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Moment Decompositions of Measure Spaces

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Consider a Souslin space X and a countable set B of bounded Borel measurable real functions defined on X. The decomposition $M(X, B)$ of the set of all Borel probability measures on X induced by the equivalence relation that makes measures P and Q equivalent if $P(f) = Q(f)$ for all f in B is represented uniquely up to an isomorphism in the category of measure convex Souslin sets (Theorem 3). Theorem 2 is used to obtain a characterization of sets of uniqueness for the moment problem connected with the decomposition $M(X, B)$ (Theorem 4). The results presented here extend results proved in Štěpán (1994) for a compact metrizable space X and a countable family B of continuous bounded functions on X.

1. Bounded Countable Moment Decompositions of Measure Spaces. For a Hausdorff topological space X we shall denote by $\mathcal{P}(X)$, $\mathcal{B}(X)$ and $C(X)$ the space of Radon probability measures, the space of bounded Borel measurable and bounded continuous real functions defined on X, respectively. Given a nonempty (countable) set $B \in \mathcal{B}(X)$ we shall denote by $M(X, B)$ the quotient space obtained from $\mathcal{P}(X)$ by the equivalence relation

$$P = Q \bmod B \text{ if and only if } P(f) = Q(f),\ f \in B,\ P, Q \in \mathcal{P}(X),$$

where $P(f) = \int f dP$ and call it a *bounded (countable) moment decomposition of* $\mathcal{P}(X)$. Recall, moreover that if

$$\begin{aligned} T : X \to E \quad &\text{is a bounded Borel measurable map from } X \\ &\text{into a complete Hausdorff locally convex space } E, \end{aligned} \tag{1}$$

then the expectation of T with respect to a measure P in $\mathcal{P}(X)$ is a point $E_P(T)$ in E for which

$$x'(E_P(T)) = \int_X x'(T)dP \text{ holds for each } x' \text{ in the topological dual } E'.$$

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The existence and uniqueness of the expectation $E_P(T)$ (assuming the boundedness of T and completeness of E) follows easily by Proposition 1.1.3 (a), p. 16 in Winkler (1985) by observing that the expectation $E_P(T)$ is equivalently defined as the barycenter of the image probability measure $TP \in \mathcal{P}(E)$, i.e. $E_P(T) = b(TP)$. Further on, for such a pair (T, E) we shall denote by $E(T) : \mathcal{P}(X) \to E$ the map which sends a measure P in $\mathcal{P}(X)$ to the expectation $E_P(T)$ in E and let

$$S(X,T,E) = \{E_P(T), P \in \mathcal{P}(X)\} \subset E$$

and

$$M(T,X,E) = \{(E(T))^{-1}(s),\ s \in S(X,T,E)\}.$$

If $M(X,B) = M(X,T,E)$ for a bounded moment decomposition of $\mathcal{P}(X)$ ($B \in \mathcal{B}(X)$) and a pair (T, E) satisfying (1), then we call (T, E) a *generator of the decomposition* $M(X, B)$ and the set $S(X, T, E)$ its convex representation in E.

There is of course a very direct way to construct such a generator:

REMARK 1.

(a) Put $E = R^B$ and endow the space with the product topology (which makes it a locally convex complete space for which the one-dimensional projections $P_f : R^B \to R$ linearly generate its topological dual E').

(b) Define $T = T^B$ from X into $E = R^B$ by $T(x) = (f(x), f \in B)$. Then (obviously) T^B satisfies requirements (1) if

$$\begin{aligned} &B \text{ is either a countable subset in } \mathcal{B}(X) \\ &\quad \text{or an arbitrary subset in } \mathcal{C}(X), \\ &\quad T^B : X \to R^B \text{ being a continuous map in the latter case} \end{aligned} \tag{2}$$

and

$$P = Q \bmod B \quad \text{if and only if } E_P(T^B) = E_Q(T^B) \quad \text{for} \quad P, Q \in \mathcal{P}(X),$$
$$\text{whence} \quad M(X,B) = M(X,T^B,R^B).$$

Observe also that topological stability is achieved by the construction (a), (b), (2) when we assume that X is a Souslin space (a continuous image of a Polish space), since according to Lemma 16 in Schwartz (1973), p. 107 and Theorem 2 below we have that $T^B(X)$ and $S(X, T^B, R^B)$ are also Souslin spaces in this case. In fact we are able to prove the following statement (see also Theorem 4 in Štěpán (1994):

