Variational formulas and explicit bounds of Poincaré-type inequalities for one-dimensional processes

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Abstract

This paper serves as a quick and elementary overview of the recent progress on a large class of Poincaré-type inequalities in dimension one. The explicit criteria for the inequalities, the variational formulas and explicit bounds of the corresponding constants in the inequalities are presented. As typical applications, the Nash inequalities and logarithmic Sobolev inequalities are examined.

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1 Introduction

The one-dimensional processes in this paper mean either one-dimensional diffusions or birth-death Markov processes. Let us begin with diffusions.

Let $L = a(x)\mathrm{d}^2/\mathrm{d}x^2 + b(x)\mathrm{d}/\mathrm{d}x$ be an elliptic operator on an interval $(0, D)$ $(D \le \infty)$ with Dirichlet boundary at 0 and Neumann boundary at D when $D < \infty$, where a and b are Borel measurable functions and a is positive everywhere. Set $C(x) = \int_0^x b/a$, here and in what follows, the Lebesgue measure $\mathrm{d}x$ is often omitted. Throughout the paper, assume that

$$Z := \int_0^D e^C/a < \infty. \tag{1.0}$$

Hence, $\mathrm{d}\mu := a^{-1}e^C\mathrm{d}x$ is a finite measure, which is crucial in the paper. We are interested in the first Poincaré inequality

$$\|f\|^2 := \int_0^D f^2\mathrm{d}\mu \le A\int_0^D f'^2 e^C := AD(f), \quad f \in \mathbb{C}_d[0, D],\ f(0) = 0, \tag{1.1}$$

where $\mathbb{C}_d$ is the set of all continuous functions, differentiable almost everywhere and having compact supports. When $D = \infty$, one should replace $[0, D]$ by $[0, D)$ but we will not mention again in what follows. Next, we are also interested in the second Poincaré inequality

$$\|f - \pi(f)\|^2 := \int_0^D (f - \pi(f))^2\mathrm{d}\mu \le \overline{A}D(f) \qquad f \in \mathbb{C}_d[0, D], \tag{1.2}$$

where $\pi(f) = \mu(f)/Z = \int f\mathrm{d}\mu/Z$. To save the notations, we use the same A (resp., $\overline{A}$) to denote the optimal constant in (1.1) (resp., (1.2)).

The aim of the study on these inequalities is looking for a criterion under which (1.1) (resp., (1.2)) holds, i.e., the optimal constant $A < \infty$ (resp., $\overline{A} < \infty$),

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and for the estimations of A (resp., $\overline{A}$). The reason why we are restricted in dimension one is looking for some explicit criteria and explicit estimates. Actually, we have dual variational formulas for the upper and lower bounds of these constants. Such explicit story does not exist in higher dimensional situation.

Next, replacing the L^2-norm on the right-hand sides of (1.1) and (1.2) with a general norm $\|\cdot\|_{\mathbb{B}}$ in a suitable Banach space (the details are delayed to §3), respectively, we obtain the following Poincaré-type inequalities

$$\|f^2\|_{\mathbb{B}} \le A_{\mathbb{B}} D(f), \quad f \in \mathscr{C}_d[0,D],\ f(0)=0. \tag{1.3}$$

$$\|(f-\pi(f))^2\| \le \overline{A}_{\mathbb{B}} D(f), \qquad f \in \mathscr{C}_d[0,D]. \tag{1.4}$$

For which, it is natural to study the same problems as above. The main purpose of this paper is to answer these problems. By using this general setup, we are able to handle with the following Nash inequalities[23]

$$\|f-\pi(f)\|^{2+4/\nu} \le A_N D(f)\|f\|_1^{4/\nu} \tag{1.5}$$

in the case of $\nu > 2$, and the logarithmic Sobolev inequality[18]:

$$\mathrm{Ent}(f^2) := \int_0^D f^2 \log\frac{f^2}{\pi(f^2)}\mathrm{d}\mu \le A_{LS} D(f). \tag{1.6}$$

To see the importance of these inequalities, define the first Dirichlet eigenvalue λ_0 and the first Neumann eigenvalue λ_1, respectively, as follows.

$$\begin{aligned}\lambda_0 &= \inf\{D(f): f\in C^1(0,D)\cap C[0,D],\ f(0)=0,\ \pi(f^2)=1\},\\ \lambda_1 &= \inf\{D(f): f\in C^1(0,D)\cap C[0,D],\ \pi(f)=0,\ \pi(f^2)=1\}.\end{aligned} \tag{1.7}$$

Then, it is clear that $\lambda_0 = 1/A$ and $\lambda_1 = 1/\overline{A}$. Furthermore, it is known that

$$\begin{aligned}&\text{The second Poincaré inequality} \Longleftrightarrow \mathrm{Var}(P_t f) \le \mathrm{Var}(f)\, e^{-2\lambda_1 t}.\\ &\text{Logarithmic Sobolev inequality} \Longleftrightarrow \mathrm{Ent}(P_t f) \le \mathrm{Ent}(f)\, e^{-2t/A_{LS}},\\ &\text{Nash inequality} \Longleftrightarrow \mathrm{Var}(P_t f) \le C\|f\|_1^2\, t^{-\nu},\end{aligned} \tag{1.8}$$

where $\|f\|_r$ is the $L^r(\mu)$-norm (cf., [8], [13], [18] and references within). It is clear now that the convergence in the first line is also equivalent to the exponential ergodicity for any reversible Markov processes with density (cf. [10]), i.e., $\|P_t(x,\cdot)-\pi\|_{\mathrm{Var}} \le C(x)e^{-\varepsilon t}$ for some constants $\varepsilon > 0$ and $C(x)$, where $P_t(x,\cdot)$ is the transition probability. The study on the existence of the equilibrium π and on the speed of convergence to equilibrium, by Bhattacharya and his cooperators, consists a fundamental contribution in the field. See for instance [2]–[6] and references within. The second line in (1.8) is correct for diffusions but incorrect in the discrete situation. In general, one has to replace "$\Longleftrightarrow$" by "$\Longrightarrow$". Here are three examples which distinguish the different inequalities.

