BOUNDS FOR TAIL PROBABILITIES OF WEIGHTED SUMS OF INDEPENDENT GAMMA RANDOM VARIABLES

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The tail probabilities of two weighted sums of independent gamma random variables are compared when the first vector of weights majorizes the second vector of weights. The conjecture that the two cumulative distribution functions cross exactly once is established in four special cases by means of the variation-diminishing property of totally positive kernels. Bounds are obtained for the location of the unique crossing point and its asymptotic behavior is determined.

1. Introduction. In this paper we continue the study of tail probabilities of weighted sums of independent, identically distributed (i.i.d.) gamma random variables begun by Diaconis (1976) and extended by Bock, Diaconis, Huffer, and Perlman (1987) [hereafter abbreviated as BDHP (1987)].

Let $Y_1, \ldots, Y_n$ be i.i.d. gamma random variables with common probability density function (pdf)

$$g_{\alpha,\beta}(y) = [\beta^\alpha \Gamma(\alpha)]^{-1} y^{\alpha-1} e^{-y/\beta}, \quad 0 < y < \infty, \tag{1}$$

where $\alpha > 0$ and $\beta > 0$ denote the shape and scale parameters, respectively. We denote this gamma distribution by $G(\alpha, \beta)$. For nonnegative weights $\theta_1, \ldots \theta_n$, the tail probabilities of the weighted sum $\sum \theta_i Y_i$ are denoted as follows:

$$\begin{aligned} F_{\boldsymbol{\theta}}(t) &= P\left[\sum \theta_i Y_i \le t\right] \\ \bar{F}_{\boldsymbol{\theta}}(t) &= P\left[\sum \theta_i Y_i \ge t\right] = 1 - F_{\boldsymbol{\theta}}(t), \end{aligned} \tag{2}$$

where $\boldsymbol{\theta} = (\theta_1, \ldots \theta_n)$ and $0 \le t < \infty$.

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Such weighted sums arise in many contexts in statistics and probability, for example as the distribution of quadratic forms $X'AX$ where X is an n-dimensional normal random vector and A is an arbitrary $n \times n$ positive semidefinite matrix. Such a quadratic form occurs, for example, as the limiting distribution of the chi-squared goodness-of-fit statistic when parameters are estimated on the basis of the ungrouped, rather than grouped, data—cf. Chernoff and Lehmann (1954). Weighted sums of exponential random variables also occur in the form $-\log(\Pi P_i^{\theta_i})$, a weighted version of the Fisher statistic for combining independent p-values $P_1, \ldots P_n$, where each P_i is uniformly distributed on (0,1) under the combined null hypothesis—cf. Good (1955) and Zelen and Joel (1959). See BDHP (1987) for additional examples.

Because the distribution of $\sum \theta_i Y_i$ cannot be expressed in a simple form, it is important to determine approximations or bounds for its tail probabilities. Much work concerning such approximations exists in the literature—cf. Johnson and Kotz (1970), Chapter 29—but little is known about bounds. One obvious question is the comparison of the tail probabilities of $\sum \theta_i Y_i$ and $\bar{\theta} \sum Y_i$, where $\bar{\theta} = n^{-1} \sum \theta_i$. This comparison is both natural (since $E(\sum \theta_i Y_i) = E(\bar{\theta} \sum Y_i)$) and potentially useful, since the tail probabilities of $\bar{\theta} \sum Y_i \sim G(n\alpha, \bar{\theta}\beta)$ are easily determined. For the reason mentioned in the next paragraph, it is appropriate to conjecture that the tail probabilities of $\bar{\theta} \sum Y_i$ provide lower bounds for those of $\sum \theta_i Y_i$.

Since $\boldsymbol{\theta} \equiv (\theta_1, \ldots, \theta_n)$ majorizes $\bar{\boldsymbol{\theta}} \equiv (\bar{\theta}, \ldots, \bar{\theta})$ [denoted by $\boldsymbol{\theta} \succ \bar{\boldsymbol{\theta}}$—cf. Marshall and Olkin (1979)], the above suggests a stronger conjecture, namely, that the tail probabilities of $\sum \theta_i Y_i$ exceed those of $\sum \eta_i Y_i$ whenever $\boldsymbol{\theta} \succ \boldsymbol{\eta} \equiv (\eta_1, \ldots, \eta_n)$. [Recall that $\boldsymbol{\theta} \succ \boldsymbol{\eta}$ requires that $\sum \theta_i = \sum \eta_i$, so that again $E(\sum \theta_i Y_i) = E(\sum \eta_i Y_i)$. Also, we shall adopt the convention that $\boldsymbol{\theta} \succ \boldsymbol{\eta}$ requires that $(\eta_1, \ldots, \eta_n)$ *not* be a permutation of $(\theta_1, \ldots, \theta_n)$.] Support for this conjecture is immediate: since

$$\text{Var}\left(\sum \theta_i Y_i\right) = \left(\sum \theta_i^2\right) \text{Var}\, Y_1 \tag{3}$$

and $\sum \theta_i^2$ is a strictly Schur-convex function of $(\theta_1, \ldots, \theta_n)$,

$$\boldsymbol{\theta} \succ \boldsymbol{\eta} \Longrightarrow \text{Var}\left(\sum \theta_i Y_i\right) > \text{Var}\left(\sum \eta_i Y_i\right). \tag{4}$$

This states that if the weights $\theta_1, \ldots, \theta_n$ are more dispersed (in the sense of majorization) than $\eta_1, \ldots, \eta_n$ about their common average, then the random variable $\sum \theta_i Y_i$ is more dispersed than $\sum \eta_i Y_i$ about their common expected value, as measured by their variances. Our basic question is whether $\sum \theta_i Y_i$ is more dispersed than $\sum \eta_i Y_i$ as measured by the stronger criterion of their tail probabilities.

In this paper we investigate two aspects of this question, requiring two different techniques. First, in Section 2 we investigate the conjecture that if $\boldsymbol{\theta} \succ \boldsymbol{\eta}$, then $F_{\boldsymbol{\theta}}(\cdot)$ and $F_{\boldsymbol{\eta}}(\cdot)$ cross exactly once on $(0, \infty)$ at a unique point t^*. If true, this conjecture (called the *Unique Crossing Conjecture*, or UCC), implies that the probability distribution of $\sum \eta_i Y_i$ is more concentrated about t^* than that of $\sum \theta_i Y_i$. Although we believe that the UCC is true in general, we are able to verify it only in the following special cases:

(a) $n = 2$ (Proposition 2.1);

(b) $n = 3$, $\alpha = 1$ (Proposition 2.5);

(c) $n \geq 3$, $\alpha \geq 1$, $\boldsymbol{\theta}$ and $\boldsymbol{\eta}$ differ in only two components (Proposition 2.3);

(d) $n \geq 3$, $\boldsymbol{\eta} = \bar{\boldsymbol{\theta}}$ (Proposition 2.7).