THEOREM 1. *Let X be a Souslin space. Then a decomposition D of $\mathcal{P}(X)$ is a bounded countable moment decomposition if and only if $D = M(X, T, E)$*

for a map

$$T : X \to E \text{ satisfying (1) such that } T(X) \text{ is a Souslin set.} \tag{3}$$

PROOF. In view of Remark 1 it suffices to consider a map T that satisfies (3) and to construct a countable set $B \in \mathcal{B}(X)$ such that $M(X,T,E) = M(B,X)$. By Theorem 2 below, $S = S(X,T,E)$ is a Souslin set and we can apply Propositions 3 and 4, p. 104, 105, in Schwartz (1973) to get a countable set $B' \subset E'$ that separates points in S. Putting $B = \{x' \circ T,\ x' \in B'\}$ we get a countable set in $\mathcal{B}(X)$ such that (for $P, Q \in \mathcal{P}(X)$)

$$P = Q \mod B \quad \text{if and only if } x'(E_P(T)) = x'(E_Q(T)),\ x' \in B'$$
$$\text{and hence if and only if} \quad E_P(T) = E_Q(T)$$

holds. ∎

Thus, the study of bounded countable moment decompositions $M(X,B)$ when X is a Souslin space is exactly the same problem as the study of $M(X,T,E)$ decompositions with generators (T,E) obeying the requirements (3). There is always a variety of ways to choose a suitable generator for a given moment decomposition.

EXAMPLE. The marginal and transshipment problem. Let $X = Y^2$ where Y is a Souslin space. Denote by π_1 and π_2 the coordinate projections from X onto $Y_1 = Y$ and $Y_2 = Y$, respectively, and by P_1 and P_2 the corresponding marginals $\pi_1 P \in \mathcal{P}(Y)$ and $\pi_2 P \in \mathcal{P}(Y)$ of a measure $P \in \mathcal{P}(X)$.

Consider decompositions $marg(X)$ and $trans(X)$ of $\mathcal{P}(X)$ into the equivalence classes of probability measures P with a fixed pair of marginals (P_1, P_2) and with a fixed difference of marginals $P_1 - P_2$, respectively. It is easy to see that both the marginal and transshipment decompositions are bounded countable moment decompositions since

$$marg(X) = M(X, bm(L)) \text{ where } bm(L) = \{f \circ \pi_1 + g \circ \pi_2, f, g \in L\},$$
$$trans(X) = M(X, bt(L)), \text{ where } bt(L) = \{f \circ \pi_1 - f \circ \pi_2, f \in L\}$$

for any L in $\mathcal{B}(Y)$ that separates measures in $\mathcal{P}(Y)$ (i.e. $P = Q \bmod L$ if and only if $P = Q,\ P, Q \in \mathcal{P}(Y)$). Obviously, such a set L may be chosen as a countable subset of $C(Y)$.

To construct nontrivial generators (T,E) satisfying (3) for $marg(X)$ and $trans(X)$ put $E = (C(Y))'$ and consider the space with its *weak** topology (i.e. $E' \cong C(Y)$) which is of course complete and locally convex as a closed set in $R^{C(Y)}$. Observe that the set $\mathcal{P}(Y)$ may be naturally embedded into E

(as a convex bounded Souslin subset). Denoting

$$T_1(y_1, y_2) = \varepsilon_{y_1},\ T_2(y_1, y_2) = \varepsilon_{y_2},\ T(y_1, y_2) = (\varepsilon_{y_1}, \varepsilon_{y_2}) \text{ for } (y_1, y_2) \in X,$$

where ε_y is the point measure supported by y, we get continuous bounded maps $T : X \to E \times E$ and $T_1 - T_2 : X \to E$ such that $T(X)$ and $(T_1 - T_2)(X)$ are obviously Souslin sets.

Straightforward calculations show that $E_P(T) = (P_1, P_2)$ and $E_P(T_1 - T_2) = P_1 - P_2$ for each $P \in \mathcal{P}(X)$. Hence, $marg(X) = M(X, T, E \times E)$ and $trans(X) = M(X, T_1 - T_2, E)$, where $(T, E \times E)$ and $(T_1 - T_2, E)$ are the generators satisfying (3). Using these generators we get the corresponding convex representations in the form $S(X, T, E \times E) = \{(P_1, P_2),\ P \in \mathcal{P}(X)\}$ and $S(X, T_1 - T_2, E) = \{P_1 - P_2, P \in \mathcal{P}(X)\}$.

2. Convex Representations of Moment Decompositions. We will denote the set of extremal points of a convex set S by ex$S$ and the closed convex hull of a set $H \subset E$ by $\overline{co}(H)$. Recall that a subset S of a locally convex space is called measure convex if for every $P \in \mathcal{P}(X)$ the barycenter $b(P)$ exists and belongs to S. From now on, we assume that the spaces $\mathcal{P}(\cdot)$ have the standard weak topology.

