Stochastic Orders and Decision under Risk
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STOCHASTIC ORDERS IN WELFARE ECONOMICS

BY MICHEL LE BRETON

GREQE

The purpose of this paper is to show why and how stochastic dominance type results are useful for setting and solving problems arising in welfare economics.

1. Introduction. One of the main economic problems that face a society is to determine what is the "best" way to distribute the total income between the members of the society. A welfarist approach to this question consists in deriving principles which can be used for ranking the distributions according to the "social welfare" they generate; we get an order on the income distributions. Social welfare depends in some way on the individual welfares. If some information is missing on individual welfares we just get a partial order on income distributions. Then it remains to develop measurement theory compatible with the partial order to perform empirical evaluations.

The present paper does not intend to be a comprehensive survey of the literature on the topic discussed above. Our main objective is to show in which way the formulation of the problem leads inexorably from a formal point of view to the derivation of a stochastic order (we shall not always present welfare orders as stochastic orders, but the reader will easily see that it is the case). In that respect, Section 2 of the paper is crucial since it collects the basic ingredients for a general statement of the problem. Our paper must be considered as a very introductory survey on the derivation of stochastic orders in the welfare economics of distribution problems. The reader who wants to have a more thorough view on the themes developed here can consult former monographs by Cowell (1977), Kakwani (1980), Nygard and Sandström (1981) and Sen (1976). There is some difficulty in writing the history of the subject. It seems however that Kolm (1967, 1969) was the first to realize the usefulness of stochastic dominance type results for financial and welfare economics. This connection was also discovered independently by Atkinson (1970).

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The paper is organized as follows. In Section 2 we give a full statement of the problem under consideration. In Section 3 we present some elements of the welfare and inequality measurement theory for the basic distribution problem. We conclude in Section 4 by providing complements and extensions.

2. Statement of the Problem. One of the main purposes of welfare economics is to elaborate tools for comparing different ways of solving a distribution problem.

From a formal point of view a distribution problem is defined by the following ingredients:

- A population of individuals described by a set N. Most of the time this set will be finite but we shall also consider the infinite case as a limit situation.
- A set of consequences describing for each individual the consequences of the distributions to him. We assume that this set is the same for all the individuals. The word "consequence" may cover many different situations: it may be a sum of money, an endowment of divisible goods, or even a number of working hours. We denote the set of consequences by C.

A distribution is an application from N to C. Consequently, the whole set of distributions is the set C^N. This set is denoted shortly by X. Given x in X and $i \in N$ $x(i)$ or x_i denotes the consequence of distribution x for i. Sometimes we restrict attention to proper subsets of X. In what follows we shall denote by X^* the subset of distributions under consideration.

Let us now illustrate by means of some examples the above abstract formulation of a distribution problem.

The first example is the basic distribution problem considered by almost all the authors in the area of inequality measurement.

EXAMPLE 1. There is a finite number of individuals, say n, thus for some labeling we have $N = \{1, \cdots, n\}$. The set of consequences C is $\mathbb{R}_+$. Here a consequence will be an income level. The set of distributions is $\mathbb{R}_+^n$. A distribution specifies the income level of each individual. We may restrict attention to the set of distributions corresponding to a fixed total income, say K. In that case X^* is the simplex $\{x \in \mathbb{R}_+^n : \sum_{i=1}^n x_i = K\}$.

EXAMPLE 2. We also assume $N = \{1, \cdots, n\}$ and we consider n different indivisible goods also labeled from 1 to n. The set of consequences C is here $\{1, \ldots, n\}$, i.e., a distribution specifies which type of good each individual receives. The set of distributions is $\{1, \cdots, n\}^n$. If we assume that there is one and only one good of each type then we restrict attention to the subset X^* of permutations over $\{1, \cdots, n\}$.

The first two examples were unidimensional. In the next example we provide a multidimensional version of this distribution problem.

EXAMPLE 3. We always consider $N = \{1, \cdots, n\}$ but now we assume that each individual receives some quantity of m different divisible goods. The set C is $\mathbb{R}^m_+$. The set of distributions is the set $\mathbb{R}^{nm}_+$: we may identify a distribution with a matrix $x = (x_{ij})$ $i = 1, \cdots, n$; $j = 1, \cdots, m$; x_{ij} being the quantity of good j received by individual i. As before, if we assume that there is a fixed amount of good j, say K^j, to be distributed then we restrict our attention to the subset X^* of matrices x satisfying $\sum_{i=1}^n x_{ij} = K^j$ for all $j = 1, \cdots, m$.

