i(J^{-1}(t)) - f_i(-J^{-1}(t))].$$

This representation is somewhat revealing in the following sense. The first term is due to the shift $\mathbf{u}'c_i$ in the r.v. X_i and the second term is due to the nonidentical and asymmetric nature of the distribution of X_i, $1 \le i \le n$. □

Remark 3.3.3. If one is interested in the symmetric case or in the i.i.d. symmetric case then Theorem 3.3.2 and Corollary 3.3.1, respectively, give better results than Theorem 3.3.3. □

3.4. WEAK CONVERGENCE OF RANK AND SIGNED RANK W.E.P.'S.

Throughout this section we shall use the notation of Sections 3.2 – 3.3 with $u = 0$. Thus, e.g., $Z_d(t)$, $Z_d^+(t)$, etc. will represent $Z_d(t, 0)$, $Z_d^+(t, 0)$, etc. of (3.2.2) and (3.3.1), i.e., for $0 \le t \le 1$,

(1) $$Z_d(t) = \Sigma_i d_{ni} I(R_{ni} \le nt), \qquad Z_d^+(t) = \Sigma_i d_{ni} I(R_{ni}^+ \le nt)s(X_{ni}),$$

$$\mathcal{V}_d(t) = \Sigma_i d_{ni} I(X_{ni} \le H_n^{-1}(t)), \qquad \mu_d(t) = \Sigma_i d_{ni} L_{ni}(t),$$

where R_{ni} (R_{ni}^+) is the rank of X_{ni} ($|X_{ni}|$) among $X_{n1}, \ldots, X_{nn}$ ($|X_{n1}|, \ldots, |X_{nn}|$).

We shall first prove the asymptotic normality of Z_d and Z_d^+ for a fixed t, say $t = v$, $0<v<1$. To begin with consider $Z_d(v)$. In the following theorem v is a fixed number in $(0, 1)$.

Theorem 3.4.1. *Suppose that* $\{X_{ni}\}$, $\{F_{ni}\}$, $\{L_{ni}\}$, L_d *are as in* (2.2a.33) and (2.2a.34). *Assume that* $\{d_{ni}\}$ *satisfy* (N1), (N2) *and that* H *is strictly increasing for each* n. *Also assume that*

(2) $$\lim_{\delta \to 0} \limsup_n [L_d(v + \delta) - L_d(v - \delta)] = 0,$$

and that there are nonnegative numbers $\ell_{ni}(v)$, $1 \le i \le n$, *such that for every* $0 < k < \infty$,

(3) $$\max_i \sup_{|t-s| \le kn^{-1/2}} n^{1/2} |L_{ni}(t) - L_{ni}(v) - (t-v)\ell_{ni}(v)| = o(1).$$

Denoting

(4) $$\tilde{d}_n(v) := n^{-1} \Sigma_i d_{ni}\ell_{ni}(v), \quad \sigma_d^2(v) := \Sigma_i (d_{ni} - \tilde{d}_n(v))^2 L_{ni}(v)(1-L_{ni}(v)),$$

assume that

(5) $$n^{1/2} |\tilde{d}_n(v)| = O(1).$$

(6) $$\liminf_n \sigma_d^2(v) > 0.$$

Then,

$$\{\sigma_d(v)\}^{-1}\{Z_d(v) - \mu_d(v)\} \xrightarrow[d]{} N(0, 1).$$

The proof of Theorem 3.4.1 is a consequence of the following *three* lemmas. In these lemmas the setup is the same as in Theorem 3.4.1.

Lemma 3.4.1. *Under the sole assumption of* (2.2a.34),

$$\sup_{0\le t\le 1} |HH_n^{-1}(t) - t| = o_p(1).$$

Proof. Upon taking $u = 0$ in (3.2.19), one obtains

$$\sup_{0\le t\le 1} |HH_n^{-1}(t) - t| \le \sup_{-\infty\le x\le +\infty} |H_n(x) - H(x)| + n^{-1} = o_p(1),$$

by (3.2.14) of Lemma 3.2.1. □

Lemma 3.4.2. *Let* $Y_d(t)$ denote the $Y_d(t, 0)$ of (2.3.1). *Then, under* (3), *for every* $\epsilon > 0$,

$$\lim_{\delta\to 0} \limsup_n P\Big(\sup_{|t-v|<\delta} | Y_d(t) - Y_d(v)| > \epsilon\Big) = 0$$