The following theorems provide topological properties of convex representations $S(X, T, E)$ attached to a countable bounded moment decomposition $M(X, B)$ via Theorem 1 (see also Theorems 1, 2 in Štěpán (1994)).

THEOREM 2. *Assume that X is a Souslin space and that the pair (T, E) is such that (3) holds. Then $S = S(X, T, E)$ is a bounded, measure convex (hence convex) Souslin set in E such that $ex(S) \subset T(X)$ and $S \subset \overline{co}(T(X))$ hold. If moreover X is a metrizable compact space and T is a continuous map then S is a compact metrizable space such that $S = \overline{co}(T(X))$ holds.*

PROOF OF THE THEOREM. Observe first that each Borel probability measure on X (resp. $T(X)$) belongs to $\mathcal{P}(X)$ (resp. $\mathcal{P}(T(X))$) by Theorem 10, p. 122 in Schwartz (1973), since both X and $T(X)$ are Souslin sets. Hence by Theorem 12, p. 39 in Schwartz (1973)

$$T \circ (\mathcal{P}(X)) = \mathcal{P}(T(X)) \text{ and } S = b \circ \mathcal{P}(T(X)) \tag{4}$$

(we let the symbol T also denote the map $P \to TP$ from $\mathcal{P}(X)$ onto $\mathcal{P}(T(X))$). Here TP denotes the measure in $\mathcal{P}(T(X))$ defined by $(TP)(B) = P(T^{-1}(B))$, $B \in \mathcal{B}(X)$. Since $\mathcal{P}(T(X))$ is obviously a Souslin set and the barycenter map $b : \mathcal{P}(T(X)) \to S$ is an affine continuous surjection by Proposition 1.1.3, p. 16 in Winkler (1985), it follows by (4) that S is also a Souslin set. According to 1.2.3 in Winkler (1985) it follows that S is contained in $\overline{co}(T(X))$, hence a bounded set in E.

Denote by $c : S \to \mathcal{P}(T(X))$ a universally measurable section $b : \mathcal{P}(T(X)) \to S$ ($b \circ c$ is the identity map on S), the existence of which follows by Theorem 13, p. 127 in Schwartz (1973). If P is a measure in $\mathcal{P}(S)$, then $Q = \int_S c(s)P(ds)$ is a well defined measure in $\mathcal{P}(T(X))$ such that $b(P) = b(Q) \in S$ (the existence of $b()$ follows again by Proposition 1.1.3, p. 16 in Winkler (1985) since we have already proved that S is a bounded set). Thus, S is a measure convex set. It follows by Corollary 1.5.5, p. 49 in Winkler (1985) that for each $x \in ex(S)$ the point measure ϵ_x is the only measure in $\mathcal{P}(S)$ with the barycenter x. Hence $ex(S) \subset T(X)$.

Finally, assume that X is a metrizable compact space and that $T : X \to E$ is a continuous map. Now it follows easily that $E(T) : \mathcal{P}(X) \to S$ is a continuous surjection. Hence S is a metrizable compact set (by Proposition 7.6.3, p. 126 in Semadeni (1971)) so that $\overline{co}(T(X)) \subset S$. ∎

The next definition and Theorem may be helpful when trying to establish the identity of two bounded countable moment decompositions. They show the role played by their convex representations.

Let S and S_1 be measure convex Souslin sets. A map $a : S \to S_1$ will be called *measure affine* if it is Borel measurable and if $a(b(P)) = b(aP)$ for each $P \in \mathcal{P}(S)$, where $aP \in \mathcal{P}(S_1)$ denotes the image of P under the map a. Note that a continuous map $a : S \to S_1$ is measure affine.

REMARK 2. A measure affine map $a : S \to S_1$ is affine, and if it is a bijection then $a^{-1} : S_1 \to S$ is measure affine too, according to Theorem 10, p. 122 in Schwartz (1973) and due to the fact that the equality $a^{-1}(b(aP)) = b(a^{-1}(aP))$ which holds for each P in $\mathcal{P}(S)$ implies that $a^{-1}(b(P_1)) = b(a^{-1}P_1)$ holds for each P_1 in $\mathcal{P}(S_1)$.