Second, in Section 3 we investigate a conjecture regarding the *location* of the unique crossing point t^* when $\boldsymbol{\eta} = \bar{\boldsymbol{\theta}}$. It has been established by Diaconis (1976) and BDHP (1987) that when $\boldsymbol{\theta} \succ \boldsymbol{\eta}$ and $n = 2$, the unique crossing point of $F_{\boldsymbol{\theta}}$ and $F_{\boldsymbol{\eta}}$ lies in the interval

$$(2\alpha\bar{\theta}\beta,\ (2\alpha+1)\bar{\theta}\beta) \tag{5}$$

(See also Proposition 2.2 in Section 2.) This implies that $F_{\boldsymbol{\theta}}(t)$ is Schur-convex in $\boldsymbol{\theta}$ when $t \leq 2\alpha\bar{\theta}\beta$ and that $\bar{F}_{\boldsymbol{\theta}}(t)$ is Schur-convex in θ when $t \geq (2\alpha+1)\bar{\theta}\beta$. For $n \geq 3$, however, BDHP (1987) obtained only that $\bar{F}_{\boldsymbol{\theta}}(t)$ is Schur-convex in $\boldsymbol{\theta}$ when

$$t \geq n(n\alpha+1)\bar{\theta}\beta \tag{6}$$

which, when $n = 2$, is a smaller interval than that implied by (5). Furthermore, they obtained no general result on the Schur-convexity of $F_{\boldsymbol{\theta}}(t)$ for $n \geq 3$; in fact, BDHP (1987, p. 394) presented a counterexample to show that no such result is possible.

In Section 3 of the present paper, we shall show that when $\boldsymbol{\eta} = \bar{\boldsymbol{\theta}}$, the unique crossing point t^* of $F_{\boldsymbol{\theta}}$ and $F_{\bar{\boldsymbol{\theta}}}$ in fact satisfies

$$t^*(\boldsymbol{\theta}, \bar{\boldsymbol{\theta}}) \sim n\alpha\bar{\theta}\beta \quad \text{as } n \to \infty \tag{7}$$

uniformly in $\boldsymbol{\theta}$ for fixed $\bar{\theta}$. In the course of this demonstration, we derive approximate bounds of the form

$$n\alpha\bar{\theta}\beta e^{-\frac{1}{2}} \stackrel{\cdot}{\leq} t^*(\boldsymbol{\theta}, \bar{\boldsymbol{\theta}}) \stackrel{\cdot}{\leq} (n\alpha+1)\bar{\theta}\beta \log[n\alpha(n-1)] \tag{8}$$

valid for all $n \geq 2$ (cf. Propositions 3.1 and 3.2 and also (50) and (60)).

As in BDHP (1987), many of our methods also can be applied to obtain bounds for tail probabilities of weighted sums of independent Weibull random variables. Furthermore, it is likely that part of our results extend to the case where some of the weights θ_i may be negative, and also to the case where $Y_1, \ldots, Y_n$ are not i.i.d. but are exchangeable with pdf of the form

$$(\Pi y_i)^{\alpha-1} h\left(\sum y_i\right)$$

for suitable functions h.

2. The Unique Crossing Conjecture. By (4), $F_{\boldsymbol{\theta}}$ and $F_{\boldsymbol{\eta}}$ cannot be identical when $\boldsymbol{\theta} \succ \boldsymbol{\eta}$. Since

$$\boldsymbol{\theta} \succ \boldsymbol{\eta} \Longrightarrow E\left(\sum \theta_i Y_i\right) = E\left(\sum \eta_i Y_i\right), \tag{9}$$

$F_{\boldsymbol{\theta}} - F_{\boldsymbol{\eta}}$ must change sign *at least once* on $(0, \infty)$. In this section we investigate the

Unique Crossing Conjecture (UCC): If $\boldsymbol{\theta} \succ \boldsymbol{\eta}$, then $F_{\boldsymbol{\theta}} - F_{\boldsymbol{\eta}}$ changes sign *exactly once* on $(0, \infty)$. This crossing occurs at a unique point $t^* \equiv t^*(\boldsymbol{\theta}, \boldsymbol{\eta})$, which is the only zero of $F_{\boldsymbol{\theta}} - F_{\boldsymbol{\eta}}$ on $(0, \infty)$.

If t^* exists, then necessarily

$$F_{\boldsymbol{\theta}}(t) - F_{\boldsymbol{\eta}}(t) \begin{cases} > 0 & \text{for } 0 < t < t^*, \\ < 0 & \text{for } t^* < t < \infty. \end{cases} \tag{10}$$

To see this, note that (4) and (9) imply that

$$\begin{aligned} E\left(\sum \theta_i Y_i - t^*\right)^2 &= \operatorname{Var}\left(\sum \theta_i Y_i\right) + \left[E\left(\sum \theta_i Y_i\right) - t^*\right]^2 \\ &> \operatorname{Var}\left(\sum \eta_i Y_i\right) + \left[E\left(\sum \eta_i Y_i\right) - t^*\right]^2 \\ &= E\left(\sum \eta_i Y_i - t^*\right)^2, \end{aligned} \tag{11}$$

hence

$$\begin{aligned} 0 &< E\left(\sum \theta_i Y_i - t^*\right)^2 - E\left(\sum \eta_i Y_i - t^*\right)^2 \\ &= \int_0^\infty \left\{P\left[\left(\sum \theta_i Y_i - t^*\right)^2 \geq u\right] - P\left[\left(\sum \eta_i Y_i - t^*\right)^2 \geq u\right]\right\} du \\ &= \int_0^\infty \{[F_{\boldsymbol{\eta}}(t^* + \sqrt{u}) - F_{\boldsymbol{\theta}}(t^* + \sqrt{u})] + [F_{\boldsymbol{\theta}}(t^* - \sqrt{u}) - F_{\boldsymbol{\eta}}(t^* - \sqrt{u})]\} du. \end{aligned}$$

If the inequalities in (10) were reversed, then (11) would be violated, so (10) must hold. (See also Remark 2.4 for the case $\alpha \geq 1$.) The result (10) shows that if the UCC is true, then the distribution of $\sum \theta_i Y_i$ is indeed more dispersed about t^* than that of $\sum \eta_i Y_i$ in the strong sense of tail probabilities.

Without loss of generality, we may set the scale parameter $\beta = 1$ for the remainder of this section. We shall establish the UCC in four special cases by means of the representation (13) below for $F_{\boldsymbol{\theta}}$. First, define

$$\begin{aligned} S_n &= \sum_{i=1}^n Y_i \\ W_i &= Y_i / S_n, \qquad i = 1, \ldots, n, \\ \mathbf{W} &= (W_1, \ldots, W_n). \end{aligned}$$

Then $\mathbf{W}$ and S_n are independent,

$$\begin{aligned} S_n &\sim G(n\alpha, 1), \\ \mathbf{W} &\sim \text{Dirichlet}(\alpha, \ldots, \alpha), \end{aligned}$$

i.e., $\mathbf{W}$ has the (exchangeable) Dirichlet distribution on the simplex

$$\Sigma^{(n)} \equiv \left\{\mathbf{w} | w_1 \geq 0, \ldots, w_n \geq 0, \sum w_i = 1\right\} \tag{12}$$

with pdf proportional to $(\Pi W_i)^{\alpha-1}$. Thus

$$\begin{aligned} F_{\boldsymbol{\theta}}(t) &= E\left\{P\left[\sum \theta_i W_i \leq t S_n^{-1} | S_n\right]\right\} \\ &= \int_0^\infty H_{\boldsymbol{\theta}}(ts^{-1}) g_{n\alpha,1}(s) ds \\ &= t \int_0^\infty H_{\boldsymbol{\theta}}(u) g_{n\alpha,1}(tu^{-1}) u^{-2} du, \end{aligned} \tag{13}$$

where

$$H_{\boldsymbol{\theta}}(u) = P\left[\sum \theta_i W_i \leq u\right]. \tag{14}$$

Note that the support of $\sum \theta_i W_i$ is the interval $(\theta_{\min}, \theta_{\max}) \subseteq (0, n\bar{\theta})$.