Proof. Apply Lemma 2.2a.2 to $\eta_{ni} = H(X_{ni})$, $G_{ni} = L_{ni}$, to obtain that $Y_d \equiv W_d$ of that lemma and that

$$\begin{aligned} P\Big(\sup_{|t-v|<\delta} | Y_d(t) - Y_d(v)| > \epsilon\Big) &\\ \le \kappa\epsilon^{-2}[L_d(v + \delta) - L_d(v - \delta)]^2 &+ P(| Y_d(v - \delta) - Y_d(v)| > \epsilon/2)\\ &+ P(| Y_d(v + \delta) - Y_d(v - \delta)| > \epsilon/4)\\ \le (\kappa + 20)\epsilon^{-2} [L_d(v + \delta) - L_d(v - \delta)], &\qquad \text{(by Chebyshev).} \end{aligned}$$

The Lemma now follows from the assumption (3). □

Lemma 3.4.3. *Under* (3), *for every* $\epsilon > 0$,

$$\limsup_n P(| Y_d(HH_n^{-1}(v)) - Y_d(v)| > \epsilon) = 0.$$

Proof. Follows from Lemmas 3.4.1 and 3.4.2 . □

Remark 3.4.1. Lemmas 3.4.2 could be deduced from Corollary 3.3.1 which gives the tightness of the process Y_d under stronger condition (C*). But here we are interested in the behavior of Y_d only in the neighborhood of one point v and the above lemma proves the contnuity of Y_d at the point v at which (3) holds. Similarly, many of the approximations that follow could of course be deduced from proofs of Theorems 3.2.1 and 3.2.2. But these theorems obtain results uniformly in $0\le t\le 1$ under rather stronger conditions than would be needed in the present case. Of course various decompositions used in their proofs will be useful here also. □

Proof of Theorem 3.4.1. In view of (3.2.9) and (**N2**), it suffices to prove that $\{\sigma_d(v)\}^{-1}\, \tilde{T}_d(v) \underset{d}{\longrightarrow} N(0, 1)$, where

(7) $$\tilde{T}_d(v) = \{\, \mathcal{V}_d(v) - \mu_d(v)\}.$$

But, from (3.2.11) applied with $\mathbf{u} = \mathbf{0}$,

$$\tilde{T}_d(v) = \mathcal{Y}_d(HH_n^{-1}(v)) + \mu_d(HH_n^{-1}(v)) - \mu_d(v), \quad \text{w.p. } 1.$$

(8) $$= \mathcal{Y}_d(v) + o_p(1) + \mu_d(HH_n^{-1}(v)) - \mu_d(v), \qquad \text{by (6).}$$

Apply the identity (3.2.33) with $\mathbf{u} = \mathbf{0}$ and Lemma 3.4.3 with $d_i \equiv n^{-1/2}$ to obtain,

(9) $$n^{1/2}[HH_n^{-1}(v) - v] = -\mathcal{Y}_1(HH_n^{-1}(v)) + o_p(1) = -\mathcal{Y}_1(v) + o_p(1).$$

Since $\mathcal{Y}_1(v) \underset{d}{\longrightarrow} N(0,\, v(1-v))$, $|\,\mathcal{Y}_1(v)| = O_p(1)$. Again, argue as for (3.2.37) with $\mathbf{u} \equiv \mathbf{0}$, $t \equiv v$ (i.e., without the supremum on the l.h.s. and with $\mathbf{u} \equiv \mathbf{0}$, $t \equiv v$), to conclude that

(10) $$\mu_d(HH_n^{-1}(v)) - \mu_d(v) = -\mathcal{Y}_1(v)\, n^{1/2}\, \tilde{d}(v) + o_p(1).$$

Combine (9), (10) to obtain

(11) $$\tilde{T}_d(v) = \mathcal{Y}_d(v) - n^{1/2}\, \tilde{d}(v)\; \mathcal{Y}_1(v) + o_p(1)$$

$$= \sum_{i=1}^{n} (d_{ni} - \tilde{d}(v))\; \{I(X_{ni} \le H^{-1}(v)) - L_{ni}(v)\} + o_p(1).$$

The theorem now follows from (6) and the fact that $\{\sigma_d(v)\}^{-1}\cdot$\{leading term in the r.h.s. of (11)\} $\underset{d}{\longrightarrow} N(0,1)$ by the L-F CLT, in view of (**N1**) and (**N2**).□

Remark 3.4.2. If $\{F_{ni}\}$ have densities $\{f_{ni}\}$ then $\ell_{ni}(v)$ can be taken to be $f_{ni}(H^{-1}(v))/h(H^{-1}(v))$, just as in (3.2.34). However, if one is interested in the asymptotic normality of linear rank statistic corresponding to the jump score function, with jump at v, then we need $\{L_{ni}\}$ to be smooth only at that jump point.