THEOREM 3. *Let X be a Souslin space and $(T, E), (T_1, E_1)$ pairs satisfying (3). Then*

(i) $M(X,T,E)$ is a finer decomposition than $M(X,T_1,E_1)$ if and only if there exists a measure affine surjection $a : S(X,T,E) \to S(X,T_1,E_1)$ such that $a \circ T = T_1$;

(ii) $M(X,T,E) = M(X,T_1,E_1)$ if and only if there exists a measure affine bijection $a : S(X,T,E) \to S(X,T_1,E_1)$ such that $a \circ T = T_1$.

REMARK 3. Note that(ii) says that if we have a fixed generator (T_o, E_o) of a bounded countable moment decomposition moment decomposition $M(X, B)$, then we may get all other generators (T, E) by putting $T = a \circ T_o$, where $a : S(X, T_o, E_o) \to S$ is a measure affine bijection and S is a bounded measure convex Souslin set in a complete locally convex space E.

Also note (see Theorem 2 in Štěpán (1994)) that if X is a compact metrizable set and if both T and T_1 are continuous maps, then (ii) reads as follows

$$\begin{aligned}&M(X,T,E) = M(X,T_1,E_1) \text{ if and only if}\\ &\text{there exists a continuous affine bijection}\\ &a: S(X,T,E) \to S(X,T_1,E_1) \text{ such that } a \circ T = T_1.\end{aligned}$$

Proof of the Theorem. Assume first that there is a map a with the properties stipulated by (i) and consider $P, Q \in P(X)$ such that $E_P(T) = E_Q(T)$. Then using the measure affinity of the map a we get

$$E_P(T_1) = b(a(TP)) = a(E_P(T)) = a(E_Q(T)) = E_Q(T_1),$$

hence $M(X,T,E)$ is a finer decomposition than $M(X,T_1,E_1)$. Assume that $M(X,T,E)$ is finer than $M(X,T_1,E_1)$. Denote the maps $E(T)$ and $E(T_1)$ by F and F_1, respectively. It is easy to see that putting

$$a(s) = F_1(F^{-1}(\{s\})) \text{ for } s \in S \ (S = S(X,T,E), S_1 = (X,T_1,E_1)) \tag{5}$$

we obtain a well-defined surjective map $a: S \to S_1$ such that $a \circ T = T_1$. This is because $F(\epsilon_x) = T(x)$ and $F_1(\epsilon_x) = T_1(x)$ hold for each $x \in X$.

It follows by Lemma 11 and 12, p. 106 in Schwartz (1973) and Theorem 1 that $graph(a) = \{(F(x), F_1(x)), x \in X\}$ is a Borel set in $S \times S_1$. Hence it follows from Corollary, p. 107 in Schwartz (1973) that a is a Borel measurable map. To verify that $a: S \to S_1$ is a measure affine map we use Theorem 13, p. 127 in Schwartz (1973) and again Theorem 1 to find universally measurable sections

$$\begin{aligned}&P_{(\cdot)}: S \to \mathcal{P}(X), Q_{(\cdot)}: S_1 \to \mathcal{P}(X) \text{ such that}\\ &E_{P_{(s)}}(T) = s \text{ and } E_{Q_{(s_1)}}(T_1) = s_1\\ &\text{hold for } s \in S \text{ and } s_1 \in S_1, \text{ respectively.}\end{aligned}$$

Thus, if P is a measure in $\mathcal{P}(S)$, then

$$m = \int_S P_{(s)} P(ds), \text{ and } n = \int_{S_1} Q_{(s_1)}(aP)(ds_1)$$

are measures in $\mathcal{P}(X)$ such that $E_m(T) = b(P)$, $E_n(T_1) = b(aP)$ and $n = \int_S Q_{(a(s))} P(ds)$ hold. Hence $E_{P_{(s)}}(T_1) = a(s) = E_{Q_{(a(s))}}(T_1)$ for $s \in S$ and therefore $E_m(T_1) = E_n(T_1) = b(aP)$. On the other hand, it follows from the assumption that $M(X,T,E)$ is a finer decomposition than $M(X,T_1,E_1)$ that $E_m(T_1) = E_{P_{(b(P))}}(T_1)$ holds. Finally, combining the above observations we get

$$b(aP) = E_m(T_1) = E_{P_{(b(P))}}(T_1) = a(bP)$$

which shows that the map a is measure affine.

To verify (ii) note that the relation (5) defines a bijective map $a : S \to S_1$ when $M(X,T,E)$ and $M(X,T_1,E_1)$ are identical decompositions; the equivalence (ii) is therefore a consequence of (i) and Remark 3. ∎

3. Sets of Uniqueness. Assume again that X is a Souslin space. A Borel set $D \subset X$ will be called a *set of uniqueness* for a bounded moment decomposition $M(X,B)$ if each member of the decomposition contains at most one measure $P \in \mathcal{P}(X)$ supported by D, i.e. $P \in \mathcal{P}(D)$. In other words D is a set of uniqueness if and only if $M(D,B) := M(D,B \mid D) = \mathcal{P}(D)$.