From (13) it follows that

$$\begin{aligned} F_{\boldsymbol{\theta}}(t) - F_{\boldsymbol{\eta}}(t) &= t \int_0^\infty [H_{\boldsymbol{\theta}}(u) - H_{\boldsymbol{\eta}}(u)] g_{n\alpha,1}(tu^{-1}) u^{-2} du \\ &= \frac{t^{n\alpha}}{\Gamma(n\alpha)} \int_0^\infty [H_{\boldsymbol{\theta}}(u) - H_{\boldsymbol{\eta}}(u)] e^{-t/u} u^{-n\alpha-1} du. \end{aligned} \tag{15}$$

Because the kernel

$$K_1(t, u) \equiv e^{-t/u}$$

is *strictly* totally positive (STP) [cf. Karlin (1968), p. 15, eqn. (9)], it is strictly variation-diminishing, i.e., the number of sign changes of $F_{\boldsymbol{\theta}} - F_{\boldsymbol{\eta}}$ on $(0, \infty)$ cannot exceed the number of sign changes of $H_{\boldsymbol{\theta}} - H_{\boldsymbol{\eta}}$ provided this latter number is finite, the sign changes of $F_{\boldsymbol{\theta}} - F_{\boldsymbol{\eta}}$ must occur at isolated crossing points, and these crossing points are the only zeroes of $F_{\boldsymbol{\theta}} - F_{\boldsymbol{\eta}}$ on $(0, \infty)$ [apply Theorem 3.1(b) on p. 21 of Karlin (1968)]. We make use of these facts to establish the UCC in several special cases.

PROPOSITION 2.1. *If $n = 2$, the UCC is valid.*

PROOF. Without loss of generality, assume that $\theta_1 > \theta_2$. Since $(\theta_1, \theta_2) \succ (\eta_1, \eta_2)$, it follows that

$$\begin{aligned}(\theta_1, \theta_2) &= (\bar{\theta}+a, \bar{\theta}-a)\\ (\eta_1, \eta_2) &= (\bar{\theta}+b, \bar{\theta}-b),\end{aligned}$$

where $\bar{\theta} \geq a > |b| \geq 0$. Since $W_1 + W_2 = 1$, it follows that

$$\begin{aligned}H_{\boldsymbol{\theta}}(u) &= P[\bar{\theta} + a(W_1 - W_2) \leq u]\\ H_{\boldsymbol{\eta}}(u) &= P[\bar{\theta} + |b|(W_1 - W_2) \leq u],\end{aligned} \tag{16}$$

where we use the fact that $W_1 - W_2$ is symmetrically distributed about 0 on $(-1,1)$ since $\mathbf{W}$ is exchangeable. Also, since W_1 has a Beta distribution, $W_1 - W_2 \equiv 2W_1 - 1$ assigns positive probability to every open subinterval of $(-1,1)$. Therefore

$$H_{\boldsymbol{\theta}}(u) - H_{\boldsymbol{\eta}}(u) \begin{cases} > 0 & \text{if } \bar{\theta} - a < u < \bar{\theta}\\ < 0 & \text{if } \bar{\theta} < u < \bar{\theta} + a\\ = 0 & \text{if } u \leq \bar{\theta} - a,\ u = \bar{\theta}, \text{ or } u \geq \bar{\theta} + a,\end{cases} \tag{17}$$

hence has exactly one sign change (at $\bar{\theta}$) on $(0,\infty)$. Thus $F_{\boldsymbol{\theta}} - F_{\boldsymbol{\eta}}$ can have at most one sign change, hence by (9), must have exactly one sign change on $(0,\infty)$. Furthermore, this sign change must occur at a unique point t^* which must be the only zero of $F_{\boldsymbol{\theta}} - F_{\boldsymbol{\eta}}$ on $(0,\infty)$. $\|$

For the case $n = 2$, a closer examination of (15) in fact yields the upper bound in (5) for $t^* \equiv t^*(\boldsymbol{\theta}, \boldsymbol{\eta})$, the unique crossing point of $F_{\boldsymbol{\theta}} - F_{\boldsymbol{\eta}}$ on $(0,\infty)$.

PROPOSITION 2.2. *(Diaconis and Perlman (1976), BDHP (1987)). When $n = 2$ and $\boldsymbol{\theta} \succ \boldsymbol{\eta}$, $t^*(\boldsymbol{\theta}, \boldsymbol{\eta}) < (2\alpha + 1)\bar{\theta}$.*

PROOF. From (16) and the symmetry of $W_1 - W_2$, $\Lambda \equiv H_{\boldsymbol{\theta}} - H_{\boldsymbol{\eta}}$ is antisymmetric about $\bar{\theta}$, i.e.,

$$\Lambda(u) = -\Lambda(2\bar{\theta} - u). \tag{18}$$

Thus, if we define

$$\varphi_t(u) = e^{-t/u} u^{-2\alpha-1}, \tag{19}$$

we obtain from (15) and (18) that

$$\begin{aligned}\frac{\Gamma(2\alpha)}{t^{2\alpha}}[F_{\boldsymbol{\theta}}(t) - F_{\boldsymbol{\eta}}(t)] &= \int_0^{\bar{\theta}} \Lambda(u)\varphi_t(u)du + \int_{\bar{\theta}}^{2\bar{\theta}} \Lambda(u)\varphi_t(u)du\\ &= \int_0^{\bar{\theta}} \Lambda(u)[\varphi_t(u) - \varphi_t(2\bar{\theta} - u)]du.\end{aligned} \tag{20}$$

When $t \geq (2\alpha+1)\bar{\theta}$ and $0 < u < \bar{\theta}$, it follows from Lemma 2.8 (at the end of this section) that $\varphi_t(u) - \varphi_t(2\bar{\theta} - u) < 0$, so by (17) we have that $F_{\boldsymbol{\theta}}(t) - F_{\boldsymbol{\eta}}(t) < 0$. By (10), this implies that $t^* < (2\alpha+1)\bar{\theta}$ as claimed. ||

Unfortunately, this method does not yield the lower bound for t^* in (5), since it is not true that $\varphi_t(u) - \varphi_t(2\bar{\theta} - u) > 0$ for every $u \in (0, \bar{\theta})$ when $t \leq 2\alpha\bar{\theta}$. Furthermore, when $n \geq 3$, $\Lambda(\cdot)$ need not be antisymmetric so the method does not immediately yield useful information about the location of t^*. An alternate approach is presented in Section 3 which does provide upper and lower bounds (though not sharp) for t^* when $\boldsymbol{\eta} = \bar{\boldsymbol{\theta}}$.

We now return to the UCC for the case $n \geq 3$. At present we cannot establish the UCC in general, so must content ourselves with four propositions (2.3, 2.5, 2.7, 2.7a) dealing with special cases of interest.

PROPOSITION 2.3. *Suppose that $n \geq 3$ and $\alpha \geq 1$. If $\boldsymbol{\theta}$ and $\boldsymbol{\eta}$ differ in exactly two components, then the UCC is valid.*

PROOF. When $n = 3$, we may assume without loss of generality that $\theta_3 = \eta_3 > 0$, so that $\boldsymbol{\theta} \succ \boldsymbol{\eta} \iff (\theta_1, \theta_2) \succ (\eta_1, \eta_2)$. Now

$$\begin{aligned} F_{\boldsymbol{\theta}}(t) &= E\{P[\theta_1 Y_1 + \theta_2 Y_2 \leq t - \theta_3 Y_3 | Y_3]\} \\ &= \int_0^\infty F_{(\theta_1,\theta_2)}(t - v) g(v) dv, \end{aligned}$$

where $g = g_{\alpha,\theta_3}$. Thus

$$F_{\boldsymbol{\theta}}(t) - F_{\boldsymbol{\eta}}(t) = \int_0^t \Delta(u) g(t-u) du, \tag{21}$$

where

$$\Delta(u) = F_{(\theta_1,\theta_2)}(u) - F_{(\eta_1,\eta_2)}(u).$$

We shall apply (21) to show that for $0 < t_1 < t_2 < \infty$,

$$F_{\boldsymbol{\theta}}(t_1) - F_{\boldsymbol{\eta}}(t_1) = 0 \implies F_{\boldsymbol{\theta}}(t_2) - F_{\boldsymbol{\eta}}(t_2) < 0, \tag{22}$$

$$F_{\boldsymbol{\theta}}(t_2) - F_{\boldsymbol{\eta}}(t_2) = 0 \implies F_{\boldsymbol{\theta}}(t_1) - F_{\boldsymbol{\eta}}(t_1) > 0. \tag{23}$$

These implications, together with the facts that $F_{\boldsymbol{\theta}} - F_{\boldsymbol{\eta}}$ is continuous and has at least one zero crossing on $(0, \infty)$, imply that $F_{\boldsymbol{\theta}}$ and $F_{\boldsymbol{\eta}}$ satisfy the UCC, hence establish the proposition when $n = 3$.