The above Theorem 3.4.1 bears strong resemblance to Theorem 1 of Dupač–Hájek (1969). The assumptions (**N1**), (**N2**) and (4) correspond to (2.2a), (2.13) and (2.2a2) of Dupač-Hájek. Condition (3) above is not quite comparable to condition (2.12) Dupač-Hájek but it appears to be less restrictive. In any case, (2.12) and (2.13) together imply the boundedness of

$\{\ell_i(v)\}$ and hence the condition (5) above. Taken together, then, the assumptions of the above theorem are somewhat weaker than those of Dupač-Hájek. On the other hand, the conclusions of the Dupač-Hájek Theorem 1 are stronger than those of the above theorem in that it asserts not only $\{Z_d(v) - \mu_d(v)\}\sigma_d^{-1}(v) \Rightarrow N(0,1)$ but also that $E[\sigma_d^{-1}(v)(Z_d(v) - \mu_d(v))]^r \longrightarrow 0$, for $r = 1, 2$, as $n \longrightarrow \infty$. However, if one is only interested in the asymptotic normality of $\{Z_d(v)\}$ then the above theorem appears to be more desirable. Moreover, in view of the decomposition (3.2.11), the proof presented below makes the role played by conditions (3) and (4) clearer.

The assumption about H being strictly increasing is not really an assumption because, without loss of generality, one may assume that $\{F_i\}$ are not flat on a common interval. For, if all $\{F_i\}$ were flat on a common interval, then deletion of this interval would not change the distribution of $R_1, \ldots, R_n$ and hence of $\{Z_d\}$. □

Next, we turn to the asymptotic normality of $Z_d^+(v)$. Again, put $\mathbf{u} = \mathbf{0}$ in the definition (3.3.1) to obtain,

$$(12) \qquad \mathscr{V}_d^+(t) = \Sigma_i\, d_{ni}\, I(|X_{ni}| \le J_n^{-1}(t))s(X_{ni}),$$

$$\mu_{ni}^+(t) = F_{ni}(J^{-1}(t)) + F_{ni}(-J^{-1}(t)) - 2F_{ni}(0), \quad 1 \le i \le n$$

$$S_d^+(t) = \Sigma_i\, d_{ni} I(|X_{ni}| \le J^{-1}(t))s(X_{ni}), \quad 0 \le t \le 1,$$

$$\mu_d^+(t) = \Sigma_i\, d_{ni}\, \mu_{ni}^+(t), \quad 0 \le t \le 1. \qquad Y_d^+ = S_d^+ - \mu_d^+.$$

Like (3.2.9), we have

$$(13) \qquad \sup_{0 \le t \le 1} |Z_d^+(t) - \mathscr{V}_d^+(t)| \le 2 \max_i |d_i|.$$

Because of (N2), it suffices to consider $\mathscr{V}_d^+$ only. Observe that

$$Y_d^+(t) = Y_d(HJ^{-1}(t)) + Y_d(H(-J^{-1}(t))) - 2Y_d(H(0)),$$

where Y_d is as in (2.3.1). Rewrite

$$(14) \qquad Y_d^+(t) = \{Y_d(HJ^{-1}(t)) - Y_d(H(0))\} - \{Y_d(H(0)) - Y_d(-J^{-1}(t))\}$$

$$= Y_{d1}^*(t) - Y_{d2}^*(t), \quad \text{say.}$$

This representation motivates the following notation as it is required in the subsequent lemma. Let $p_i := F_i(0)$, $q_i := 1 - p_i$ and define for $0 \le t \le 1$,

$$
\begin{aligned}
(15) \qquad L^+_{i1}(t) &:= \{F_i(J^{-1}(t)) - p_i\}/q_i, && q_i > 0,\\
&= 0, && q_i = 0;\\
L^+_{i2}(t) &:= \{p_i - F_i(-J^{-1}(t))\}/p_i, && p_i > 0,\\
&= 0, && p_i = 0;\ 1 \le i \le n.
\end{aligned}
$$