Observe that the concept of a set of uniqueness is crucial when one is trying to characterize extremal (simplicial) measures in moment problems connected with bounded moment decompositions $M(X,B)$. See Štěpán (1979), Linhartová (1991), Beneš (1992), Štěpán (1993) in the context of marginal and transshipment problems.

Choquet theory and Theorem 1 may be used to get a characterization of sets of uniqueness (see also Theorem 3 in Štěpán (1994)).

THEOREM 4. *Let a pair (T,E) satisfying (3) generate a bounded countable moment decomposition $M(X,B)$. Then a Borel set $D \subset X$ is a set of uniqueness for $M(X,B)$ if and only if*

(a) the restriction of the map T to the set D is an injective map and

(b) $S(D,T,E) := S(D,T \mid D,E)$ is a simplex with $exS(D,T,E) = T(D)$.

REMARK 4. According to Theorem 2, $S = S(D,T,E)$ is a bounded measure convex Souslin set in a locally convex space E. For such a set S the set of extremal points $ex(S)$ is universally measurable and S is a simplex if and only if every element $s \in S$ is the barycenter of one and only one measure $P \in \mathcal{P}(S)$ such that $P(ex\ S) = 1$ according to Proposition 1.4.2(b), (c), p. 39 in Winkler (1985). By Theorem 2 again, $S = S(D,T,E)$ is a metrizable compact convex set if $D \subset X$ is a compact metrizable set and T is a continuous map of X into E (i.e. bounded on D). Such a set S is a simplex if and only if the cone $C = R^+(S \times \{1\}) \subset E \times R$ is lattice in its own order, according to Theorem 23.6.5, p. 420 in Semadeni (1971). Theorem 4 thus provides a purely algebraic characterization of compact metrizable sets of uniqueness $D \subset X$ for bounded countable moment decompositions $M(X,B)$ where B is a set of continuous functions.

PROOF OF THEOREM 4. Observe first that

$$\begin{aligned}&D \text{ is a set of uniqueness for } M(X,B) \text{ if and only if}\\ &E(T) : \mathcal{P}(D) \to S(D) := S(D,T,E) \text{ is a bijective map}\end{aligned}$$

since $M(D,B) = M(D,T,E) := M(D,T \mid D,E)$. Hence

D is a set of uniqueness if and only if both
$T : \mathcal{P}(D) \to \mathcal{P}(TD)$ and $b : \mathcal{P}(T(D)) \to S(D)$ are bijective maps

and therefore, according to Remark 4, D is a set of uniqueness if and only if (a) and (b) hold. ∎

COROLLARY. *Let $M(X,T,E)$ be a decomposition generated by (T,E) satisfying (3) such that E has a finite dimension n. Then a Borel set $D \subset X$ is a set of uniqueness for $M(X,T,E)$ if and only if*

(a) the restriction of the map T to the set D is an injective map and

(b) $T(D)$ is a set of affinely independent points in E.

Hence, if D is a set of uniqueness then card$(D) \leq n+1$.

To derive this Corollary from Theorem 4 just observe that extremal points in a bounded measure affine Souslin simplex S are affinely independent according to Remark 4 and that card$(D) =$ card$(T(D)) \leq n+1$ when (a) and (b) hold.

Recalling the marginal and transshipment decompositions of $\mathcal{P}(X)$ introduced in the Example of Section 1, we may consider the generators $(T, E \times E)$ and $(T_1 - T_2, E)$, respectively, and try to apply Theorem 4 when searching for sets of uniqueness in $marg(X)$ and $trans(X)$. The map T is injective and $T(S) = ex(S(X,T,E \times E))$, so that a Borel set $D \subset X$ is a set of uniqueness for $marg(X)$ if and only if the set of all available pairs of marginals $\{(P_1,P_2), P \in \mathcal{P}(D)\}$, is a simplex. The map $T_1 - T_2$ is injective when avoiding the diagonal $dg(X)$ in X. Also, it is not difficult to show that

$$(T_1 - T_2)(X - dg(X)) = ex(S(X - dg(X), T_1 - T_2, E)).$$

Hence, the sets of uniqueness for $trans(X)$ are Borel sets D disjoint from the diagonal (cf. Beneš (1992)) such that $S(D, T_1 - T_2, E)$ is a simplex.

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