	Ergodicity	2nd Poincaré	LogS	L^1-exp.	Nash
$b(x)=0$ $a(x)=x^\gamma$	$\gamma>1$	$\gamma\ge 2$	$\gamma>2$	$\gamma>2$	$\gamma>2$
$b(x)=0$ $a=x^2\log^\gamma x$	$\checkmark$	$\gamma\ge 0$	$\gamma\ge 1$	$\gamma>1$	$\times$
$a(x)=1$ $b(x)=-b$	$\checkmark$	$\checkmark$	$\times$	$\times$	$\times$

Table 1.1, Examples: Diffusions on $[0,\infty)$

Here in the first line, "LogS" means the logarithmic Sobolev inequality, "L^1-exp." means the L^1-exponential convergence which will not be discussed in this paper. "$\surd$" means always true and "$\times$" means never true, with respect to the parameters. Once known the criteria presented in this paper, it is easy to check Table 1.1 except the L^1-exponential convergence.

The remainder of the paper is organized as follows. In the next section, we review the criteria for (1.1) and (1.2), the dual variational formulas and explicit estimates of A and $\overline{A}$. Then, we extend partially these results to Banach spaces first for the Dirichlet case and then for the Neumann one. For a very general setup of Banach spaces, the resulting conclusions are still rather satisfactory. Next, we specify the results to Orlicz spaces and finally apply to the Nash inequalities and logarithmic Sobolev inequality.

Since each topic discussed subsequently has a long history and contains a large number of publications, it is impossible to collect in the present paper a complete list of references. We emphasize on recent progress and related references only. For the applications to the higher dimensional case and much more results, the readers are urged to refer to the original papers listed in References, and the informal book [13], in particular.

2 Ordinary Poincaré inequalities

In this section, we introduce the criteria for (1.1) and (1.2), the dual variational formulas and explicit estimates of A and $\overline{A}$.

To state the main results, we need some notations. Write $x \wedge y = \min\{x, y\}$ and similarly, $x \vee y = \max\{x, y\}$. Define

$$\begin{aligned}
\mathcal{F} &= \big\{f \in C[0,D] \cap C^1(0,D) : f(0)=0,\ f'|_{(0,D)} > 0\big\},\\
\widetilde{\mathcal{F}} &= \big\{f \in C[0,D] : f(0)=0, \text{there exists} x_0 \in (0,D] \text{so that}\\
&\qquad f = f(\cdot \wedge x_0),\ f \in C^1(0,x_0) \text{and} f'|_{(0,x_0)} > 0\big\},\\
\mathcal{F}' &= \big\{f \in C[0,D] : f(0)=0, f|_{(0,D)} > 0\big\},\\
\widetilde{\mathcal{F}}' &= \big\{f \in C[0,D] : f(0)=0, \text{there exists} x_0 \in (0,D] \text{so that}\\
&\qquad f = f(\cdot \wedge x_0) \text{and} f|_{(0,x_0)} > 0\big\}.
\end{aligned} \tag{2.1}$$

Here the sets $\mathcal{F}$ and $\mathcal{F}'$ are essential, they are used, respectively, to define below the operators of single and double integrals, and are used for the upper bounds. The sets $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{F}}'$ are less essential, simply the modifications of $\mathcal{F}$ and $\mathcal{F}'$, respectively, to avoid the integrability problem, and are used for the lower bounds. Define

$$\begin{aligned}
I(f)(x) &= \frac{e^{-C(x)}}{f'(x)} \int_x^D \big[fe^C/a\big](u)\mathrm{d}u, \qquad f \in \mathcal{F},\\
II(f)(x) &= \frac{1}{f(x)} \int_0^x \mathrm{d}y e^{-C(y)} \int_y^D \big[fe^C/a\big](u)\mathrm{d}u, \qquad f \in \mathcal{F}'.
\end{aligned} \tag{2.2}$$

The next result is taken from [12; Theorems 1.1 and 1.2]. The word "dual" below means that the upper and lower bounds are interchangeable if one exchanges the orders of "sup" and "inf" with a slight modification of the set $\mathcal{F}$ (resp., $\mathcal{F}'$) of test functions.

Theorem 2.1. *Let* (1.0) *hold. Define* $\varphi(x)=\int_0^x e^{-C}$ *and* $B=\sup_{x\in(0,D)}\varphi(x)\int_x^D \frac{e^C}{a}$. *Then, we have the following assertions.*

(1) *Explicit criterion:* $A<\infty$ *iff* $B<\infty$.

(2) *Dual variational formulas:*

$$\begin{aligned} A &\le \inf_{f\in\mathscr{F}'}\sup_{x\in(0,D)} I\!I(f)(x)=\inf_{f\in\mathscr{F}}\sup_{x\in(0,D)} I(f)(x),\\ A &\ge \sup_{f\in\widetilde{\mathscr{F}}'}\inf_{x\in(0,D)} I\!I(f)(x)=\sup_{f\in\widetilde{\mathscr{F}}}\inf_{x\in(0,D)} I(f)(x). \end{aligned}\tag{2.3}$$

The two inequalities all become equalities whenever both a *and* b *are continuous on* $[0,D]$.

(3) *Approximating procedure and explicit bounds:*

(a) *Define* $f_1=\sqrt{\varphi}$, $f_n=f_{n-1}I\!I(f_{n-1})$ *and* $D_n=\sup_{x\in(0,D)} I\!I(f_n)\,(x)$. *Then* D_n *is decreasing in* n *and* $A\le D_n\le 4B$ *for all* $n\ge 1$.