Observe that $\mu^+_i(v) = q_i L^+_{i1}(v) - p_i L^+_{i2}(v)$, $1 \le i \le n$. Also define

$$
\begin{aligned}
(16) \qquad L^+_i(t) &:= q_i L^+_{i1}(t) + p_i L^+_{i2}(t) = P(|X_i| \le J^{-1}(t)), && 1 \le i \le n,\\
L^+_{d1}(t) &:= \Sigma_i\, d_i^2\, q_i\, L^+_{i1}(t),\ \ L^+_{i1}(t) := \Sigma_i\, d_i^2\, p_i\, L^+_{i2}(t), && 0 \le t \le 1.
\end{aligned}
$$

Argue as for the proof of Lemma 2.2a.2 and use the triangle and the Chebychev inequalitites to conclude

Lemma 3.4.4. *For every* $\epsilon > 0$ *and* $0 < v < 1$ *fixed,*

$$
\begin{aligned}
&P\Big(\sup_{|t-v|<\delta} |Y^*_{dj}(t) - Y^*_{dj}(v)| > \epsilon\Big)\\
(17) \qquad &\le (\kappa + 20)\epsilon^{-2}\,[L^+_{dj}(v+\delta) - L^+_{dj}(v-\delta)], \qquad j = 1, 2
\end{aligned}
$$

where κ *does not depend on* ϵ, δ *or any other underlying quantities.* □

Theorem 3.4.2. *Let* $X_{n1}, \ldots, X_{nn}$ *be independent r.v.'s with respective continuous d.f.'s* $F_{n1}, \ldots, F_{nn}$ *and* $d_{n1}, \ldots, d_{nn}$ *be real numbers. Assume that* $\{d_{ni}\}$ *satisfy* (N1), (N2). *In addition, assume the following.*

With $\{L^+_{dj}\}$ *as in* (16), *for* v *fixed in* (0, 1),

$$(18) \qquad \lim_{\delta\to 0} \limsup_n |L^+_{dj}(v+\delta) - L^+_{dj}(v-\delta)| = 0, \qquad j = 1, 2.$$

(19) *There exist numbers* $\{\ell^+_{ij}(v),\ 1 \le i \le n;\ j = 1, 2\}$ *such that for all* $0 < k < \infty$, $j = 1, 2$,

$$\max_i \sup_{|t-v| \le kn^{-1/2}} n^{1/2} |L^+_{ij}(t) - L^+_{ij}(v) - (t-v)\ell^+_{ij}(v)| = o(1).$$

With

$$(20) \qquad \tilde{d}^+_n(v) := n^{-1}\Sigma_i\, d_{ni}\{q_i \ell^+_{i1}(v) - p_i \ell^+_{i2}(v)\},$$

$$\tau^2(v) := \Sigma_i \{d_{ni}^2[L_{ni}^+(v)-\{\mu_{ni}^+(v)\}^2] + (\tilde{d}_n^+(v))^2 L_{ni}^+(v)(1-L_{ni}^+(v)) - $$
$$- 2d_{ni}\,\tilde{d}_n^+(v)\,\mu_{ni}^+(v)\,(1-L_{ni}^+(v))\},$$

(21a) $$\liminf_n \tau^2(v) > 0.$$

(21b) $$\limsup_n n^{1/2}|\tilde{d}_n^+(v)| < \infty.$$

Then,

(22) $$\{\tau(v)\}^{-1}[Z_d^+(v) - \mu_d^+(v)] \xrightarrow[d]{} N(0, 1)$$

where μ_d^+ *is as in* (12).

Proof. The proof of this theorem is similar to that of Theorem 3.4.1 so we shall be brief. To begin with, by (13) and (**N2**) it suffices to prove that $\{\tau(v)\}^{-1}\,\tilde{T}_d^+(v) \xrightarrow[d]{} N(0, 1)$, where $\tilde{T}_d^+(v) := \mathcal{T}_d^+(v) - \mu_d^+(v)$.

Apply Lemma 3.4.1 above to the r.v.'s $|X_{n1}|, \ldots, |X_{nn}|$, to conclude that

$$\sup_{0\le t\le 1} |J(J_n^{-1}(t)) - t| = o_p(1).$$

From this, (14), (17) and (18),

$$\tilde{T}_d^+(v) = Y_d^+(JJ_n^{-1}(v)) + \mu_d^+(JJ_n^{-1}(v)) - \mu_d^+(v).$$
$$= Y_d^+(v) + [\mu_d^+(JJ_n^{-1}(v)) - \mu_d^+(v)] + o_p(1).$$