By Proposition 2.1 and (10), there exists $t_0 \in (0, \infty)$ such that

$$\Delta(u) \begin{cases} > 0 & \text{if } 0 < u < t_0 \\ = 0 & \text{if } u = t_0 \\ < 0 & \text{if } t_0 < u < \infty. \end{cases}$$

If $F_{\boldsymbol{\theta}}(t_1) - F_{\boldsymbol{\eta}}(t_1) = 0$, then since $g > 0$ on $(0, \infty)$, (21) implies that $t_1 > t_0$. Thus

$$\begin{aligned} F_{\boldsymbol{\theta}}(t_2) - F_{\boldsymbol{\eta}}(t_2) &< \int_0^{t_0} \Delta(u) g(t_2 - u) du + \int_{t_0}^{t_1} \Delta(u) g(t_2 - u) du \\ &\leq \frac{g(t_2 - t_0)}{g(t_1 - t_0)} \left[\int_0^{t_0} \Delta(u) g(t_1 - u) du + \int_{t_0}^{t_1} \Delta(u) g(t_1 - u) du \right] \\ &= \frac{g(t_2 - t_0)}{g(t_1 - t_0)} [F_{\boldsymbol{\theta}}(t_1) - F_{\boldsymbol{\eta}}(t_1)] \\ &= 0. \end{aligned}$$

The second inequality follows because $\alpha \geq 1 \Longrightarrow g_{\alpha,\beta}(\cdot)$ is log concave $\Longrightarrow g(t-u)$ is totally positive of order two (TP_2) [cf. Karlin (1968), p. 32]. Thus (22) is valid, and (23) is established in similar fashion.

For $n \geq 4$, the proposition is established by a similar argument, using induction on n. ||

We remark that the kernel

$$K_2(t, u) \equiv I_{[0,t]}(u) g(t - u)$$

is TP_2 [cf. Karlin (1968), p. 16], hence Theorem 3.1(a) of Karlin (1968), p. 21, together with (21), shows that $F_{\boldsymbol{\theta}} - F_{\boldsymbol{\eta}}$ has at most one sign change on $(0, \infty)$. However, $K(t, u)$ is not *strictly* TP_2, so Karlin's Theorem 3.1(b) cannot be applied to conclude that the crossing point is *unique*, hence the need for a direct demonstration of this fact.

REMARK 2.4. If $\boldsymbol{\theta} \succ \boldsymbol{\eta}$ and $\alpha \geq 1$, we may apply Proposition 2.3 to deduce that $F_{\boldsymbol{\theta}}(t) - F_{\boldsymbol{\eta}}(t)$ is positive for sufficiently small $t > 0$ and negative for sufficiently large t. This follows from the fundamental majorization result that if $\boldsymbol{\theta} \succ \boldsymbol{\eta}$, then there exists a finite sequence

$$\boldsymbol{\theta} \equiv \boldsymbol{\psi}_0 \succ \boldsymbol{\psi}_1 \succ \ldots \succ \boldsymbol{\psi}_k \equiv \boldsymbol{\eta}$$

such that $\boldsymbol{\psi}_i$ and $\boldsymbol{\psi}_{i+1}$ differ in exactly two components [cf. Marshall and Olkin (1979), p. 21]. By Proposition 2.3 and (10), each difference $F_{\boldsymbol{\psi}_i}(t) - F_{\boldsymbol{\psi}_{i+1}}(t)$ must be positive for small $t > 0$ and negative for large t, hence the same must be true of the sum

$$\sum_{i=1}^{k-1} \left[F_{\boldsymbol{\psi}_i}(t) - F_{\boldsymbol{\psi}_{i+1}}(t) \right] \equiv F_{\boldsymbol{\theta}}(t) - F_{\boldsymbol{\eta}}(t).$$

Finally, we note that this implies that if the number of sign changes of $F_{\boldsymbol{\theta}} - F_{\boldsymbol{\eta}}$ on $(0, \infty)$ is finite, then this number must be *odd*. ||

REMARK 2.4A. The referee has pointed out that the argument for Proposition 2.3 also shows the following fact when $\alpha \geq 1$: if the UCC is valid for the vectors $\boldsymbol{\theta}$

and $\boldsymbol{\eta}$ in $\mathbb{R}^n$, then it remains valid for the vectors $(\boldsymbol{\theta}, \mathbf{c})$ and $(\boldsymbol{\eta}, \mathbf{c})$ in $\mathbb{R}^{n+k}$, where $\mathbf{c} = (c_1, \ldots, c_k)$ with $c_i > 0$. ||

PROPOSITION 2.5. *If $n = 3$ and $\alpha = 1$, the UCC is valid.*

Proof. Since $\alpha = 1$, $\mathbf{W} \equiv (W_1, W_2, W_3)$ is uniformly distributed on the 3-simplex $\Sigma^{(3)}$ (cf. (12)) with vertices (1,0,0), (0,1,0), (0,0,1). Let $h_{\boldsymbol{\theta}}$ denote the pdf of $\sum \theta_i W_i$, i.e.,

$$h_{\boldsymbol{\theta}}(u) = \frac{d}{du} H_{\boldsymbol{\theta}}(u)$$

(cf. (14)). Then $h_{\boldsymbol{\theta}}$ is a continuous triangular density function with support $(\theta_{\min}, \theta_{\max})$; $h_{\boldsymbol{\theta}}$ increases linearly on $(\theta_{\min}, \theta_{\text{med}})$ and decreases linearly on $(\theta_{\text{med}}, \theta_{\max})$, where $\theta_{\min}$, θ_{med}, and $\theta_{\max}$ denote the minimum, median, and maximum of $(\theta_1, \theta_2, \theta_3)$. Similarly, $h_{\boldsymbol{\eta}}$ is a triangular density function that increases linearly on $(\eta_{\min}, \eta_{\text{med}})$ and decreases linearly on $(\eta_{\text{med}}, \eta_{\max})$. Since

$$\boldsymbol{\theta} \succ \boldsymbol{\eta} \Longrightarrow \theta_{\min} \leq \eta_{\min} \leq \eta_{\max} \leq \theta_{\max}, \tag{24}$$

it follows that $h_{\boldsymbol{\theta}} - h_{\boldsymbol{\eta}}$ changes sign at most twice. But

$$H_{\boldsymbol{\theta}}(u) - H_{\boldsymbol{\eta}}(u) = \int_0^\infty [h_{\boldsymbol{\theta}}(v) - h_{\boldsymbol{\eta}}(v)] I_{[0,u]}(v) dv \tag{25}$$

and the kernel

$$K_3(u, v) \equiv I_{[0,u]}(v)$$

is totally positive of every order [Karlin (1968), p. 16] hence is TP_3. Thus by (25) and Theorem 3.1(a) of Karlin (1968), p. 21, $H_{\boldsymbol{\theta}} - H_{\boldsymbol{\eta}}$ changes sign at most twice, so $F_{\boldsymbol{\theta}} - F_{\boldsymbol{\eta}}$ changes sign at most twice and has at most two zeroes on $(0, \infty)$, which must coincide with the crossing points (cf. the discussion following (15)). But the final sentence in Remark 2.4 implies that the number of sign changes of $F_{\boldsymbol{\theta}} - F_{\boldsymbol{\eta}}$ must be odd, hence cannot exceed one. Thus $F_{\boldsymbol{\theta}} - F_{\boldsymbol{\eta}}$ must have exactly one sign change and exactly one zero, i.e., the UCC is valid in this case. ||