(b) *Fix* $x_0\in(0,D)$. *Define*
$f_1^{(x_0)}=\varphi(\cdot\wedge x_0)$, $\quad f_n^{(x_0)}=f_{n-1}^{(x_0)}(\cdot\wedge x_0)I\!I\big(f_{n-1}^{(x_0)}(\cdot\wedge x_0)\big)$
and $C_n=\sup_{x_0\in(0,D)}\inf_{x\in(0,D)} I\!I\big(f_n^{(x_0)}(\cdot\wedge x_0)\big)(x)$. *Then* C_n *is increasing in* n *and* $A\ge C_n\ge B$ *for all* $n\ge 1$.

We mention that the explicit estimates "$B\le A\le 4B$" were obtained previously in the study on the weighted Hardy's inequality by [22].

We now turn to study $\overline{A}$, for which it is natural to assume that

$$\int_0^D e^{-C(s)}\mathrm{d}s\int_0^s a(u)^{-1}e^{C(u)}\mathrm{d}u=\infty.\tag{2.4}$$

Theorem 2.2. *Let* (1.0) *and* (2.4) *hold and set* $\bar f=f-\pi(f)$. *Then, we have the following assertions.*

(1) *Explicit criterion:* $\overline{A}<\infty$ *iff* $B<\infty$, *where* B *is given by Theorem* 1.1.

(2) *Dual variational formulas:*

$$\sup_{f\in\widetilde{\mathscr{F}}}\inf_{x\in(0,D)} I(\bar f)(x)\le\overline{A}\le\inf_{f\in\mathscr{F}}\sup_{x\in(0,D)} I(\bar f)(x).\tag{2.5}$$

The two inequalities all become equalities whenever both a *and* b *are continuous on* $[0,D]$.

(3) *Approximating procedure and explicit bounds:*

(a) *Define* $f_1=\sqrt{\varphi}$, $f_n=\bar f_{n-1}I\!I(\bar f_{n-1})$ *and* $\overline{D}_n=\sup_{x\in(0,D)} I\!I(\bar f_n)\,(x)$. *Then* $\overline{A}\le\overline{D}_n\le 4B$ *for all* $n\ge 1$.

(b) *Fix* $x_0\in(0,D)$. *Define*
$f_1^{(x_0)}=\varphi(\cdot\wedge x_0)$, $\quad f_n^{(x_0)}=\bar f_{n-1}^{(x_0)}(\cdot\wedge x_0)I\!I\big(\bar f_{n-1}^{(x_0)}(\cdot\wedge x_0)\big)$
and $\overline{C}_n=\sup_{x_0\in(0,D)}\inf_{x\in(0,D)} I\!I\big(\bar f_n^{(x_0)}(\cdot\wedge x_0)\big)(x)$. *Then* $\overline{A}\ge\overline{C}_n$ *for all* $n\ge 2$. *By convention,* $1/0=\infty$.

Part (1) of the theorem is taken from [11; Theorem 3.7]. The upper bound in (2.5) is due to [16]. The other parts are taken from [12; Theorems 1.3 and 1.4].

Finally, we consider inequality (1.2) on a general interval $(p, q)\,(-\infty \le p < q \le \infty)$. When p (resp., q) is finite, at which the Neumann boundary condition is endowed. We adopt a splitting technique. The intuitive idea goes as follows: Since the eigenfunction corresponding to $\overline{A}$, if exists, must change signs, it should vanish somewhere in the present continuous situation, say θ for instance. Thus, it is natural to divide the interval (p, q) into two parts: (p, θ) and (θ, q). Then, one compares $\overline{A}$ with the optimal constants in the inequality (1.1), denoted by $A_{1\theta}$ and $A_{2\theta}$, respectively, on (θ, q) and (p, θ) having the common Dirichlet boundary at θ. Actually, we do not care about the existence of the vanishing point θ. Such θ is unknown, even if it exists. In practice, we regard θ as a reference point and then apply an optimization procedure with respect to θ. We now redefine $C(x) = \int_\theta^x b/a$. Again, since it is in the ergodic situation, we assume the following (non-explosive) conditions:

$$
\begin{aligned}
&Z_{1\theta} := \int_\theta^q e^C/a < \infty, \qquad Z_{2\theta} := \int_p^\theta e^C/a < \infty \\
&\int_p^\theta e^{-C(s)}\mathrm{d}s \int_s^\theta e^C/a = \infty \ \text{ if } p = -\infty \quad \text{and} \\
&\int_\theta^q e^{-C(s)}\mathrm{d}s \int_\theta^s e^C/a = \infty \ \text{ if } q = \infty
\end{aligned}
\tag{2.6}
$$

for some (equivalently, all) $\theta \in (p, q)$. Corresponding to the intervals (θ, q) and (p, θ), respectively, we have constants $B_{1\theta}$ and $B_{2\theta}$, given by Theorem 1.1.

Theorem 2.3. *Let* (2.6) *hold. Then, we have*

(1) $\inf_{\theta\in(p,q)} \big(A_{1\theta} \wedge A_{2\theta}\big) \le \overline{A} \le \sup_{\theta\in(p,q)} \big(A_{1\theta} \vee A_{2\theta}\big)$.

(2) *Let θ be the medium of μ, then* $\big(A_{1\theta} \vee A_{2\theta}\big)/2 \le \overline{A} \le A_{1\theta} \vee A_{2\theta}$.

In particular, $\overline{A} < \infty$ *iff* $B_{1\theta} \vee B_{2\theta} < \infty$.

Comparing the variational formulas (2.3) and (2.5) with the classical variational formulas given in (1.7), one sees that there are no common points. This explains why the new formulas (2.3) and (2.5) have not appeared before. The key here is the discover of the formulas rather than their proofs, which are usually simple due to the advantage of dimension one. As an illustration, here we present parts of the proofs.