The distribution problem described as Example 3 is the classical problem in microeconomic theory if we consider all the goods produced in the economy. We may also include tasks or activities in the list of goods.

In the last example we consider an infinite number of individuals. This situation is unrealistic but can be considered as the limit of arbitrary large finite populations.

EXAMPLE 4. There is a continuum of individuals described by the interval $[0,1]$, i.e. $N = [0,1]$. As in Example 1, we consider $C = \mathbb{R}_+$ and thus $X = \mathbb{R}_+^{[0,1]}$. In fact, we limit our attention to distributions which are Lebesgue integrable, i.e. we consider X^* being the set $\mathbb{L}^1(0,1)$. Sometimes we even restrict our attention to distributions which are essentially bounded, i.e., X^* becomes the set $\mathbb{L}^\infty(0,1)$.

We end the list of examples here, but it is clear that we can find many other types of distributions problems by considering mixtures of the above problems.

Given a distribution problem and, and thus a set X^*, welfare economics addresses the following question: how to rank the different elements of X^* according to welfare. To formulate the issue we have first to precise how each individual evaluates the different consequences and thus the different distributions (assuming that each individual is selfish, i.e., just concerned with the consequences of the distribution relevant for himself.)

We assume that the welfare of individual i over C is described by a numerical function v_i defined over C (v_i is the utility function of i). Given x in X^* and keeping in mind the selfishness assumption, the welfare level of individual i in X is $v_i(x_i)$. By V we denote the set of utility functions over C. A profile of utility functions is the specification for each individual of a utility funciton, i.e., formally an element of the product space V^N, denoted shortly by U. Given x in X^* and u in U, we generate a profile w of utility levels in $\mathbb{R}^N$ according to the formula $w_i = v_i(x_i)$.

The problem we face now is very simple: given a profile of individual welfare levels, how to aggregate these numbers to define a social welfare level. A social welfare function is a numerical function W defined over $\mathbb{R}^N$ reflecting the way chosen by the social decision maker to solve the problem stated above. We shall not discuss this difficult subject and refer the reader to Hammond (1976) and Sen (1973) for a concise analysis of the interpersonal comparison of utilities in the welfare economics of a distribution problem.

Given W and a profile u of utility functions, the social decision maker obtains a complete ranking of the elements in X^*. Unfortunately, to do that the social decision maker needs to know or specify the profile u of utility functions. In general, he has only partial information (or makes some assumptions) on it. We shall assume that this partial knowledge is represented by a subset U^* of U, i.e. he only knows (or assumes) that the profile of utility functions is within U^*.

Given W and U^* we define a binary relation $\gtrsim_{W,U^*}$ over X^* as follows. Let $x, y \in X^*$; then

$$x \gtrsim_{W,U^*} y \text{ iff } W(u(x)) \text{ for all } u \in U^*. \tag{1}$$

It is easy to see that $\gtrsim_{W,U^*}$ is a preorder, i.e., is reflexive and transitive. We denote by $>_{W,U^*}$ (resp. $\sim_{W,U^*}$) its asymmetric (resp. symmetric) component. In order to have more information on the social welfare preorder $\gtrsim_{W,U^*}$ we need to make some assumptions on W and U^*.

To make the notations and arguments easier, I will consider in the remainder of this section the distribution problem presented in Example 1 (the careful reader will convince himself that most of the arguments can be extended to more general problems).

The following two assumptions are common to almost all the papers in the area and will be kept (unless otherwise stated) through the paper.

Assumption 1. For all $w = (w_1, \cdots, w_n) \in \mathbb{R}^n$:

$$W(w_1, \cdots, w_n) = \sum_{i=1}^{n} w_i.$$

Assumption 1 means that the social welfare function considered is the utilitarian one, i.e., the social welfare is the sum of individual welfares.

Assumption 2. U^* is a subset of the diagonal of U, i.e., for every $u = (v_1, \cdots, v_n) \in U^*$ we have $v_i = v_j \ \forall i, j$.

Assumption 2 means that the individuals are assumed to be identical. This assumption is very demanding and its relaxation will be discussed in Section 4.