Again, apply arguments like those that yielded (9) to $\{|X_{ni}|\}$ to obtain

$$n^{1/2}[JJ_n^{-1}(v) - v] = -Y_1^*(v) + o_p(1),$$

where $Y_1^*(v)$ is as in (3.3.19) with $t = v$ and $\mathbf{u} = \mathbf{0}$. Consequently,

$$\tilde{T}_d^+(v) = Y_d^+(v) - n^{1/2}\tilde{d}_n^+(v)Y_1^*(v) + o_p(1) = K_d^+(v) + o_p(1)$$

where

$$K_d^+(v) = Y_d^+(v) - n^{1/2}\tilde{d}_n^+(v)\,Y_1^*(v)$$
$$= \Sigma_i \{d_{ni}[I(J(|X_{ni}|) \le v)\,s(X_{ni}) - \mu_{ni}^+(v)]$$
$$- \tilde{d}_n^+(v)[I(J(|X_{ni}|) \le v) - L_{ni}^+(v)]\}.$$

Note that $\mathrm{Var}(K_d^+(v)) = \tau^2(v)$. The proof of the theorem is now completed by using the L-F CLT which is justified, in view of (N1), (N2), and (21a). □

Remark 3.4.3. Observe that if $\{F_i\}$ are symmetric about 0 then $\mu_i^+ \equiv 0 \equiv \tilde{d}_n^+$ and $\tau^2(v) = \Sigma_i\, d_{ni}^2\, L_{ni}^+(v)$. □

Remark 3.4.4. An alternative proof of (22), using the techniques of Dupač and Hájek (op. cit.), appears in Koul and Staudte, Jr. (1972a). Thus comments like those in Remark 3.4.1 are appropriate here also. □

Next, we turn to the ***weak convergence*** of $\{Z_d\}$ and $\{Z_d^+\}$. These results will be stated without proofs as their proofs are consequences of the results of the previous sections in this chapter.

Therorem 3.4.3. (***Weak convergence of*** Z_d). *Let* $X_{n1}, \ldots, X_{nn}$ *be independent r.v.'s with respective continuous d.f.'s* $F_{n1}, \ldots, F_{nn}$. *With notation as in* (2.2a.33), *assume that* (N1), (N2), (C*) *hold. In addition assume the following:*

(23) *There are measurable functions* $\{\ell_{ni},\ 1 \le i \le n\}$ *on* $[0, 1]$, *such that for all* $0 < k < \infty$,

$$\max_i \sup_{|t-s| \le kn^{-1/2}} n^{1/2} |L_{ni}(t) - L_{ni}(s) - (t-s)\ell_{ni}(s)| = 0$$

Moreover, assume that

$$\text{(24)} \qquad \limsup_n \sup_{0\le t\le 1} n^{1/2} |\tilde{d}_n(t)| < \infty,$$

$$\text{(25)} \qquad \lim_{\delta\to 0} \limsup_n \sup_{|t-s|<\delta} n^{1/2} |\tilde{d}_n(t) - \tilde{d}_n(s)| = 0,$$

$$\text{(26)} \qquad \liminf_n \sigma^2(t) > 0, \quad 0 < t < 1.$$

Finally, with $K_d(t) := \Sigma_i\,(d_{ni} - \tilde{d}_n(t))\{I(X_{ni} \le H^{-1}(t)) - L_{ni}(t)\}$, *assume that*

$$\text{(27)} \qquad \mathcal{C}(t, s) = \lim_n \mathrm{Cov}(K_d(t), K_d(s))$$
$$= \lim_n \Sigma_i\,(d_{ni} - \tilde{d}_n(t))(d_{ni} - \tilde{d}_n(s)) L_{ni}(s)(1 - L_{ni}(t)),$$

exists for all $0 \le s \le t \le 1$.

Then, $Z_d - \mu_d \Rightarrow$ *to a mean zero, covariance* $\mathcal{C}$ *continuous Gaussian process on* $[0, 1]$, *tied down at* 0 *and* 1. □

Remark 3.4.5. In (23), without loss of generality it may be assumed that $n^{-1}\Sigma_i \ell_{ni}(s) = 1,\ 0 \le s \le 1$. For, if (23) holds for some $\{\ell_{ni},\ 1 \le i \le n\}$, then it also holds for $\{\ell^*_{ni},\ 1 \le i \le n\}$, $\ell^*_{ni}(s) := n^{1/2}[L_{ni}(s+n^{-1/2}) - L_{ni}(s)]$, $1 \le i \le n,\ 0 \le s \le 1$. Because $n^{-1}\Sigma_i L_{ni}(s) \equiv s,\ n^{-1}\Sigma_i \ell^*_{ni}(s) \equiv 1$. □