Proof of the upper bound in (2.5).

Originally, the assertion was proved in [16] by using the coupling methods. Here we adopt the analytic proof given in [9].

Let $g \in C[0, D] \cap C^1(0, D)$, $\pi(g) = 0$ and $\pi(g^2) = 1$. Then, for every $f \in \mathcal{F}$

with $\pi(f) \geq 0$, we have

$$\begin{aligned}
1 &= \frac{1}{2}\int_0^D \pi(\mathrm{d}x)\pi(\mathrm{d}y)[g(y)-g(x)]^2 \\
&= \int_{\{x\leq y\}} \pi(\mathrm{d}x)\pi(\mathrm{d}y)\bigg(\int_x^y \frac{g'(u)\sqrt{f'(u)}}{\sqrt{f'(u)}}\mathrm{d}u\bigg)^2 \\
&\leq \int_{\{x\leq y\}} \pi(\mathrm{d}x)\pi(\mathrm{d}y)\int_x^y \frac{g'(u)^2}{f'(u)}\mathrm{d}u\int_x^y f'(\xi)\mathrm{d}\xi \\
&\qquad \text{(by Cauchy-Schwarz inequality)} \\
&= \int_{\{x\leq y\}} \pi(\mathrm{d}x)\pi(\mathrm{d}y)\int_x^y g'(u)^2 e^{C(u)}\frac{e^{-C(u)}}{f'(u)}\mathrm{d}u\big[f(y)-f(x)\big] \\
&= \int_0^D a(u)g'(u)^2\pi(\mathrm{d}u)\frac{Ze^{-C(u)}}{f'(u)}\int_0^u \pi(\mathrm{d}x)\int_u^D \pi(\mathrm{d}y)\,\big[f(y)-f(x)\big] \\
&\leq D(g)\sup_{u\in(0,D)}\frac{Ze^{-C(u)}}{f'(u)}\int_0^u \pi(\mathrm{d}x)\int_u^D \pi(\mathrm{d}y)\,\big[f(y)-f(x)\big] \\
&\leq D(g)\sup_{x\in(0,D)} I(f)(x) \quad \big(\text{since} \pi(f)\geq 0\big).
\end{aligned}$$

Thus, $D(g)^{-1} \leq \sup_{x\in(0,D)} I(\bar{f})(x)$, and so

$$\overline{A} = \sup_{g:\,\pi(g)=0,\ \pi(g^2)=1} D(g)^{-1} \leq \sup_{x\in(0,D)} I(\bar{f})(x).$$

This gives us the required assertion:

$$\overline{A} \leq \inf_{f\in\mathscr{F}}\sup_{x\in(0,D)} I(\bar{f})(x).$$

The proof of the sign of the equality holds for continuous a and b needs more work, since it requires some more precise properties of the corresponding eigenfunctions. □

Proof of the explicit upper bound "$A \leq 4B$".

As mentioned before, this result is due to [22]. Here we adopt the proof given in [11], as an illustration of the power of our variational formulas.

Recall that $B = \sup_{x\in(0,D)}\int_0^x e^{-C}\int_x^D e^C/a$. By using the integration by parts formula, it follows that

$$\begin{aligned}
\int_x^D \frac{\sqrt{\varphi}\,e^C}{a} &= -\int_x^D \sqrt{\varphi}\,\mathrm{d}\bigg(\int_\bullet^D \frac{e^C}{a}\bigg) \\
&\leq \frac{B}{\sqrt{\varphi(x)}} + \frac{B}{2}\int_x^D \frac{\varphi'}{\varphi^{3/2}} \leq \frac{2B}{\sqrt{\varphi(x)}}.
\end{aligned} \tag{2.1}$$

Hence

$$I\big(\sqrt{\varphi}\big)(x) = \frac{e^{-C(x)}}{\big(\sqrt{\varphi}\big)'(x)}\int_x^D \frac{\sqrt{\varphi}e^C}{a} \leq \frac{e^{-C(x)}\sqrt{\varphi(x)}}{(1/2)e^{-C(x)}}\cdot\frac{2B}{\sqrt{\varphi(x)}} = 4B$$

as required. □

3 Extension; Banach spaces

Starting from this section, we introduce the recent results obtained in [14] and [15], but we will not point out time by time subsequently.

In this section, we study the Poincaré-type inequality (1.3). Clearly, the Banach spaces used here can not be completely arbitrary since we are dealing with a topic of hard mathematics. ¿From now on, let $(\mathbb{B}, \|\cdot\|_{\mathbb{B}}, \mu)$ be a Banach space of functions $f: [0, D] \to \mathbb{R}$ satisfying the following conditions:

$$\begin{aligned} &(1) \quad 1 \in \mathbb{B};\\ &(2) \quad \mathbb{B} \text{is ideal: If} h \in \mathbb{B} \text{and} |f| \le |h|, \text{then} f \in \mathbb{B};\\ &(3) \quad \|f\|_{\mathbb{B}} = \sup_{g \in \mathcal{G}} \int_0^D |f| g \mathrm{d}\mu,\\ &(4) \quad \mathcal{G} \ni g_0 \text{with} \inf g_0 > 0, \end{aligned} \tag{3.1}$$

where $\mathcal{G}$ is a fixed set, to be specified case by case later, of non-negative functions on $[0, D]$. The first two conditions mean that $\mathbb{B}$ is rich enough and the last one means that $\mathcal{G}$ is not trivial, it contains at least one strictly positive function. The third condition is essential in this paper, which means that the norm $\|\cdot\|_{\mathbb{B}}$ has a "dual" representation. A typical example of the Banach space is $\mathbb{B} = L^r(\mu)$, then $\mathcal{G} =$ the unit ball in $L_+^{r'}(\mu)$, $1/r + 1/r' = 1$.