From Assumption 2 we deduce that there exists a subset V^* of V such that $u = (v_1, \cdots, v_n) \in U^*$ if and only if for some v in V^*, $v_i = v$, $\forall i$. Under the above two assumptions the preorder $\succsim_{W,U^*}$ defined in (1) will be denoted shortly by $\succsim_{V^*}$ and is defined by: Let $x, y \in \mathbb{R}_+^n$

$$x \succsim_{V^*} y \text{ iff } \sum_{i=1}^{n} v(x_i) \geq \sum_{i=1}^{n} v(y_i) \quad \forall v \in V^*. \tag{2}$$

We observe that the social welfare preorder $\succsim_{V^*}$ is symmetric, i.e., for all $x \in \mathbb{R}_+^n$ and all permutations σ over $\{1, \cdots, n\}$ $x \sim_{V^*} (x_{\sigma(1)}, \cdots, x_{\sigma(n)})$. Equivalently if we denote by P_x the probability measure $\frac{1}{n}\sum_{i=1}^{n} \delta_{x_i}$ where δ_{x_i} is the Dirac mass in x_i, we have $P_x = P_y \Rightarrow x \sim_{V^*} y$. In fact (2) can alternatively be formulated as follows:

$$x \succsim_{V^*} y \text{ iff } \int_{\mathbf{R}} v(t) P_x(dt) \geq \int_{\mathbf{R}} v(t) P_y(dt), \quad \forall v \in V^*. \tag{3}$$

The above expression illustrates the strong connection between the problem stated here and the questions considered in the stochastic dominance area. To make the connection even stronger, we need to extend the framework a little bit.

Until now we have considered social welfare preorders defined over $\mathbb{R}_+^n$, i.e., we have compared distributions involving the same number of individuals. We may want to compare distributions involving different and arbitrary population sizes in which case the social welfare preorder $\succsim$ is defined over $\bigcup_{n \geq 1} \mathbb{R}_+^n$. Given x in $\mathbb{R}_+^n$ and r in $\mathbb{N}$, $x^{(r)}$ will denote the vector in $\mathbb{R}_+^{nr}$ defined as $(x_1, \cdots, x_1, \cdots, x_n, \cdots, x_n)$, i.e., where each component x_i in x has been replicated r times. We shall say that $\succsim$ satisfies Dalton's Population Principle if for all $x \in \bigcup_{n \geq 1} \mathbb{R}_+^n$ and all $r \in \mathbb{N}$ $x^{(r)} \sim x$. It is very easy to see that if $\succsim$ is symmetric and satisfies Dalton's Population Principle, then for all x, $y \in \bigcup_{n \geq 1} \mathbb{R}_+^n$, $P_x = P_y \Rightarrow x \sim y$. Consequently if we consider a social welfare preorder satisfying the above two properties, we can reduce it to a stochastic preorder. Roughly speaking, what really matters in this case when comparing two distributions is the proportion of individuals at each income level. The preorder $\succsim_{V^*}$ is of this type if in Assumption 1 we consider mean welfare instead of total welfare.

It is very useful to realize that the social welfare preorder is of the stochastic dominance type, but the reader must keep in mind that this connection has been established under some specific assumptions. Anyway, this connection will be useful, and we conclude this section by recalling two standard results in the area. First observe that the preorder $\precsim_{V^*}$ defined in (3) is meaningful for arbitrary bounded (i.e. with compact support) probability measures if we assume that functions in V^* are bounded over bounded subsets of $\mathbb{R}$. The two results below provide alternative characterizations of $\succsim_{V^*}$ when V^* is the set of increasing and concave functions over $\mathbb{R}$.

Given a probability measure P over $\mathbb{R}$ we denote by F_p its distribution function.

THEOREM 1. *Let P and Q be two bounded probability measures. Then*

$$P \succsim_{V^*} Q \text{ iff } \int_0^z F_P(t)dt) \leq \int_0^z F_Q(t)dt \qquad \forall z \in \mathbb{R}.$$

To state the next result, let us recall that given a probability measure P over $\mathbb{R}$ there exists a unique (up to the equivalence almost everywhere) random variable over $[0,1]$ which is increasing, right continuous and has P as probability law. This is the so-called right-continuous inverse of F_P, and it is denoted by F_P^{-1}.