Remark 3.4.6. Conditions (C*), (N1) and (24) may be replaced by the condition (B), because, in view of the previous remark,

$$n^{1/2}|\tilde{d}_n(t)| = |n^{-1/2}\,\Sigma_i\, d_{ni}\ell_{ni}(t)| \le n^{1/2}\max_i |d_{ni}|,\ 0 \le t \le 1. \qquad \square$$

Remark 3.4.7. In the case F_{ni} have density f_{ni}, one can choose $\ell_{ni} = f_{ni}(H^{-1})/n^{-1}\Sigma_j f_{nj}(H^{-1}),\ 1 \le i \le n$. □

Remark 3.4.8. In the case $F_{ni} \equiv F$, F a continuous and strictly increasing d.f., $L_{ni}(t) \equiv t$, $\ell_{ni}(t) \equiv 1$, so that (C*) and (23) – (26) are trivially satisfied. Moreover, $\mathcal{C}(s,t)$, $= s(1-t),\ 0 \le s \le t \le 1$, so that (27) is satisfied. Thus Theorem 3.4.3 includes Theorem V.3.5.1 of Hájek and Sidák (1967). □

Theorem 3.4.4. (*Weak convergence of* Z_d^+). *Let* $X_{n1}, \dots, X_{nn}$ *be independent* r.v.'s *with respective* d.f.'s $F_{n1}, \dots, F_{nn}$ *and let* $d_{n1}, \dots, d_{nn}$ *be real numbers. Assume that* (N1) *and* (N2) *hold and that the following hold.*

(28) *With* L^+_{dj} *as in* (16),

$$\lim_{\delta\to 0} \limsup_n \sup_{0\le t\le 1-\delta} [L^+_{dj}(t+\delta) - L^+_{dj}(t)] = 0, \qquad j = 1, 2.$$

(29) *There are measurable functions* $\ell^+_{ij},\ 1 \le i \le n,\ j = 1, 2$ *on* $[0, 1]$ *such that for any* $0 < k < \infty$,

$$\max_i \sup_{|t-s|\le kn^{-1/2}} n^{1/2}|L^+_{ij}(t) - L^+_{ij}(s) - (t-s)\ell^+_{ij}(s)| = o(1).$$

(30) *With* $\tilde{d}^+_n$ *as in* (20),

$$\limsup_n \sup_{0\le t\le 1} n^{1/2}|\tilde{d}^+_n(t)| < \infty,$$

$$(31) \qquad \lim_{\delta\to 0} \limsup_n \sup_{|t-s|<\delta} n^{1/2}|\tilde{d}^+_n(t) - \tilde{d}^+_n(s)| = 0.$$

(32) *With* τ^2 *as in* (20),

$$\liminf_n \tau^2(t) > 0, \qquad 0 < t < 1.$$

(33) *With* $K_d^+(t)$ *as in the proof of Theorem* 3.4.2,

$$\lim_n \operatorname{Cov}(K_d^+(s), K_d^+(t)) = C^+(s,t) \text{ exists}, \qquad 0 \le s \le t \le 1.$$

Then, $Z_d^+ - \mu_d^+ \Rightarrow$ *to a continuous mean zero covariance* C^+ *Gaussian process, tied down at* 0. □

Remark 3.4.9. Remarks 3.4.5 through 3.4.7 are applicable here also, with appropriate modifications. □

Remark 3.4.10. Suppose that $F_{ni} \equiv F$, F continuous, and $d_{ni} \equiv n^{-1/2}$. Then

$$\sup_{0 \le t \le 1} |Z_d^+(t) - \mu_d^+(t)| = \sup_{0<x<\infty} n^{1/2} |\{H_n(x) - H_n(0)\} - \{H_n(0) - H_n(-x)\} - \{F(x) - F(0)\} - \{F(0) - F(-x)\}|$$

which is precisely the statistic τ_n^* proposed by Smirnov (1947) to test the hypothesis of symmetry about F. Smirnov considered only the null distribution. Theorem 3.4.4 allows one to study its asymptotic distribution under fairly general independent alternatives.

If $\{d_{ni}\}$ are arbitrary, subject to (**N1**) and (**N2**), then $\sup\{|Z_d^+(t) - \mu_d^+(t)|;\ 0 \le t \le 1\}$ may be considered a generalized Smirnov statistic for testing the hypothesis of symmetry. □□