The optimal constant $\overline{A}$ in (1.3) can be expressed as a variational formula as follows.

$$A_{\mathbb{B}} = \sup\left\{\frac{\|f^2\|_{\mathbb{B}}}{D(f)} : f \in \mathbb{C}_{\mathrm{d}}[0, D], f(0) = 0,\ 0 < D(f) < \infty\right\}. \tag{3.2}$$

Clearly, this formula is powerful mainly for the lower bounds of $\overline{A}$. However, the upper bounds are more useful in practice but much harder to handle. Fortunately, for which we have quite complete results.

Define $\varphi(x) = \int_0^x e^{-C}$ as before and let

$$\begin{aligned} &B_{\mathbb{B}} = \sup_{x \in (0,D)} \varphi(x) \big\|I_{(x,D)}\big\|_{\mathbb{B}}, \qquad C_{\mathbb{B}} = \sup_{x \in (0,D)} \frac{\big\|\varphi(x \wedge \cdot)^2\big\|_{\mathbb{B}}}{\varphi(x)},\\ &D_{\mathbb{B}} = \sup_{x \in (0,D)} \frac{\big\|\sqrt{\varphi}\, \varphi(x \wedge \cdot)^2\big\|_{\mathbb{B}}}{\sqrt{\varphi(x)}}. \end{aligned} \tag{3.3}$$

Theorem 3.1. *Let* (1.0) *and* (3.1) *hold. Then we have the following assertions.*

(1) *Explicit criterion:* $A_{\mathbb{B}} < \infty$ *iff* $B_{\mathbb{B}} < \infty$.

(2) *Variational formulas for the upper bounds:*

$$\begin{aligned} A_{\mathbb{B}} &\le \inf_{f \in \mathcal{F}'} \sup_{x \in (0,D)} f(x)^{-1} \big\|f \varphi(x \wedge \cdot)\big\|_{\mathbb{B}}\\ &\le \inf_{f \in \mathcal{F}} \sup_{x \in (0,D)} \frac{e^{-C(x)}}{f'(x)} \|f I_{(x,D)}\|_{\mathbb{B}}. \end{aligned} \tag{3.4}$$

(3) *Approximating procedure and explicit bounds: Let $B_{\mathbb{B}} < \infty$. Define $f_0 = \sqrt{\varphi}$, $f_n(x) = \|f_{n-1}\varphi(x \wedge \cdot)\|_{\mathbb{B}}$ and $D_{\mathbb{B}}(n) = \sup_{x\in(0,D)} f_n/f_{n-1}$ for $n \geq 1$. Then, $D_{\mathbb{B}}(n)$ is decreasing in n and*

$$B_{\mathbb{B}} \leq C_{\mathbb{B}} \leq A_{\mathbb{B}} \leq D_{\mathbb{B}}(n) \leq D_{\mathbb{B}} \leq 4B_{\mathbb{B}} \tag{3.5}$$

for all $n \geq 1$.

We are now going to sketch the proof of the second variational formula in (3.4), from which the explicit upper bound $A_{\mathbb{B}} \leq 4B_{\mathbb{B}}$ follows immediately, as we did at the end of the last section. The explicit estimates "$B_{\mathbb{B}} \leq A_{\mathbb{B}} \leq 4B_{\mathbb{B}}$" were previously obtained in [7] in terms of the weighted Hardy's inequality [22]. The lower bounds follows easily from (3.2).

Sketch of the proof of the second variational formula in (3.4).

The starting point is the variational formula for A (cf. (2.3)):

$$A \leq \inf_{f\in\mathcal{F}} \sup_{x\in(0,D)} \frac{e^{-C(x)}}{f'(x)} \int_x^D \frac{fe^C}{a} = \inf_{f\in\mathcal{F}} \sup_{x\in(0,D)} \frac{e^{-C(x)}}{f'(x)} \int_x^D f\mathrm{d}\mu.$$

Fix $g > 0$ and introduce a transform as follows.

$$b \to b/g, \qquad a \to a/g > 0. \tag{3.6}$$

Under which, $C(x)$ is transformed into

$$C_g(x) = \int_0^x \frac{b/g}{a/g} = C(x).$$

This means that the function C is invariant of the transform, and so is the Dirichlet form $D(f)$. The left-hand side of (1.1) is changed into

$$\int_0^D f^2 g e^C/a = \int_0^D f^2 g\mathrm{d}\mu.$$

At the same time, the constant A is changed into

$$A_g \leq \inf_{f\in\mathcal{F}} \sup_{x\in(0,D)} \frac{e^{-C(x)}}{f'(x)} \int_x^D fg\mathrm{d}\mu.$$

Making supremum with respect to $g \in \mathcal{G}$, the left-hand side becomes

$$\sup_{g\in\mathcal{G}} \int_0^D f^2 g\mathrm{d}\mu = \|f^2\|_{\mathbb{B}}$$

and the constant becomes

$$\begin{aligned} A_{\mathbb{B}} = \sup_g A_g &\leq \sup_g \inf_f \sup_x \frac{e^{-C(x)}}{f'(x)} \int_x^D fg\mathrm{d}\mu \leq \inf_f \sup_g \sup_x \\ &= \inf_f \sup_x \frac{e^{-C(x)}}{f'(x)} \sup_g \int_0^D fI_{(x,D)} g\mathrm{d}\mu. \\ &= \inf_f \sup_x \frac{e^{-C(x)}}{f'(x)} \|fI_{(x,D)}\|_{\mathbb{B}}. \end{aligned}$$

We are done! Of course, more details are required for completing the proof. For instance, one may use $g + 1/n$ instead of g to avoid the condition "$g > 0$" and then pass limit. □

The lucky point in the proof is that "sup inf ≤ inf sup", which goes to the correct direction. However, we do not know at the moment how to generalize the dual variational formula for lower bounds, given in the second line of (2.3), to the general Banach spaces, since the same procedure goes to the opposite direction.