THEOREM 2. *Let P and Q be two bounded probability measures over $\mathbb{R}$. Then*

$$P \succsim_{V^*} Q \text{ iff } \int_0^p F_P^{-1}(t)(dt) \geq \int_0^p F_Q^{-1}(t)(dt) \qquad \forall p \in [0,1].$$

The second result is less known than Theorem 1. A proof has been given by Atkinson (1970) for specific cases. A general proof is provided in Le Breton (1986). We refer the reader to Mosler and Scarsini (1991) for an overview of the state of the art in the stochastic dominance area.

3. Inequality Measurement and Theory of Majorization. We have already precised that most of the attention of economists in the area concentrated on the distribution problem presented in Example 1. The purpose of this section is to provide a detailed analysis of this problem according to the principles discussed in Section 2.

We shall consider $X^* = \mathbb{R}^n_{++}$, i.e., we don't consider income distributions where some individuals receive nothing. Given x in $\mathbb{R}^n_{++}$ we denote by $\mu(x)$ the mean of x and by $\tilde{x}$ the increasing rearrangement of X. Consider the following three preorders over $\mathbb{R}^n_{++}$. Let $x, y \in \mathbb{R}^n_{++}$.

$$x \succsim_{GL} y \text{ iff } \sum_{i=1}^{k} \tilde{x}_i \geq \sum_{i=1}^{k} \tilde{y}_i \ \forall\, k = 1, \cdots, n \tag{4}$$

$$x \succsim_{L} y \text{ iff } x \succsim_{GL} y \text{ and } \mu(x) = \mu(y) \tag{5}$$

$$x \succsim_{LR} y \text{ iff } \frac{x}{\mu(x)} \succsim_{GL} \frac{y}{\mu(y)}. \tag{6}$$

The preorder defined in (4) has a direct economic interpretation. Given an income distribution, we order the incomes from the lower to the greater, and we calculate the total income possessed by the poorest, the two poorest, and so on. The preorder $\succsim_{GL}$ declares x better than y if the vector of cumulated incomes corresponding to x is greater than the vector of cumulated incomes corresponding to y.

We deduce the following result directly from Theorem 2.

THEOREM 3. *Let $x, y \in \mathbb{R}^n_{++}$ and V^* be the set of increasing and concave numerical functions over $\mathbb{R}_+$. Then*

$$x \succsim_{GL} y \text{ iff } x \succsim_{V^*} y.$$

Thus the preorder $\succsim_{GL}$ coincides with the social welfare preorder generated by the class of increasing and concave utility functions, i.e. when the social decision maker just assumes that utility (resp. marginal utility) is increasing (resp. decreasing) with income. A direct proof of Theorem 3 can be found in Marshall and Olkin (1979). This result has been rediscovered in economics by Shorrocks (1983), and we use the subscript GL for "Generalized Lorenz," a terminology introduced by this author.

If a social decision maker considers the class V^* of utility functions defined in Theorem 3 as a social welfare measure, he must choose a numerical function which preserves the preorder $\succsim_{GL}$ in the strict sense. Then a social welfare measure will be a continuous, strictly increasing and strictly Schur-concave numerical function over $\mathbb{R}^n_{++}$. It is easy to show that it will be symmetric.

Until now we have been concerned with the measurement of social welfare. Now we are going to see how this question is related to the measurement of inequality. Intuitively, the social welfare generated by a distribution x depends on two components: the size of the income to be shared and its distribution between the individuals. Inequality measurement is just concerned with the second component. If we keep in mind the relation with social welfare, we get the following interpretation. Given x and y in $\mathbb{R}^n_{++}$ we shall say that the inequality has been reduced going from x to y iff $\mu(x) = \mu(y)$ and $W(x) \geq W(y)$.

Roughly speaking, inequality measurement corresponds (up to change of sign) to welfare measurement when we restrict to a simplex of income distributions. Otherwise stated, there is no general principle for guiding inequality measurement when we have to compare on welfare grounds income distributions with different means. In that respect, the preorder $\underset{\sim}{>}_L$ (L for Lorenz) defined in (5) is the relevant preorder for inequality measurement. This suggests the following definition.

DEFINITION 1. An inequality measure over $\mathbb{R}^n_{++}$ is a numerical function which is continuous and strictly Schur-convex.