It seems likely that Proposition 2.5 remains valid for $n \geq 4$. When $\alpha = 1$, both $h_{\boldsymbol{\theta}}$ and $h_{\boldsymbol{\eta}}$ are univariate B-splines (cf. Karlin, Micchelli, and Rinott (1986)) with knots $\theta_1, \ldots, \theta_n$ and $\eta_1, \ldots, \eta_n$, respectively. *We conjecture that the integrated B-splines $H_{\boldsymbol{\theta}}$ and $H_{\boldsymbol{\eta}}$ cross exactly once whenever $\boldsymbol{\theta} \succ \boldsymbol{\eta}$.* By the argument following (15), the validity of this conjecture would imply the validity of the UCC when $\alpha = 1$. We have been able to establish this conjecture when $\boldsymbol{\theta}$ and $\boldsymbol{\eta}$ differ in exactly two components. In fact, for *every* $0 < \alpha < \infty$, $H_{\boldsymbol{\theta}}$ and $H_{\boldsymbol{\eta}}$ cross exactly once when $n = 2$ (recall (17)), while when $\alpha \geq 1$ and $n \geq 3$ the following result obtains:

PROPOSITION 2.6. *(Diaconis and Perlman (1976)). Suppose that $n \geq 3$ and $\alpha \geq 1$. If $\boldsymbol{\theta} \succ \boldsymbol{\eta}$ and differ in exactly two components, then $H_{\boldsymbol{\theta}}$ and $H_{\boldsymbol{\eta}}$ cross exactly once.* ||

Our proof of this result is similar to that of Proposition 2.3 though somewhat longer, hence is omitted. Note that Proposition 2.3 follows from Proposition 2.6 by the argument following (15).

Now fix $\alpha = 1$ and consider the above conjecture regarding the single crossing of the integrated B-splines $H_{\boldsymbol{\theta}}$ and $H_{\boldsymbol{\eta}}$ when $\boldsymbol{\theta} \succ \boldsymbol{\eta}$ and differ in *more* than two components. *This conjecture is valid when* $n = 3$. To see this, one first shows that (4), (9), and (10) remain true with $Y_1, \ldots, Y_n$ replaced by $W_1, \ldots, W_n$ and F replaced by H. Then, by applying Proposition 2.6 rather than Proposition 2.3, it may be shown as in Remark 2.4 that the number of crossings of $H_{\boldsymbol{\theta}}$ and $H_{\boldsymbol{\eta}}$ must be *odd*. But it has been established in the proof of Proposition 2.5 that $H_{\boldsymbol{\theta}}$ and $H_{\boldsymbol{\eta}}$ can cross at most twice, hence they must cross exactly once as conjectured. (Note again that Proposition 2.5 follows from this result by the argument following (15).)

D.L. Ragozin has obtained some convincing numerical evidence that the above conjecture is valid when $n = 4$.

In the final proposition of this section, we return to the UCC for $F_{\boldsymbol{\theta}}$ and $F_{\boldsymbol{\eta}}$ when $\boldsymbol{\eta} = \bar{\boldsymbol{\theta}} \equiv (\bar{\theta}, \ldots \bar{\theta})$.

PROPOSITION 2.7. *(Diaconis and Perlman (1976), Shaked (1980)). If* $\boldsymbol{\eta} = \bar{\boldsymbol{\theta}}$, *the UCC is valid for every* $n \geq 2$.

PROOF. When $\boldsymbol{\eta} = \bar{\boldsymbol{\theta}}$, $\sum \eta_i W_i = \bar{\theta} \sum W_i = \bar{\theta}$, so

$$H_{\bar{\boldsymbol{\theta}}}(u) = \begin{cases} 0 & \text{if } u < \bar{\theta}, \\ 1 & \text{if } u > \bar{\theta}. \end{cases}$$

Thus $H_{\boldsymbol{\theta}} - H_{\bar{\boldsymbol{\theta}}}$ has exactly one sign change, so by (15), $F_{\boldsymbol{\theta}} - F_{\bar{\boldsymbol{\theta}}}$ has at most one sign change, hence exactly one sign change and exactly one zero on $(0, \infty)$. ||

We are grateful to the referee for pointing out the following extension of Proposition 2.7:

PROPOSITION 2.7a. *If* $\boldsymbol{\eta} = (1 - \lambda)\bar{\boldsymbol{\theta}} + \lambda \boldsymbol{\theta}$ *for* $0 \leq \lambda < 1$, *the UCC is valid for every* $n \geq 2$.

PROOF. Since $H_{\boldsymbol{\eta}}(u) = H_{\boldsymbol{\theta}}(\lambda^{-1}(u - (1 - \lambda)\bar{\theta}))$ when $\lambda > 0$, $H_{\boldsymbol{\theta}} - H_{\boldsymbol{\eta}}$ has exactly one sign change (at $u = \bar{\theta}$), so the argument for Proposition 2.7 remains applicable. ||

The following lemma was needed in the proof of Proposition 2.2.

LEMMA 2.8. *Define* $\varphi_t(u)$ *by (19). If* $t \geq (2\alpha + 1)\bar{\theta}$ *and* $0 < u < \bar{\theta}$, *then* $\varphi_t(u) < \varphi_t(2\bar{\theta} - u)$.

PROOF. As u increases from 0 to $\bar{\theta}$, $b \equiv \bar{\theta} u^{-1} - 1$ decreases from ∞ to 0. Since $t \geq (2\alpha + 1)\bar{\theta}$, the desired inequality will follow from the inequality

$$\frac{2\bar{\theta} - u}{u} < \exp\left\{ \frac{2\bar{\theta}(\bar{\theta} - u)}{u(2\bar{\theta} - u)} \right\},$$

which is equivalent to

$$1 + 2b < \exp\left\{\frac{2b(1+b)}{1+2b}\right\}$$

and therefore to the inequality

$$f(b) \equiv \frac{2b(1+b)}{1+2b} - \log(1+2b) > 0 \tag{26}$$

for $b > 0$. But $f(0) = 0$ while

$$\frac{1}{4}(1+2b)^2 f'(b) = b^2 > 0,$$

hence (26) holds. ||

3. Location of the Unique Crossing Point When $\eta = \bar{\theta}$. It is a consequence of Proposition 2.7 that $F_{\boldsymbol{\theta}}$ and $F_{\bar{\boldsymbol{\theta}}}$ have a unique crossing point $t^* \equiv t^*(\boldsymbol{\theta}, \bar{\boldsymbol{\theta}}) \in (0, \infty)$ and that (10) holds when $\boldsymbol{\eta} = \bar{\boldsymbol{\theta}}$. In this section we present partial results regarding the location of t^*. Once again, without loss of generality we may assume that $\beta = 1$.