4 Neumann Case; Orlicz Spaces

In the Neumann case, the boundary condition becomes $f'(0) = 0$, rather than $f(0) = 0$. Then $\lambda_0 = 0$ is trivial. Hence, we study λ_1 (called spectral gap of L), that is the inequality (1.2). We now consider its generalization (1.4). Naturally, one may play the same game as in the last section extending (2.5) to the Banach spaces. However, it does not work this time. Note that on the left-hand side of (1.4), the term $\pi(f)$ is not invariant under the transform (3.6). Moreover, since $\pi(\bar{f}) = 0$, it is easy to check that for each fixed $f \in \mathscr{F}$, $I(\bar{f})(x)$ is positive for all $x \in (0, D)$. But this property is no longer true when $\mathrm{d}\mu$ is replaced by $g\mathrm{d}\mu$. Our goal is to adopt the splitting technique explained in Section 2.

Let $\theta \in (p, q)$ be a reference point and let $A_{\mathbb{B}}^{k\theta}$, $B_{\mathbb{B}}^{k\theta}$, $C_{\mathbb{B}}^{k\theta}$, $D_{\mathbb{B}}^{k\theta}$ $(k = 1, 2)$ be the constants defined in (3.2) and (3.3) corresponding to the intervals (θ, q) and (p, θ), respectively. By Theorem 3.1, we have

$$B_{\mathbb{B}}^{k\theta} \le C_{\mathbb{B}}^{k\theta} \le A_{\mathbb{B}}^{k\theta} \le D_{\mathbb{B}}^{k\theta} \le 4B_{\mathbb{B}}^{k\theta}, \qquad k = 1, 2.$$

Theorem 4.1. *Let* (2.6) *and* (3.1) *hold. Then, we have the following assertions.*

(1) *Explicit criterion:* $\overline{A}_{\mathbb{B}} < \infty$ *iff* $B_{\mathbb{B}}^{1\theta} \vee B_{\mathbb{B}}^{2\theta} < \infty$.

(2) *Estimates:*

$$\max\Big\{\frac{1}{2}\big(A_{\mathbb{B}}^{1\theta} \wedge A_{\mathbb{B}}^{2\theta}\big),\ K_\theta\big(A_{\mathbb{B}}^{1\theta} \vee A_{\mathbb{B}}^{2\theta}\big)\Big\} \le \overline{A}_{\mathbb{B}} \le A_{\mathbb{B}}^{1\theta} \vee A_{\mathbb{B}}^{2\theta},$$

where K_θ *is a constant.*

It is the position to consider briefly the discrete case, i.e., the birth-death process. Let b_i $(i \ge 0)$ be the birth rates and a_i $(i \ge 1)$ be the death rates of the process. Define

$$\mu_0 = 1, \quad \mu_n = \frac{b_0 \cdots b_{n-1}}{a_1 \cdots a_n}, \quad Z = \sum_{n=0}^{\infty} \mu_n, \quad \pi_n = \frac{\mu_n}{Z},\ n \ge 1.$$

Consider a Banach space $(\mathbb{B}, \|\cdot\|_{\mathbb{B}}, \mu)$ of functions $E := \{0, 1, 2, \cdots\} \to \mathbb{R}$ satisfying (3.1). Define

$$\varphi_i = \sum_{j=1}^{i} \frac{1}{\mu_j a_j},\ i \ge 1; \qquad B_{\mathbb{B}} = \sup_{i \ge 1} \varphi_i \big\|I_{\{i, i+1, \cdots\}}\big\|_{\mathbb{B}}.$$

Clearly, the inequalities (1.3) and (1.4) are meaningful with a slight modification.

Theorem 4.2. *Consider birth-death processes with state space E. Assume that $Z < \infty$.*

(1) *Explicit criterion for (1.3): $A_{\mathbb{B}} < \infty$ iff $B_{\mathbb{B}} < \infty$.*

(2) *Explicit bounds for $A_{\mathbb{B}}$: $B_{\mathbb{B}} \le A_{\mathbb{B}} \le 4B_{\mathbb{B}}$.*

(3) *Explicit criterion for (1.4): Let the birth-death process be non-explosive:*

$$\sum_{i=0}^{\infty} \frac{1}{\mu_i b_i} \sum_{j=0}^{i} \mu_j = \infty. \tag{4.1}$$

Then $\overline{A}_{\mathbb{B}} < \infty$ iff $B_{\mathbb{B}} < \infty$.

(4) *Estimates for $\overline{A}_{\mathbb{B}}$: Let $E_1 = \{1, 2, \cdots\}$ and let c_1 and c_2 be two constants such that $|\pi(f)| \le c_1 \|f\|_{\mathbb{B}}$ and $|\pi(f I_{E_1})| \le c_2 \|f I_{E_1}\|_{\mathbb{B}}$ for all $f \in \mathbb{B}$. Then,*

$$\begin{aligned} &\max\left\{\|1\|_{\mathbb{B}}^{-1},\ \left(1 - \sqrt{c_2(1-\pi_0)\|1\|_{\mathbb{B}}}\,\right)^2\right\} A_{\mathbb{B}} \\ &\quad \le \overline{A}_{\mathbb{B}} \le \left(1 + \sqrt{c_1 \|1\|_{\mathbb{B}}}\,\right)^2 A_{\mathbb{B}}. \end{aligned} \tag{4.2}$$

Similarly, one can handle the birth-death processes on $\mathbb{Z}$.

An interesting point here is that the first lower bound in (4.2) is meaningful only in the discrete situation.