The continuity requirement is introduced for eliminating measures which make differences between arbitrarily close distributions. The strict-Schur convexity requirement is the order-preserving requirement with respect to $\underset{\sim}{>}_L$. It is easy to see that an inequality measure will be symmetric. We denote by I_n the set of inequality measures over $\mathbb{R}^n_{++}$.

In order to provide an alternative and meaningful interpretation of the preorer $\underset{\sim}{>}_L$ we introduce:

DEFINITION 2. Let $x, y \in \mathbb{R}^n_{++}$ and assume that $y_i \leq y_j$. We say that x is deduced from y by a Pigou-Dalton transfer between i and j if there exists t such that $0 \leq t \leq y_j - y_i$, $x_i = y_j + t$, $y_i = y_j - t$ and $y_k = x_k$ for $k \neq i, j$.

A Pigou-Dalton transfer is thus a transfer from some individual to an individual with a lower income. The following result, which is part of a theorem by Hardy, Littlewood and Polya (1934), has been rediscovered in economics by Dasgupta, Sen and Starrett (1973).

THEOREM 4. *Let $x, y \in \mathbb{R}^n_{++}$. Then $x \underset{\sim}{>}_L y$ if x is obtained from y by a finite sequence of Pigou-Dalton transfers.*

As shown by Fei and Fields (1978) a suitable version of Theorem 4 can be obtained even if we further assume the transfers to be rank-preserving. Theorem 4 is very interesting since it shows that the strict Schur-convexity requirement can be equivalently formulated as a principle of transfers. We stop the analysis of $\underset{\sim}{>}_L$ here, and we refer the reader to Marshall and Olkin (1979) for further equivalent expressions of that preorder.

In Definition 1 we have introduced inequality measures. Most of the time we shall not be interested in the inequality measure itself but in the ranking over $\mathbb{R}^n_{++}$ generated by the inequality measure. This leads to the following definition.

DEFINITION 3. An inequality preorder over $\mathbb{R}^n_{++}$ is a complete preorder $\underset{\sim}{>}$ over $\mathbb{R}^n_{++}$ which is continuous (i.e. the sets $\{y \in \mathbb{R}^n_{++} : y > x\}$ and $\{y \in$

$\mathbb{R}^n_{++} : x > y\}$ are open for all $x \in \mathbb{R}^n_{++}$) and strictly Schur-convex (i.e. for all $x, y \in \mathbb{R}^n_{++} x >_L y \Rightarrow x > y$).

We denote by $\mathcal{P}^i_n$ the set of inequality preorders over $\mathbb{R}^n_{++}$.

An inequality measure as defined has no implications when comparing distributions with different means. Many authors think that this is not sufficiently restrictive and propose to introduce further axioms for comparing distributions with different means. To this effect they normalize income distributions to avoid the difficulty discussed above, and they apply to normalized income distributions the principles previously defined. There are different ways for rescaling income distributions. The most popular method consists in dividing every component of the income distribution by the mean income. A relative inequality measure over $\mathbb{R}^n_{++}$ is an inequality measure over $\mathbb{R}^n_{++}$ which is homogeneous of degree 0. It is easy to see that equivalently a relative inequality measure is a continuous numerical function over $\mathbb{R}^n_{++}$ which preserves the preorder $\gtrsim_{LR}$ (LR for Lorenz Relative) in the strict sense.

We refer the reader to Kolm (1976) for a discussion of the implications of the above normalization. In particular he discusses another normalization he calls the absolute one. More recently some authors like Pfingsten (1986) proposed intermediate normalizations.

To illustrate the above developments, let us consider a standard construction initiated by Atkinson (1970). Consider a social welfare measure W over $\mathbb{R}^n_{++}$. Assume that, given x in $\mathbb{R}^n_{++}$ there exists a unique positive real number $e(x)$ such that $W(x) = W((e(x), \cdots, e(x)))$. Define I over $\mathbb{R}^n_{++}$ by $I(x) + 1 - (e(x)/\mu(x))$. It is easy to see that I is an inequality measure. However, it is not relative unless that W is homogeneous of degree 1.

This construction and more generally the connections between welfare and inequality measurement have received attention by many authors (see among others Blackorby and Donaldson (1978) (1980), Dutta and Esteban (1988), Ebert (1987), and Trannoy (1987)).