Our results are based on the following alternate representation of $F_{\boldsymbol{\theta}}$ (compare to (13)):

$$\begin{aligned} F_{\boldsymbol{\theta}}(t) &= E\left\{P[S_n \le t\left(\sum \theta_i W_i\right)^{-1} |\mathbf{W}]\right\} \\ &\equiv E\{G\left(t^{-1}\sum \theta_i W_i\right)\} \end{aligned} \tag{27}$$

where, for $0 < u < \infty$, $G \equiv G_{n\alpha}$ is defined by

$$G(u) = P[S_n \le u^{-1}] = \int_0^{u^{-1}} g_{n\alpha,1}(s)ds. \tag{28}$$

Since

$$G'(u) = -\frac{1}{\Gamma(n\alpha)} u^{-n\alpha-1} e^{-u^{-1}}, \tag{29}$$

$$G''(u) = \frac{1}{\Gamma(n\alpha)} u^{-n\alpha-2} e^{-u^{-1}} [(n\alpha+1) - u^{-1}], \tag{30}$$

it is immediate that G is strictly decreasing on $[0, \infty)$, strictly concave on $[0, (n\alpha + 1)^{-1}]$, and strictly convex on $[(n\alpha+1)^{-1}, \infty)$. Thus, for every $u \in [0, \infty)$ there exists a unique line L_u tangent to the graph of G at $(u, G(u))$, the equation of which is given by

$$L_u(v) = G(u) + G'(u)(v - u), \quad 0 \le v < \infty. \tag{31}$$

When $u < (n\alpha+1)^{-1}$, L_u is tangent to the graph of G from above, while when $u > (n\alpha+1)^{-1}$, L_u is tangent from below.

Because G is strictly concave on $[0,(n\alpha+1)^{-1}]$, clearly $L_u(0) > G(0) \equiv 1$ when $0 < u \le (n\alpha+1)^{-1}$, while

$$L_\infty(0) = G(\infty) - \lim_{u\to\infty}[uG'(u)] = 0.$$

Thus since

$$\frac{d}{du}[L_u(0)] = -uG''(u) < 0$$

for $u > (n\alpha+1)^{-1}$, there exists a unique positive number

$$\hat{u} \equiv \hat{u}_{n\alpha} > (n\alpha+1)^{-1} \tag{32}$$

such that $L_{\hat{u}}(0) = 1 = G(0)$. From (31) the point $\hat{u}$ is the unique positive solution to

$$G(\hat{u}) = 1 + \hat{u}G'(\hat{u}). \tag{33}$$

The line $L_{\hat{u}}$ is tangent to the graph of G at $(\hat{u}, G(\hat{u}))$ and elsewhere lies strictly below this graph, except at the point (0,1) where they coincide. In fact, for every point $u \ge \hat{u}$, L_u is tangent to the graph at $(u, G(u))$ and elsewhere lies strictly below the graph, i.e.,

$$u \ge \hat{u} \Longrightarrow L_u(v) \begin{cases} = G(v) & \text{if } v = u, \\ < G(v) & \text{if } v \ne 0,\ u. \end{cases} \tag{34}$$

If we set $t = \bar{\theta}u^{-1}$ with $u \ge \hat{u}$, it follows from (27), (34), and the linearity of $L_u(\cdot)$ that

$$\begin{aligned} F_{\boldsymbol{\theta}}(t) &= E\left\{G\left(u\bar{\theta}^{-1}\sum \theta_i W_i\right)\right\} \\ &> E\left\{L_u\left(u\bar{\theta}^{-1}\sum \theta_i W_i\right)\right\} \\ &= L_u\left(E(u\bar{\theta}^{-1}\sum \theta_i W_i)\right) \\ &= L_u\left(u\bar{\theta}^{-1}\left(\sum \theta_i\right) EW_1\right) \\ &= L_u(u) \\ &= G(u) \\ &= P[\bar{\theta}S_n \le t] \\ &= F_{\bar{\boldsymbol{\theta}}}(t). \end{aligned} \tag{35}$$

Here we have used the exchangeability of $(W_1,\ldots,W_n)$ and the fact that $\sum W_i = 1$. Since $u \ge \hat{u}$ iff $t \le \bar{\theta}\hat{u}^{-1}$, we have derived the following result:

PROPOSITION 3.1. *Let $\hat{u} \equiv \hat{u}_{n\alpha}$ denote the unique positive solution to (33), where $G \equiv G_{n\alpha}$ is given by (28). Then $F_{\boldsymbol{\theta}}(t) > F_{\bar{\boldsymbol{\theta}}}(t)$ whenever $t \leq \bar{\theta}\hat{u}_{n\alpha}^{-1}$, i.e.,*

$$t^*(\boldsymbol{\theta}, \bar{\boldsymbol{\theta}}) > \bar{\theta}\hat{u}_{n\alpha}^{-1} \tag{36}$$ ||

Thus the *lower* tail probabilities of $\sum \theta_i Y_i$ are bounded below by those of $\bar{\theta} \sum Y_i$ for sufficiently small t. In order to show that the same is true of the *upper* tail probabilities when t is sufficiently large, we must modify (34) to show that $L_u(v)$ lies *above* the graph of G, at least for v in the support of $t^{-1} \sum \theta_i W_i$, for sufficiently small u.

For each $u \in (0, (n\alpha + 1)^{-1})$ there exists a unique point $v(u) > (n\alpha + 1)^{-1}$ such that $L_u(v(u)) = G(v(u))$, i.e., such that

$$G(v(u)) = G(u) + (v(u) - u)G'(u). \tag{37}$$

For each u it is clear that

$$L_u(v) \begin{cases} = G(v) & \text{if } v = u \text{ or } v = v(u) \\ > G(v) & \text{if } 0 < v < v(u) \text{ and } v \neq u. \end{cases} \tag{38}$$

For $u \in (0, (n\alpha + 1)^{-1})$ the function $v(u)$ is strictly decreasing and satisfies

$$v(0) = \infty, \quad v((n\alpha + 1)^{-1} = (n\alpha + 1)^{-1}.$$

Therefore, $v(u)/u$ strictly decreases from ∞ to 1, so there exists a unique point

$$\tilde{u} \equiv \tilde{u}_{n\alpha,n} < (n\alpha + 1)^{-1} \tag{39}$$

such that

$$v(\tilde{u}) = n\tilde{u}. \tag{40}$$

Hence

$$u \leq \tilde{u} \Longrightarrow u\bar{\theta}^{-1} \sum \theta_i W_i \leq n\tilde{u} = v(\tilde{u}) \leq v(u), \tag{41}$$

so for $t = \bar{\theta}u^{-1}$ it follows from (38) and (41) that

$$\begin{aligned} F_{\boldsymbol{\theta}}(t) &= E\left\{G\left(u\bar{\theta}^{-1} \sum \theta_i W_i\right)\right\} \\ &< E\left\{L_u\left(u\bar{\theta}^{-1} \sum \theta_i W_i\right)\right\} \\ &= F_{\bar{\boldsymbol{\theta}}}(t) \end{aligned} \tag{42}$$

as in (35). Because $u \leq \tilde{u}$ iff $t \geq \bar{\theta}\tilde{u}^{-1}$, we thus have the following result:

PROPOSITION 3.2. *Let $\tilde{u} \equiv \tilde{u}_{n\alpha,n}$ denote the unique solution to (40) in the interval $(0,(n\alpha+1)^{-1})$, where $v(u)$ is defined by (37). Then $\bar{F}_{\theta}(t) > \bar{F}_{\bar{\theta}}(t)$ whenever $t \geq \bar{\theta}\tilde{u}_{n\alpha,n}^{-1}$, i.e.,*

$$t^*(\theta,\bar{\theta}) < \bar{\theta}\tilde{u}_{n\alpha,n}^{-1}. \tag{43}$$

||

To be useful, of course, Propositions 3.1 and 3.2 require estimates for $\hat{u}_{n\alpha}^{-1}$ and $\tilde{u}_{n\alpha,n}^{-1}$. To estimate the former, set $\nu = n\alpha$ and $x = \hat{u}_{\nu}^{-1}$, then use (29) to rewrite (33) as

$$\int_0^x s^{\nu-1}e^{-s}ds = \Gamma(\nu) - x^{\nu}e^{-x} \tag{44}$$

or, equivalently,

$$\int_x^{\infty} s^{\nu-1}e^{-s}ds = x^{\nu}e^{-x} \tag{45}$$

The substitution $w = s - x$ converts (45) to

$$\int_0^{\infty}\left(1+\frac{w}{x}\right)^{\nu}e^{-w}\frac{dw}{x+w} = 1, \tag{46}$$

then integration by parts yields

$$\int_0^{\infty}\left(1+\frac{w}{x}\right)^{\nu}e^{-w}dw = \nu+1. \tag{47}$$

This integral strictly decreases from ∞ to 1 as x increases from 0 to ∞, hence x is the unique solution to (47).

A rough lower bound for $x \equiv \hat{u}_{\nu}^{-1}$ is obtained by expressing (47) as

$$\Gamma(\nu+1)E(W^{-1}+x^{-1})^{\nu} = \nu+1 \tag{48}$$

where $W \sim G(\nu+1,1)$, then applying Jensen's inequality to obtain

$$\hat{u}_{\nu}^{-1} > \left[\left(\frac{\nu+1}{\Gamma(\nu+1)}\right)^{1/\nu} - \frac{1}{\nu+1}\right]^{-1} \sim (e-1)^{-1}\nu \doteq 0.58\nu \text{ as } \nu\to\infty. \tag{49}$$

A sharper bound may be obtained when $\nu > 1$ (in fact, equality holds when $\nu = 1$):

$$\hat{u}_{\nu}^{-1} > \left[\left(\frac{\nu+1}{\Gamma(\nu+1)}\right)^{1/\nu} - \frac{1}{\nu}\right]^{-1} \sim (e-1)^{-1}\nu \doteq 0.58\nu \text{ as } \nu\to\infty. \tag{49a}$$

A lengthier argument yields a better bound when $\nu > 1$:

$$\hat{u}_{\nu}^{-1} > e^{-(\nu-1)/2\nu}\nu^{\nu/(\nu+1)} \sim e^{-1/2}\nu \doteq 0.61\nu \text{ as } \nu\to\infty. \tag{50}$$

These bounds are crude, however, as seen from the following result and tabulation. (We warmly thank Russell Millar for suggesting this approach.)

PROPOSITION 3.3. *$\hat{u}_\nu^{-1} \sim \nu$ as $\nu \to \infty$.*

PROOF. By (45), $x \equiv \hat{u}_\nu^{-1}$ is the unique solution to the equation

$$f_\nu(x) \equiv \int_1^\infty y^{\nu-1} e^{-x(y-1)} dy - 1 = 0. \tag{51}$$

Since $f_\nu(\cdot)$ is strictly decreasing on $(0, \infty)$,

$$f_\nu(a) > 0 \Longrightarrow x > a. \tag{52}$$

Now define

$$\begin{aligned} f_\nu^*(x) &= x^\nu e^{-x} f_\nu(x)/\Gamma(\nu) \\ &= \frac{1}{\Gamma(\nu)} \int_x^\infty s^{\nu-1} e^{-s} ds - \frac{x^\nu e^{-x}}{\Gamma(\nu)} \\ &= P[G(\nu, 1) \geq x] - \frac{x^\nu e^{-x}}{\Gamma(\nu)}. \end{aligned} \tag{53}$$

Then for any $0 < \delta < 1$, it follows from the Law of Large Numbers and Stirling's formula that

$$\begin{aligned} f_\nu^*((1-\delta)\nu) &= P[G(\nu, 1)/\nu \geq 1 - \delta] - \frac{((1-\delta)\nu)^\nu e^{-(1-\delta)\nu}}{\Gamma(\nu)} \\ &\sim 1 - \sqrt{\frac{\nu}{2\pi}}((1-\delta)e^\delta)^\nu \\ &\to 1 \end{aligned} \tag{54}$$

as $\nu \to \infty$, since $(1-\delta)e^\delta < 1$. Thus, by (52) and (54),

$$\liminf_{\nu \to \infty} (\nu \hat{u}_\nu)^{-1} \geq 1,$$

while $(\nu \hat{u}_\nu)^{-1} < (\nu + 1)/\nu$ by (32), hence the asserted result follows. $\|$

The following tabulation obtained by Russell Millar indicates that the convergence of $(\nu \hat{u}_\nu)^{-1}$ to 1 is slow:

ν	$(\nu\hat{u}_\nu)^{-1}$
1	1.000
2	.809
5	.728
10	.730
20	.756
100	.840
1,000	.930
10,000	.973
100,000	.990

By Propositions 3.1 and 3.3, the crossing point $t^*(\boldsymbol{\theta}, \bar{\boldsymbol{\theta}})$ satisfies

$$\liminf_{n\alpha \to \infty} \frac{t^*(\boldsymbol{\theta}, \bar{\boldsymbol{\theta}})}{n\alpha\bar{\theta}} \geq 1, \quad \text{uniformly in } \boldsymbol{\theta}. \tag{55}$$

In order to estimate $\tilde{u}_{n\alpha,n}^{-1}$, set $\nu = n\alpha$ and $z = \tilde{u}_{\nu,n}^{-1}$, then combine (37) and (40) to obtain

$$\int_{z/n}^{z} s^{\nu-1} e^{-s} ds = (n-1) z^\nu e^{-z}. \tag{56}$$

The substitution $y = s/z$ converts (56) to

$$\int_{1/n}^{1} y^{\nu-1} e^{z(1-y)} dy = n - 1. \tag{57}$$

Since the integral strictly increases from $\nu^{-1}(1 - n^{-\nu})(< n-1)$ to ∞ as z increases from 0 to ∞, z is the unique solution to (57).

To apply Jensen's inequality, express (57) as

$$\frac{n^\nu - 1}{\nu n^\nu} E e^{z(1-Y)} = n - 1, \tag{58}$$

where Y is a random variable with pdf $p(y)$ given by

$$p(y) = \begin{cases} \frac{\nu n^\nu}{n^\nu - 1} y^{\nu-1} & \text{if } \frac{1}{n} < y < 1 \\ 0 & \text{otherwise.} \end{cases} \tag{59}$$

Then

$$EY = \frac{\nu(n^{\nu+1} - 1)}{(\nu+1)(n^{\nu+1} - n)},$$

hence

$$\begin{aligned} n-1 &> \frac{n^{\nu}-1}{\nu n^{\nu}} \exp\left\{ z\left[1-\frac{\nu(n^{\nu+1}-1)}{(\nu+1)(n^{\nu+1}-n)}\right]\right\} \\ &= \frac{n^{\nu}-1}{\nu n^{\nu}} \exp\left\{ \frac{z}{\nu+1}\left[1-\frac{\nu(n-1)}{(n^{\nu+1}-n)}\right]\right\}. \end{aligned}$$

It can be verified that the term in square brackets is positive, hence by taking logarithms we obtain the following upper bound for $z \equiv \tilde{u}_{\nu,n}^{-1}$:

$$\begin{aligned} \tilde{u}_{\nu,n}^{-1} &< (\nu+1)\left[1-\frac{\nu(n-1)}{n^{\nu+1}-n}\right]^{-1} \log\left[\frac{\nu(n-1)}{1-n^{-\nu}}\right] \\ &\sim (\nu+1)\log[\nu(n-1)] \end{aligned} \tag{60}$$

as $n \to \infty$ or $\nu \to \infty$. But (60) is not sharp, as the next result indicates.