Orlicz spaces. The results obtained so far can be specialized to Orlicz spaces. The idea also goes back to [7]. A function $\Phi : \mathbb{R} \to \mathbb{R}$ is called an N-function if it is non-negative, continuous, convex, even (i.e., $\Phi(-x) = \Phi(x)$) and satisfies the following conditions:

$$\Phi(x) = 0 \ \text{ iff } \ x = 0, \qquad \lim_{x \to 0} \Phi(x)/x = 0, \qquad \lim_{x \to \infty} \Phi(x)/x = \infty.$$

In what follows, we assume the following growth condition (or Δ_2-condition) for Φ:

$$\sup_{x \gg 1} \Phi(2x)/\Phi(x) < \infty \ \left(\iff \sup_{x \gg 1} x\Phi'_-(x)/\Phi(x) < \infty\right),$$

where Φ'_- is the left derivative of Φ. Corresponding to each N-function, we have a complementary N-function:

$$\Phi_c(y) := \sup\{x|y| - \Phi(x) : x \ge 0\}, \qquad y \in \mathbb{R}.$$

Alternatively, let φ_c be the inverse function of Φ'_-, then $\Phi_c(y) = \int_0^{|y|} \varphi_c$ (cf. [24]).

Given an N-function and a finite measure μ on $E := (p, q) \subset \mathbb{R}$, define an Orlicz space as follows:

$$L^{\Phi}(\mu) = \left\{f\ (E \to \mathbb{R}) : \int_E \Phi(f) \mathrm{d}\mu < \infty\right\}, \quad \|f\|_{\Phi} = \sup_{g \in \mathcal{G}} \int_E |f| g \mathrm{d}\mu, \tag{4.3}$$

where $\mathcal{G} = \Big\{ \} \geq \prime : \int_{\mathcal{E}} \oplus_{\rfloor}(\})\mathrm{d}\mu \leq \infty \Big\}$, which is the set of non-negative functions in the unit ball of $L^{\Phi_c}(\mu)$. Under Δ_2-condition, $(L^{\Phi}(\mu), \|\cdot\|_{\Phi}, \mu)$ is a Banach space. For this, the Δ_2-condition is indeed necessary. Clearly, $L^{\Phi}(\mu) \ni 1$ and is ideal. Obviously, $(L^{\Phi}(\mu), \|\cdot\|_{\Phi}, \mu)$ satisfies condition (3.1) and so we have the following result.

Corollary 4.1. *For any N-function Φ satisfying the growth condition, if* (1.0) *(resp.,* (2.6)*) holds, then Theorem* 3.1 *(resp.,* 4.1*) is available for the Orlicz space $(L^{\Phi}(\mu), \|\cdot\|_{\Phi}, \mu)$.*

5 Nash inequality and Sobolev-type inequality

It is known that when $\nu > 2$, the Nash inequality (1.5):

$$\|f - \pi(f)\|^{2+4/\nu} \leq A_N D(f) \|f\|_1^{4/\nu}$$

is equivalent to the Sobolev-type inequality:

$$\|f - \pi(f)\|_{\nu/(\nu-2)}^2 \leq A_S D(f),$$

where $\|\cdot\|_r$ is the $L^r(\mu)$-norm. Refer to [1], [8] and [26]. This leads to the use of the Orlicz space $L^{\Phi}(\mu)$ with $\Phi(x) = |x|^r/r$, $r = \nu/(\nu-2)$:

$$\big\|(f - \pi(f))^2\big\|_{\Phi} \leq \overline{A}_\nu D(f). \tag{5.1}$$

The results in this section were obtained in [19], based on the weighted Hardy's inequalities.

Define $C(x) = \int_\theta^x b/a$, $\mu(m,n) = \int_m^n e^C/a$ and

$$\varphi^{1\theta}(x) = \int_\theta^x e^{-C} \qquad B_\nu^{1\theta} = \sup_{x>\theta} \varphi^{1\theta}(x)\, \mu(x,q)^{(\nu-2)/\nu},$$

$$\varphi^{2\theta}(x) = \int_x^\theta e^{-C} \qquad B_\nu^{2\theta} = \sup_{x<\theta} \varphi^{2\theta}(x)\, \mu(p,x)^{(\nu-2)/\nu}.$$

Here $B_\nu^{k\theta}$ $(k=1,2)$ is specified from $B_{\mathbb{B}}$ given in (3.3) with $\mathbb{B} = L^{\Phi}((\theta,q),\mu)$ or $\mathbb{B} = L^{\Phi}((p,\theta),\mu)$, since $\|\cdot\|_{\Phi} = (r')^{1/r'}\|\cdot\|_r$, $1/r + 1/r' = 1$.

Theorem 5.1. *Let* (2.6) *hold and $\nu > 2$.*

(1) *Explicit criterion: Nash inequality (equivalently,* (5.1)*) holds on (p,q) iff $B_\nu^{1\theta} \vee B_\nu^{2\theta} < \infty$.*

(2) *Explicit bounds:*

$$\max\left\{\frac{1}{2}\big(B_\nu^{1\theta} \wedge B_\nu^{2\theta}\big),\ \left[1 - \left(\frac{Z_{1\theta} \vee Z_{2\theta}}{Z_{1\theta} + Z_{2\theta}}\right)^{1/2+1/\nu}\right]^2 \big(B_\nu^{1\theta} \vee B_\nu^{2\theta}\big)\right\} \tag{5.2}$$
$$\leq \overline{A}_\nu \leq 4\big(B_\nu^{1\theta} \vee B_\nu^{2\theta}\big).$$

In particular, if θ is the medium of μ, then

$$\big[1 - (1/2)^{1/2+1/\nu}\big]^2 \big(B_\nu^{1\theta} \vee B_\nu^{2\theta}\big) \leq \overline{A}_\nu \leq 4\big(B_\nu^{1\theta} \vee B_\nu^{2\theta}\big).$$