3.1. Structure of $\mathcal{I}_n$ and $\mathcal{P}^i_n$. The sets $\mathcal{I}_n$ and $\mathcal{P}^i_n$ defined above are relatively large (the introduction of a scale invariance property does not alter this qualitative feature). It is necessary to explore the structure of these sets at different levels. We shall concentrate our attention on $\mathcal{P}^i_n$.

One way to evaluate the "richness" of the set $\mathcal{P}^i_n$ is to have some idea about the differences of elements in the set. Imagine two experts having their respective inequality preorders: they will rank two distributions in the same way if the distributions are Lorenz ordered but will in general disagree otherwise. This raises the following question; is it possible to aggregate their opinions in a satisfactory way? Of course the answer depends on what we mean by satisfactory aggregation.

Consider a triple (Y, m, D) where Y is an arbitrary set, m is an integer, and D is a subset of the set of complete preorders over Y. An aggregation rule for the triple (Y, m, D) is a mapping ψ from D^n into D. Given $\succsim$ in D and $A \subset Y$ we denote by $\succsim|_A$ the restriction of $\succsim$ to A. The elements of D^n are denoted by $[\succsim]$ and the image of $[\succsim]$ by ψ, i.e. $\psi([\succsim])$ is denoted by $\succsim$.

The following definition states a standard concept of aggregation for that type of problem.

DEFINITION 4. Let (Y, m, D) be a triple as defined above. An aggregation ψ rule for (Y, m, D) is of the Arrow-type if it satisfies the following properties

(1) $\forall[\succsim], [\succsim'] \in D^m$, $\quad \forall x, y \in Y$, if $\succsim_i |_{\{x,y\}} = \succsim'_i |_{\{x,y\}}$ for all $i = 1, \cdots, m$, then $\succsim |_{\{x,y\}} = \succsim' |_{\{x,y\}}$,

(2) $\forall[\succsim] \in D^m$, $\forall x, y \in Y$, if $x >_i y$ for all $i = 1, \cdots, m$ then $x > y$,

(3) There is no j in $\{1, \cdots, m\}$ such that $\forall[\succsim] \in D^m$, $\forall x, y \in Y$, if $x >_j y$, then $x > y$.

We refer the reader to the pioneering work of Arrow (1963) for a discussion on the three axioms in Definition 4. The following result proved by Le Breton and Trannoy (1987) gives a negative answer to the question raised earlier.

THEOREM 5. *For every $m \geq 2$ there is no Arrow-type aggregation rules for the triple $(\mathbb{R}^n_{++}, \mathcal{P}^*_n, m)$.*

3.2. More on Inequality Measurement. We have just seen that the sets of inequality measures and preorders are rather large, and we would like to design a small family of inequality measures (and, if possible, just one) which would be superior in some sense. To this effect we need to consider further properties in addition to the two appearing in Definition 1. Many current inequality measures have been axiomatized, as for instance the Theil measure (Foster (1983)) and the Gini measure (Thon (1982), Trannoy (1986)). We must also remark that the Gini measure has been extended in some directions (see among others Donaldson and Weymark (1980), Weymark (1981), and Yitzhaki (1983)).

Among the additional properties we may require, we would like to insist on one of them called the decomposability property. Since this property concerns the sensitivity of the inequality measure to the decomposition of the whole population into subpopulations, we have to consider inequality measures defined over $\bigcup_{n \geq 1} \mathbb{R}^n_{++}$. We have already discussed this issue in Section

1. We shall say that an inequality measure I over $\bigcup_{n\geq 1} \mathbb{R}^n_{++}$ satisfies the Dalton population principle if for all $n \geq 1$, $x \in \mathbb{R}^n_{++}$ and $r \geq 1$ $I(x^{(r)}) = I(x)$. The decomposability property is also a property related to the variable population framework. If we divide the whole population N into two subgroups N_1 and N_2, can we relate the inequality in the whole population to the inequality within each subgroup and to the inequality between the subgroups? Roughly speaking, an inequality measure will be said to be decomposable if the overall inequality can be expressed as a function of the within group inequalities, the group means, and group sizes. Many authors (Bourguignon (1979), Cowell (1980), Cowell and Kuga (1981a) (1981b), Russell (1985), Shorrocks (1980)) have explored this question. A major contribution of Shorrocks (1984) is to show that this general form of decomposability could be expressed up to some increasing transformation in an additive way. Precisely he proves that if I is a decomposable inequality measure over $\bigcup_{n\geq 1} \mathbb{R}^n_{++}$ such that $I(\bar{x}) = 0$ for all $x \in \mathbb{R}^n$ ($\bar{x}$ being the vector in $\mathbb{R}^n_{++}$ with all components equal to $\mu(x)$), then there exists another inequality measure J over $\bigcup_{n\geq 1} \mathbb{R}^n_{++}$ such that $J(x) = F(I(x), \mu(x), n(x))$ ($n(x)$ being the dimension of x) with $F(I, \mu, n)$ continuous and strictly increasing in I, $F(0, \mu, n) = 0$ for all μ, n, and J satisfying