PROPOSITION 3.4. *$\tilde{u}_{\nu,n}^{-1} \sim \nu$ as $n \to \infty$ with α fixed.*

PROOF. By (57), $z \equiv \tilde{u}_{\nu,n}^{-1}$ is the unique positive solution to the equation

$$f_{\nu,n}(z) \equiv \int_{\frac{1}{n}}^{1} y^{\nu-1} e^{z(1-y)} dy - (n-1) = 0. \tag{61}$$

Since $f_{\nu,n}(\cdot)$ is strictly increasing,

$$f_{\nu,n}(a) > 0 \Longrightarrow z < a. \tag{62}$$

Now define

$$\begin{aligned} f_{\nu,n}^{*}(z) &= z^{\nu}e^{-z} f_{\nu,n}(z)/\Gamma(\nu) \\ &= P\left[\frac{z}{n} \le G(\nu,1) \le z\right] - \frac{(n-1)z^{\nu}e^{-z}}{\Gamma(\nu)} \end{aligned} \tag{63}$$

Then for any $0 < \delta < 1$ and $n \ge 2$,

$$\begin{aligned} f_{\nu,n}^{*}((1+\delta)\nu) &= P\left[\frac{1+\delta}{n} \le \frac{G(\nu,1)}{\nu} \le 1+\delta\right] - \frac{(n-1)((1+\delta)\nu)^{\nu}e^{-(1+\delta)\nu}}{\Gamma(\nu)} \\ &\sim 1-(n-1)\sqrt{\frac{\nu}{2\pi}}((1+\delta)e^{-\delta})^{\nu} \\ &\to 1 \end{aligned} \tag{64}$$

as $n \to \infty$, since $\nu = n\alpha$ and $(1+\delta)e^{-\delta} < 1$. Thus (62) and (64) together imply that

$$\limsup_{n\to\infty} (\nu \tilde{u}_{\nu,n})^{-1} \le 1,$$

while $(\nu\tilde{u}_{\nu,n})^{-1} > (\nu+1)/\nu$ by (39), so the result follows. ||

Once again, the convergence of $(\nu\tilde{u}_{\nu,n})^{-1}$ to 1 is slow. The following tabulations are given for the case $\alpha = 1$, so $n = \nu$:

ν	$(\nu\tilde{u}_{\nu,\nu})^{-1}$
2	1.88
5	1.92
10	1.87
20	1.72
100	1.39
1,000	1.14
10,000	1.05
100,000	1.02

Propositions 3.2 and 3.4 imply that for fixed α the crossing point $t^*(\boldsymbol{\theta},\bar{\boldsymbol{\theta}})$ satisfies

$$\limsup_{n\to\infty} \frac{t^*(\boldsymbol{\theta},\bar{\boldsymbol{\theta}})}{n\alpha\bar{\theta}} \le 1, \quad \text{uniformly in } \boldsymbol{\theta}. \tag{65}$$

Thus, by (55) and (65), we have derived the following approximation for the unique crossing point of $F_{\boldsymbol{\theta}}$ and $F_{\bar{\boldsymbol{\theta}}}$:

Proposition 3.5. *$t^*(\boldsymbol{\theta},\bar{\boldsymbol{\theta}}) \sim n\alpha\bar{\theta}$ as $n \to \infty$, uniformly in $\boldsymbol{\theta}$ for fixed $\bar{\theta}$ and α.* ||

The relation (7) in Section 1 follows immediately from this proposition.

The normal approximation to the distribution of $\sum Y_i$ suggests the conjecture that

$$t^*(\boldsymbol{\theta},\bar{\boldsymbol{\theta}}) = n\alpha\bar{\theta} + 0((n\alpha)^{1/2}) \quad \text{as } n\alpha \to \infty \tag{66}$$

uniformly in $\boldsymbol{\theta}$ for fixed $\bar{\theta}$. To support this conjecture, consider the behavior of $t^*(\boldsymbol{\theta},\bar{\boldsymbol{\theta}})$ when

$$\boldsymbol{\theta} = \boldsymbol{\theta}(n,k) \equiv (\underbrace{\frac{n}{k}\cdots,\frac{n}{k}}_{k},\underbrace{0,\ldots,0}_{n-k}), \tag{67}$$

where $k \in \{1,\ldots,n-1\}$. For each real number c, define

$$t_{n\alpha}(c) = n\alpha + c(n\alpha)^{1/2}. \tag{68}$$

Since $\bar{\theta} = 1$ for each k,

$$
\begin{aligned}
F_{\bar{\boldsymbol{\theta}}}(t_{n\alpha}(c)) &= P\left\{(n\alpha)^{-1/2}[G(n\alpha,1)-n\alpha] \leq c\right\} \\
&\to \Phi(c)
\end{aligned} \tag{69}
$$

as $n\alpha \to \infty$, where Φ denotes the standard normal distribution function.

Now consider the following two special cases:

(i) $k\alpha \to k_0 < \infty$: here, as $n\alpha \to \infty$,

$$
\begin{aligned}
F_{\boldsymbol{\theta}}(t_{n\alpha}(c)) &= P[G(k\alpha,1) \leq k\alpha + ck\alpha(n\alpha)^{-1/2}] \\
&\to P[G(k_0,1) \leq k_0].
\end{aligned} \tag{70}
$$

(ii) $k\alpha \to \infty$, $k/n \to \gamma \in [0,1)$: here,

$$
\begin{aligned}
F_{\boldsymbol{\theta}}(t_{n\alpha}(c)) &= P\left\{(k\alpha)^{-1/2}[G(k\alpha,1)-k\alpha] \leq c(k/n)^{1/2}\right\} \\
&\to \Phi(c\gamma^{1/2}).
\end{aligned} \tag{71}
$$

Thus, if we define

$$
c_0 = \begin{cases} \Phi^{-1}(P[G(k_0,1) \leq k_0]) & \text{in case (i),} \\ 0 & \text{in case (ii),} \end{cases} \tag{72}
$$

it follows from (69)–(71) that

$$
\lim_{n\alpha \to \infty}[F_{\boldsymbol{\theta}}(t_{n\alpha}(c)) - F_{\bar{\boldsymbol{\theta}}}(t_{n\alpha}(c))] \begin{cases} > 0 & \text{if } c < c_0 \\ = 0 & \text{if } c = c_0 \\ < 0 & \text{if } c > c_0. \end{cases} \tag{73}
$$

This implies that when (67) holds, then

$$
t^*(\boldsymbol{\theta}, \bar{\boldsymbol{\theta}}) = t_{n\alpha}(c_0) + o((n\alpha)^{1/2}) \quad \text{as} \quad n\alpha \to \infty \tag{74}
$$

in both special cases (i) and (ii), hence the conjecture (66) holds in these cases. In fact, in case (ii) it is true that

$$
t^*(\boldsymbol{\theta}, \bar{\boldsymbol{\theta}}) = n\alpha + o((n\alpha)^{1/2}) \quad \text{as} \quad n\alpha \to \infty.
$$

(This approach gives no information for the third special case where $\gamma = 1$, e.g., $k = n - 1$.)

The conjecture (66) likely follows from an appropriate uniform exponential bound for the tail probabilities of the weighted sums $\sum \theta_i Y_i$, but we do not pursue this here.

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