We now consider birth-death processes with state space $\{0,1,2,\cdots\}$. Define

$$\varphi_i=\sum_{j=1}^{i}\frac{1}{\mu_j a_j},\ i\geq 1;\qquad B_\nu=\sup_{i\geq 1}\varphi_i\bigg(\sum_{j=i}^{\infty}\mu_j\bigg)^{(\nu-2)/\nu}.$$

Theorem 5.2. *For birth-death processes, let* (4.1) *hold and assume that* $Z<\infty$. *Then, we have*

$$\max\left\{\left(\frac{2}{\nu Z^{\nu/2-1}}\right)^{2/\nu},\ \left[1-\left(\frac{Z-1}{Z}\right)^{1/2+1/\nu}\right]^2\right\}B_\nu\leq \overline{A}_\nu\leq 16B_\nu. \tag{5.3}$$

Hence, when $\nu>2$, *the Nash inequality holds iff* $B_\nu<\infty$.

6 Logarithmic Sobolev inequality

The starting point of the study is the following observation.

$$\frac{2}{5}\big\|(f-\pi(f))^2\big\|_\Phi\leq \mathcal{L}(f)\leq\frac{51}{20}\big\|(f-\pi(f))^2\big\|_\Phi, \tag{6.1}$$

where $\Phi(x)=|x|\log(1+|x|)$, $\mathcal{L}(f)=\sup_{c\in\mathbb{R}}\mathrm{Ent}\big((f+c)^2\big)$ and $\mathrm{Ent}(f)=\int_{\mathbb{R}}f\log\frac{f}{\pi(f)}\mathrm{d}\mu$, $f\geq 0$. Refer to [7] and [17; page 247], which go back to [25]. A modification of the coefficients is made in [12]. The observation leads to the use of the Orlicz space $\mathbb{B}=L^\Phi(\mu)$ with $\Phi(x)=|x|\log(1+|x|)$. The results in this section were obtained in [20], based again on the weighted Hardy's inequalities. Refer also to [21] for the related study.

Define

$$\begin{aligned}
&C(x)=\int_\theta^x e^C,\qquad \mu(m,n)=\int_m^n e^C/a;\\
&\varphi^{1\theta}(x)=\int_\theta^x e^{-C},\qquad \varphi^{2\theta}(x)=\int_x^\theta e^{-C};\\
&M(x)=x\left[\frac{2}{1+\sqrt{1+4x}}+\log\left(1+\frac{1+\sqrt{1+4x}}{2x}\right)\right];\\
&B_\Phi^{1\theta}=\sup_{x\in(\theta,q)}\varphi^{1\theta}(x)M(\mu(\theta,x)),\qquad B_\Phi^{2\theta}=\sup_{x\in(p,\theta)}\varphi^{2\theta}(x)M(\mu(x,\theta)).
\end{aligned}\tag{6.2}$$

Again, here $B_\Phi^{k\theta}$ $(k=1,2)$ is specified from $B_{\mathbb{B}}$ given in (3.3).

Theorem 6.1. *Let* (2.6) *hold.*

(1) *Explicit criterion: The logarithmic Sobolev inequality on* $(p,q)\subset\mathbb{R}$ *holds iff*

$$\begin{aligned}
&\sup_{x\in(\theta,q)}\mu(x,q)\log\frac{1}{\mu(x,q)}\int_\theta^x e^{-C}<\infty\quad\text{and}\\
&\sup_{x\in(p,\theta)}\mu(p,x)\log\frac{1}{\mu(p,x)}\int_x^\theta e^{-C}<\infty
\end{aligned}\tag{6.3}$$

hold for some (equivalently, all) $\theta\in(p,q)$.

(2) *Explicit bounds: Let $\bar\theta$ be the root of $B_\Phi^{1\theta} = B_\Phi^{2\theta}$, $\theta \in [p, q]$. Then, we have*

$$\frac{1}{5} B_\Phi^{1\bar\theta} \le A_{LS} \le \frac{51}{5} B_\Phi^{1\bar\theta}. \tag{6.4}$$

By a translation if necessary, assume that $\theta = 0$ is the medium of μ. Then, we have

$$\frac{(\sqrt{2}-1)^2}{5}\big(B_\Phi^{1\theta} \vee B_\Phi^{2\theta}\big) \le A_{LS} \le \frac{51}{5}\big(B_\Phi^{1\theta} \vee B_\Phi^{2\theta}\big). \tag{6.5}$$

We now consider birth-death processes with state space $\{0, 1, 2, \cdots\}$. Define

$$\varphi_i = \sum_{j=1}^{i} \frac{1}{\mu_j a_j}, \; i \ge 1; \qquad B_\Phi = \sup_{i \ge 1} \varphi_i M(\mu[i, \infty)),$$

where $\mu[i, \infty) = \sum_{j \ge i} \mu_j$ and $M(x)$ is defined in (6.2).

Theorem 6.2. *For birth-death processes, let* (4.1) *hold and assume that $Z < \infty$. Then, we have*

$$\begin{aligned} &\frac{2}{5} \max\left\{\frac{\sqrt{4Z+1}-1}{2}, \left(1 - \frac{Z_1 \Psi^{-1}(Z_1^{-1})}{Z \Psi^{-1}(Z^{-1})}\right)^2\right\} B_\Phi \\ &\quad \le A_{LS} \le \frac{51}{5}\Big(1 + \Psi^{-1}(Z^{-1})\Big)^2 B_\Phi, \end{aligned}$$

where $Z_1 = Z - 1$ and Ψ^{-1} is the inverse function of Ψ: $\Psi(x) = x^2 \log(1 + x^2)$. In particular, $A_{LS} < \infty$ iff

$$\sup_{i \ge 1} \varphi_i \, \mu[i, \infty) \log \frac{1}{\mu[i, \infty)} < \infty.$$

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