$$J(x, y) = J(x) + J(y) + J(\bar{x}, \bar{y}) \text{ for all } x \in \mathbb{R}^n_{++} \text{ and all } y \in \mathbb{R}^m_{++}. \quad (7)$$

He then proves that the only solutions to the functional equation (7) are given by

$$J(x) = \sum_{i=1}^{n(x)} (\phi(x_i) - \phi(\mu(x))) \text{ where } \phi \text{ is a strictly convex function over } \mathbb{R}^n_{++}.$$

Further, he shows that if I satisfies the Dalton population principle, then the transformation F defined above is independent of n. Finally, he shows that if I is relative then F is independent of μ and there exists a parameter $c \in \mathbb{R}$ such that

$$J(x) = \begin{cases} \dfrac{1}{n}\dfrac{1}{c(c-1)}\displaystyle\sum_{i=1}^{n}\left(\left(\frac{x_i}{\mu}\right)^c - 1\right) & \text{if } c \neq 0, 1 \\ \dfrac{1}{n}\displaystyle\sum_{i=1}^{n}\frac{x_i}{\mu}\operatorname{Log}\left(\frac{x_i}{\mu}\right) & \text{if } c = 1 \\ \dfrac{1}{n}\displaystyle\sum_{i=1}^{n}\operatorname{Log}\left(\frac{\mu}{x_i}\right) & \text{if } c = 0 \end{cases}$$

We just get the generalized entropy family. We refer the reader to Shorrocks (1984) for further discussion of this important issue.

4. Extensions and Variations. The purpose of this last section is to give a quick and general idea on the recent lines of research in the area of welfare economics and inequality measurement.

4.1. Some Extensions. The first extension concerns the distribution problem presented as Example 4. This extension is of some interest since many authors use continuous distributions in empirical research. Le Breton (1988a, 1988b) gives an extension of the theory presented in Section 3 to this continuous framework. The connection with the stochastic dominance literature is quite easy to derive. Indeed under the natural version of Assumption 1 (precisely $w(u) = \int_0^1 u(t)dt$) we see that the only relevant information on distributions we need to make welfare comparisons is their probability laws over $\mathbb{R}_+$.

The second extension deals with the multidimensional distribtion problem presented as Example 3 of Section 2. This question has been explored, among others, by Atkinson and Bourguignon (1982) and Kolm (1977) and is of obvious economic interest. Suppose we want to compare two bounded bi-dimensional distributions, say F and G, whose supports are contained in $\mathbb{R}_+^2$ (i.e. in the notations of Example 3, $m = 2$). We denote by F_1 (resp. F_2) the distribution function of the first (resp. second) marginal of F. Here the set of consequences is $\mathbb{R}_+^2$; we introduce the class V^* of utility function defined by $V \in V^*$ iff V is class C^4 with $V_1, V_2 > 0$, $V_{12} < 0$, $V_{11}, V_{22} < 0$, $V_{112}, V_{122} > 0$ and $V_{1122} < 0$ in every point of $\mathbb{R}_+^2$ (the symbol V_{112} is used for $\partial^3/\partial^2 x_1 \partial x_2$ and so on).

The following result has been proved by Atkinson and Bourguignon (1982).

Theorem 6. *$F \succsim_{V^*} G$ iff*

$$\int_0^{x_1} F_1(s)ds \le \int_0^{x_1} G_1(s)ds \qquad \forall x_1 \in \mathbb{R}_+$$
$$\int_0^{x_2} F_2(s)ds \le \int_0^{x_2} G_2(s)ds \qquad \forall x_2 \in \mathbb{R}_+$$

and

$$\int_o^{x_1} \int_o^{x_2} F(s,t)ds\, dt \le \int_o^{x_1} \int_o^{x_2} G(s,t)ds\, ds \qquad \forall x_1, x_2 \in \mathbb{R}_+.$$

Theorem 6 is in the spirit of Theorem 1. However, the class of utility functions defined by V^* is difficult to interpret.

The last extension consists of exploring what happens if we drop Assumption 2 in Section 2, i.e., the identify of the individuals. If for instance we interpret the individuals as households we may wish to discriminate between households with respect to the family size. We may also argue that variables like age, sex, or health for instance affect utility in some way. Suppose for the sake of simplicity that individuals differ only by some characteristic, say age, and that their utility depends on their age and income. Formally, we have thus to compare bi-dimensional distributions as before. However, there is something new with respect to the bi-dimensional problem discussed before, in the sense that the first marginal is the same for all the distributions. We refer the reader to Atkinson and Bourguignon (1983) for the derivation of stochastic dominance results in this context and their applications.

4.2. Some Variations. Let us now go back to the distribution problem examined in Section 3. We explore the consequences of restricting the class of utility functions. However, instead of modifying this assumption in an ad hoc way, we are going to modify the transfer principle stated in Theorem 4.

This transfer principle is very weak since it only requires sensitivity of the inequality measure when we transfer an amount of income from an individual to an individual with lower income. Consider now a composite transfer involving four individuals 1, 2, 3, and 4 with incomes satisfying $x_1 < x_2 < x_3 < x_4$ and $x_2 - x_1 = x_4 - x_3$. Individual 4 receives t units from individual 3 and individual 1 receives t units from individual 2 ($t < (x_2 - x_1)/2$). The Pigou-Dalton transfer principle is useless in that situation. However, we may argue that the progressive transfer which occurs for lower incomes must be weighted more than the regressive one which occurs in the top and thus that the inequality has been reduced. Foster and Shorrocks (1987) introduce a new transfer principle which is a slight modification of the example above and prove that it is equivalent to the third degree stochastic dominance rule. It must be noted that their result is strongly connected to a deep and former result by Fishburn (1982). (For further discussion on the connections between stochastic orders and transfer principles we refer to Fishburn and Willig (1984).)

Some related ideas occur in the area of poverty measurement initiated by Sen (1976) and explored by Atkinson (1987), Donaldson and Weymark (1986), Foster and Shorrocks (1988a) (1988b), Kundu and Smith (1983) and Thon (1979) among others. In terms of transfer principle we require in that area that poverty is not affected by transfer between people whose incomes are higher than some level called the poverty line. In terms of distributions, this consists in concentrating the probability mass above the poverty line on this point. A unification of inequality and poverty measurement in the framework of stochastic dominance has been developed in Le Breton (1988c).

4.3. Some Consequences. It appears useful to conclude this paper by showing quickly how these ideas relate to some familiar ideas concerning tax systems. From a formal point of view, a tax system is an application G from $\mathbb{R}_+$ into $\mathbb{R}_+$: if the pre-tax income of an individual is x its post-tax income will be $G(x)$ (the tax paid is $x - G(x)$). A tax system G will be rank-preserving if $x < y$ implies $G(x) < G(y)$.

If we start from some income distribution $(x_1, \cdots, x_n)$ a tax system generates a new distribution $(G(x_1, \cdots, G(x_n))$. Remark that if the tax is not purely distributed then $\sum_{i=1}^n x_i$ is different from $\sum_{i=1}^n G(x_i)$ so in comparing pre-tax distributions with post-tax ones we shall need to choose a normalization rule (see the discussion in Section 3).

The following definition will be useful:

DEFINITION 5. A tax system G over $\mathbb{R}_{++}$ will be said progressive if $G(x)/x$ decreases with x.

The following result due to Jackobson (1976) has been concisely formulated by Eichhorn, Funke, and Richter (1984) and Thon (1987).

THEOREM 7. *$(G(x_1), G(x_2), \cdots, G(x_n))$ Lorenz relative dominates $(x_1, \cdots, x_n)$ for every $n \geq 2$ and every $(x_1, \cdots, x_n)$ in $\mathbb{R}_+^n$ iff G is progressive and rank preserving.*

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HOSPICE DE LA VIELLE CHARITÉ
2 RUE DE LA CHARITÉ
F-13002 MARSEILLE, FRANCE